Wilson line-based action for gluodynamics: quantum corrections

In collaboration with P. Kotko, A. Stasto.

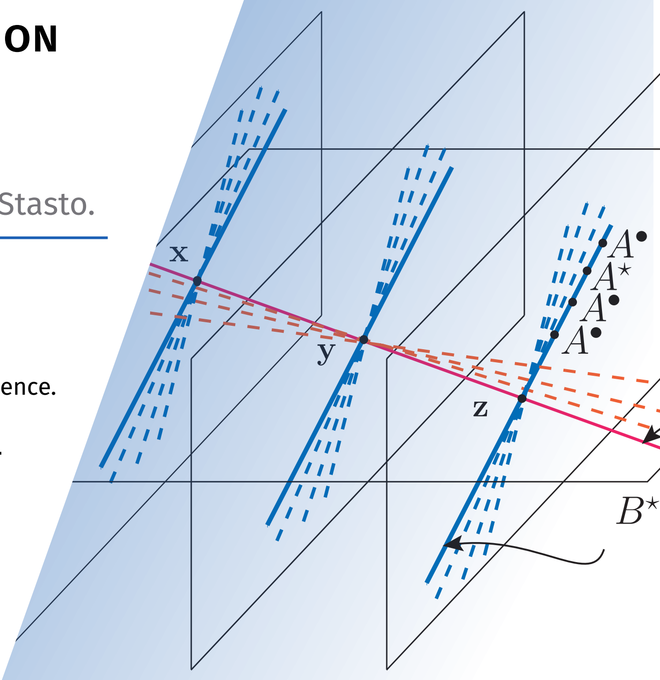
Hiren Kakkad
kakkad@agh.edu.pl

NCN GRANT 2021/41/N/ST2/02956
Faculty of Physics and Applied Computer Science.

XXX Cracow Epiphany - 2024

January 12, 2023

AGH University of Science and Technology



AGENDA

1. Highlights from P. Kotko's talk
2. Tree Amplitudes
3. Quantum Corrections
4. Summary
5. Outlook

1. Highlights from P. Kotko's talk

Z-field Action

Structure of the new action

$$S\left[Z^{\bullet}, Z^{\star}\right] = \int dx^{+} \Big\{ \mathcal{L}_{-+} + \mathcal{L}_{--++} + \mathcal{L}_{--+++} + \mathcal{L}_{--++++} + \dots$$

$$+ \mathcal{L}_{---++} + \mathcal{L}_{---+++} + \mathcal{L}_{---++++} + \dots$$

$$\vdots$$

$$+ \mathcal{L}_{---\cdots-++} + \mathcal{L}_{---\cdots-+++} + \mathcal{L}_{---\cdots-++++} + \dots \Big\}$$

- No three point interaction vertices.
- No $(+\cdots+)$, $(-\cdots-)$, $(-+\cdots+)$, and $(-\cdots-+)$ vertices.

Z-field Action

Structure of the new action

$$S[Z^{\bullet}, Z^{\star}] = \int dx^{+} \Big\{ \mathcal{L}_{-+} + \boxed{\mathcal{L}_{--++} + \mathcal{L}_{--+++} + \mathcal{L}_{--++++} + \dots \quad \text{MHV}}$$

$$+ \mathcal{L}_{---++} + \mathcal{L}_{---+++} + \mathcal{L}_{---++++} + \dots$$

$$\vdots$$

$$+ \mathcal{L}_{---\cdots-++} + \mathcal{L}_{---\cdots-+++} + \mathcal{L}_{---\cdots-++++} + \dots \Big\}$$

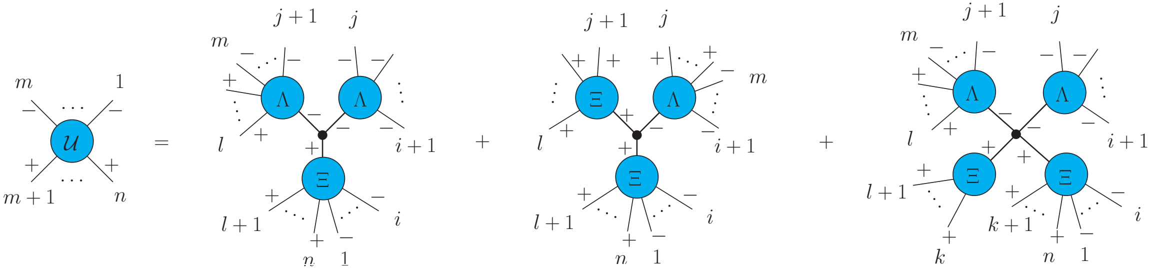
$\overline{\text{MHV}}$

- No three point interaction vertices.
- No $(+\cdots+)$, $(-\cdots-)$, $(-+\cdots+)$, and $(-\cdots-+)$ vertices.

Z-field Action

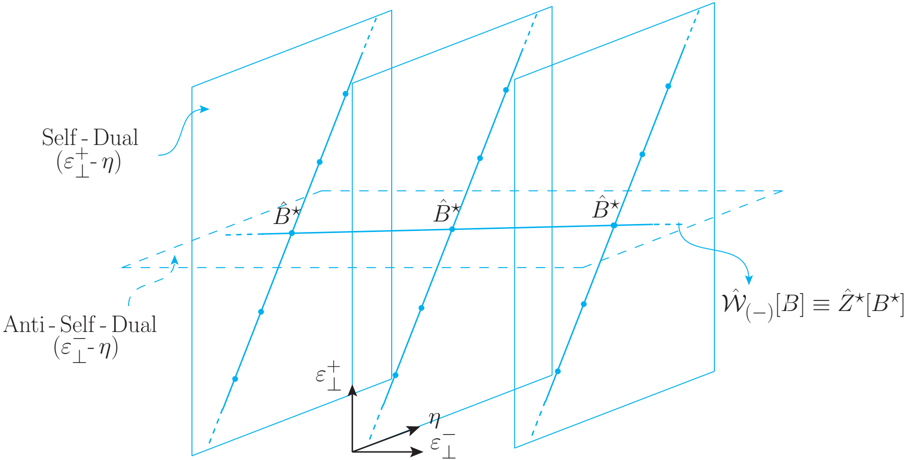
Interaction Vertices:

$$\mathcal{L}_{\underbrace{-\,\cdots\,-}_{m}\underbrace{+\,\cdots\,+}_{n-m}} = \int d^3\mathbf{p}_1 \dots d^3\mathbf{p}_n \, \mathcal{U}^{b_1\dots b_n}_{-\cdots-+\cdots+}(\mathbf{p}_1,\cdots\mathbf{p}_n) \prod_{i=1}^{m} Z^{\star}_{b_i}(x^+;\mathbf{p}_i) \prod_{j=1}^{n-m} Z^{\bullet}_{b_j}(x^+;\mathbf{p}_j)\,.$$

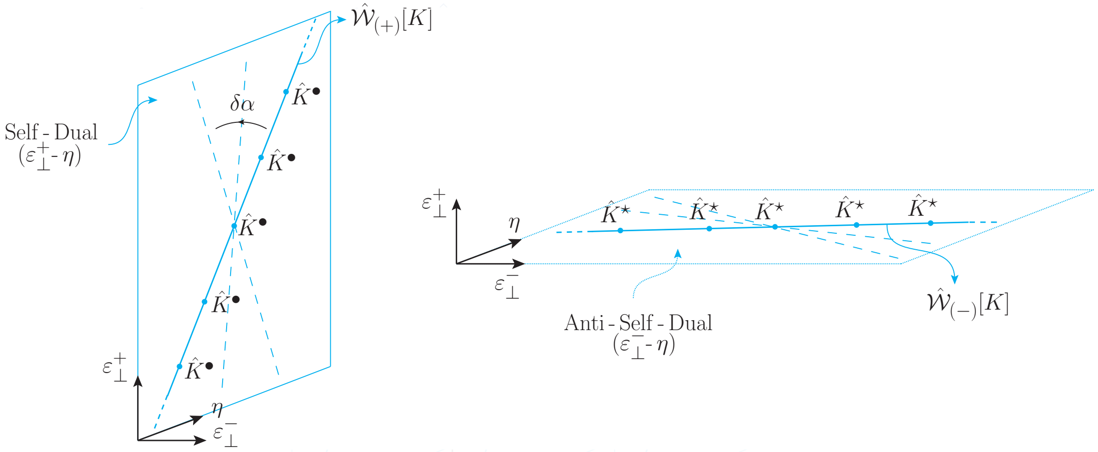


Z-field Wilson Line

Geometrical Representation of $\hat{Z}^{\star}[A^{\bullet}, A^{\star}]$:



Geometric Representation of $\mathcal{W}^a_{(\pm)}[K](x)$:



2. Tree Amplitudes

Tree Amplitudes: Delannoy Numbers

No. of diagrams

$A^{+-}_{n,m}$	2	3	4	5	...
2	1	1	1	1	MHV
3	1	3	5	7	NMHV
4	1	5	13		NNMHV
5	1	7			NNNMHV

Delannoy Numbers

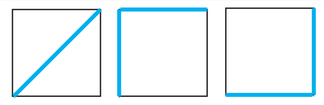
(n,m)	0	1	2	3
0	1	1	1	1
1	1	3	5	7
2	1	5	13	25
3	1	7	25	63

Delannoy Numbers

Definition:

The Delannoy numbers D(a,b) are the number of lattice paths from (0,0) to (b,a) in which only east (1, 0), north (0, 1), and northeast (1, 1) steps are allowed.

Example:



The 3 paths to go from $(0,0)$ to $(1,1)$ using just three moves: east $\rightarrow$, north $\uparrow$, and north-east $\nearrow$.

Tree Amplitudes: Delannoy Numbers

No. of diagrams

$A^{+-}_{n,m}$	2	3	4	5	...
2	1	1	1	1	MHV
3	1	3	5	7	NMHV
4	1	5	13		NNMHV
5	1	7			NNNMHV

Delannoy Numbers

(n,m)	0	1	2	3
0	1	1	1	1
1	1	3	5	7
2	1	5	13	25
3	1	7	25	63

The correspondence

$$\# \, A^{(n+m-4\,\mathrm{Tree})}_{\underbrace{-\cdots-}_{m-2}\underbrace{+\cdots+}_{n-2}} = D(n,m) = \sum_{i=0}^{\min(n,m)} \binom{m}{i}\binom{n+m-i}{m} = \sum_{i=0}^{\min(n,m)} 2^i \binom{m}{i}\binom{n}{i}.$$

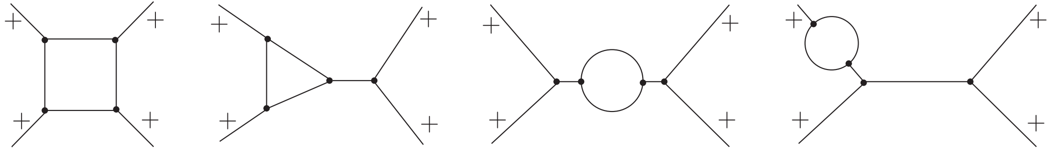
3. Quantum Corrections

Quantum Corrections - Loops

Immediate issues

- Missing terms in the action.
- One-loop amplitudes with all external gluons having positive helicities $(++\cdots+)$ cannot be calculated within the Z-field action, since every vertex has at least two $+$ and two $-$ helicity fields.

All plus one-loop gluon amplitudes: Each vertex is $(++-)$



Quantum Corrections - Loops

Immediate issues

- Missing terms in the action.
- One-loop amplitudes with all external gluons having positive helicities $(+ + \cdots +)$ cannot be calculated within the Z-field action, since every vertex has at least two $+$ and two $-$ helicity fields.

All plus one-loop gluon amplitudes is a rational function of the spinor products:

$$\mathcal{A}_n^{\text{one-loop}}(+ + \cdots +) = g^n \sum_{1 \leq i < j < k < l \leq n} \frac{\langle ij \rangle [jk] \langle kl \rangle [li]}{\langle 1n \rangle \langle n\,(n-1) \rangle \langle (n-1)\,(n-2) \rangle \ldots \langle 21 \rangle}\,.$$

[Z. Bern, D. A. Kosower - 1992][Z. Kunszt, A. Signer, Z. Trocsanyi - 1994]

Quantum Corrections - Loops

The technique of Effective Action to systematically develop loop amplitudes.

One-Loop Effective action

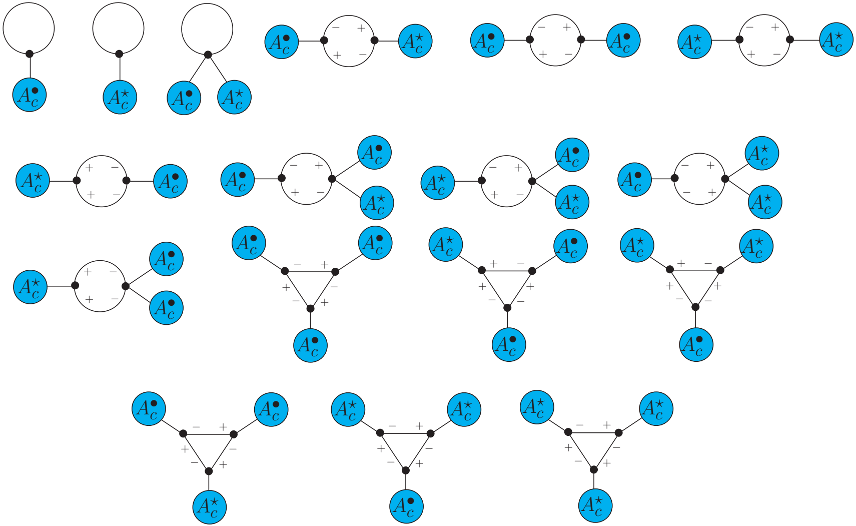
[Bryce S DeWitt - 1981]

$$\mathcal{Z}_{\mathrm{YM}}[J] = \int [dA]\, e^{i\left(S_{\mathrm{YM}}[A] + \int d^4x\, \mathrm{Tr} \hat{J}_i(x) \hat{A}^i(x)\right)} ,$$

- Expand the action, up to second order in fields, around the classical solution $A_c[J]$.
- The higher order terms are necessary for corrections beyond one loop.
- The linear term vanishes due to the classical equations of motion, whereas the integration over the quadratic term gives

$$\mathcal{Z}_{\mathrm{YM}}[J] \approx \exp\left\{ i S_{\mathrm{YM}}[A_c] + i \int d^4x\, \mathrm{Tr}\, \hat{J}_i(x)\, \hat{A}_c^i(x) - \frac{1}{2} \mathrm{Tr} \ln \left(\frac{\delta^2 S_{\mathrm{YM}}[A_c]}{\delta \hat{A}^i(x) \delta \hat{A}^j(y)} \right) \right\}$$

The diagrammatic content, up to 3-point, of the log term:



Loops: Straightforward approach

$$\mathcal{Z}_{\mathrm{YM}}[J] \approx \exp\left\{ i\, S_{\mathrm{YM}}[A_c] + i \int d^4x \,\mathrm{Tr}\, \hat{J}_i(x)\, \hat{A}^i_c(x) - \frac{1}{2}\mathrm{Tr}\ln\left(\frac{\delta^2 S_{\mathrm{YM}}[A_c]}{\delta \hat{A}^i(x)\delta \hat{A}^j(y)}\right)\right\}$$

$$\downarrow$$

$$\mathcal{Z}[J] \approx \exp\left\{ i\, S[Z_c] + i \int d^4x \,\mathrm{Tr}\, \hat{J}_i(x)\, \hat{A}^i_c[Z_c](x) - \frac{1}{2}\mathrm{Tr}\ln\left(\frac{\delta^2 S_{\mathrm{YM}}[A_c[Z_c]]}{\delta \hat{A}^i(x)\delta \hat{A}^j(y)}\right)\right\}$$

Loops: Straightforward approach

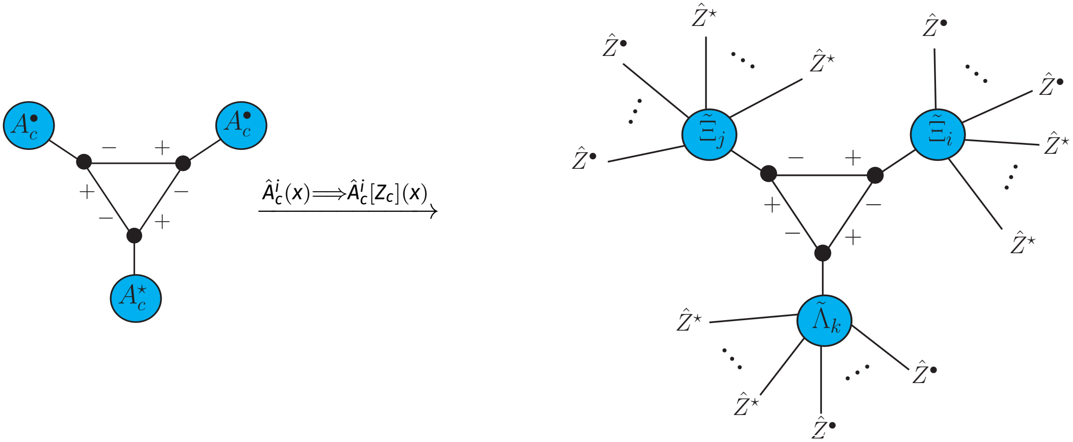
$$\mathcal{Z}_{\mathrm{YM}}[J] \approx \exp\left\{ i\, \underbrace{S_{\mathrm{YM}}[A_c]} + i\int d^4x\, \mathrm{Tr}\, \hat{J}_i(x)\, \hat{A}^i_c(x) \underbrace{-\frac{1}{2}\mathrm{Tr}\ln\left(\frac{\delta^2 S_{\mathrm{YM}}[A_c]}{\delta\hat{A}^i(x)\delta\hat{A}^j(y)}\right)} \right\}$$

$$\hat{A}^i_c(x) \Longrightarrow \hat{A}^i_c[Z_c](x)$$

$$\mathcal{Z}[J] \approx \exp\left\{ i\, \overbrace{S[Z_c]} + i\int d^4x\, \mathrm{Tr}\, \hat{J}_i(x)\, \hat{A}^i_c[Z_c](x) \overbrace{-\frac{1}{2}\mathrm{Tr}\ln\left(\frac{\delta^2 S_{\mathrm{YM}}[A_c[Z_c]]}{\delta\hat{A}^i(x)\delta\hat{A}^j(y)}\right)} \right\}$$

Loops: Straightforward approach

Change in the log term: Diagrammatically



Loops: Straightforward approach

$$\mathcal{Z}[J] \approx \exp\left\{ i\,S[Z_c] + i\int d^4x\,\mathrm{Tr}\,\hat{J}_i(x)\,\hat{A}^i_c[Z_c](x) - \frac{1}{2}\mathrm{Tr}\ln\left(\frac{\delta^2 S_{\mathrm{YM}}[A_c[Z_c]]}{\delta\hat{A}^i(x)\delta\hat{A}^j(y)}\right)\right\}$$

Tested

- Computed four point $(++++)$, $(+++-)$,$(+---)$, and $(----)$ one-loop amplitudes.
- Used the same approach to successfully develop loops in the MHV action.

[H. Kakkad, P. Kotko, A. Stasto, 2022]

Quantum corrections to the MHV action

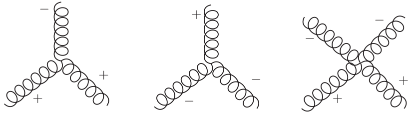
$$S_{\mathrm{YM}}\left[A^\bullet, A^\star\right] = \int dx^+ \left(\mathcal{L}_{+-} + \mathcal{L}_{++-} + \mathcal{L}_{+--} + \mathcal{L}_{++--}\right) .$$

Transformation:

$$\{A^\bullet, A^\star\} \to \{B^\bullet, B^\star\}$$

$$\mathcal{L}_{+-} + \mathcal{L}_{++-} \longrightarrow \mathcal{L}_{+-}$$

Interaction vertices



MHV action: Action with MHV vertices

$$S_{\mathrm{MHV}}\left[B^\bullet, B^\star\right] = \int dx^+ \left(\mathcal{L}_{+-} + \mathcal{L}_{--+} + \cdots + \mathcal{L}_{--+\cdots+} + \dots\right)$$

MHV action: Quantum corrections

$$\mathcal{Z}[J] \approx \exp\left\{ i\, S_{\mathrm{MHV}}[B_c] + i \int d^4x \, \mathrm{Tr}\, \hat{J}_i(x)\, \hat{A}^i_c[B_c](x) - \frac{1}{2}\mathrm{Tr}\ln\left(\frac{\delta^2 S_{\mathrm{YM}}[A_c[B_c]]}{\delta \hat{A}^i(x)\delta \hat{A}^j(y)}\right)\right\}$$

Loops: Straightforward approach

$$\mathcal{Z}[J] \approx \exp\left\{ i\,S[Z_c] + i \int d^4x \,\mathrm{Tr}\, \hat{J}_i(x)\, \hat{A}^i_c[Z_c](x) - \frac{1}{2}\mathrm{Tr}\ln\left(\frac{\delta^2 S_{\mathrm{YM}}[A_c[Z_c]]}{\delta\hat{A}^i(x)\delta\hat{A}^j(y)}\right)\right\}$$

Merit

- Systematic approach to efficiently compute pure gluonic amplitudes up to one-loop.

Drawback

- The interaction vertices of our new action are not explicit in the loop.

Loops with Z-field action vertcies

Previous approach

$$\mathcal{Z}_{\mathrm{YM}}[J] \xrightarrow{\mathrm{OLEA}} \mathcal{Z}_{\mathrm{YM}}^{\mathrm{one-loop}}[A_c[J]] \xrightarrow{\hat{A}^{\bullet}[Z^{\bullet},Z^{\star}],\,\hat{A}^{\star}[Z^{\bullet},Z^{\star}]} \mathcal{Z}^{\mathrm{one-loop}}[Z_c[J]]$$

Change: Reverse the order of operations

$$\mathcal{Z}[J] = \int [dA]\, e^{i\left(S_{\mathrm{YM}}[A]+\int d^4x\,\mathrm{Tr}\hat{J}_j(x)\hat{A}^j(x)\right)} \longrightarrow \int [dZ]\, e^{i\left(S[Z]+\int d^4x\,\mathrm{Tr}\hat{J}_j(x)\hat{A}^j[Z](x)\right)}$$

Notice:

$$\int d^4x\,\mathrm{Tr}\hat{J}_j(x)\hat{A}^j(x) \longrightarrow \int d^4x\,\mathrm{Tr}\hat{J}_j(x)\hat{A}^j[Z](x)$$

Loops with Z-field action vertcies

One-loop approximation:

$$S[Z] + \int d^4x \, \mathrm{Tr} \hat{J}_i(x)\hat{A}^i[Z](x) = S[Z_c] + \int d^4x \, \mathrm{Tr} \hat{J}_i(x)\hat{A}^i[Z_c](x)$$
$$+ \int d^4x \, \mathrm{Tr} \left(\hat{Z}^i(x) - \hat{Z}^i_c(x)\right) \left(\frac{\delta S[Z_c]}{\delta \hat{Z}^i(x)} + \int d^4y \, \hat{J}_k(y) \frac{\delta \hat{A}^k[Z_c](y)}{\delta \hat{Z}^i(x)} \right)$$
$$+ \frac{1}{2} \int d^4x d^4y \, \mathrm{Tr} \left(\hat{Z}^i(x) - \hat{Z}^i_c(x)\right) \left(\frac{\delta^2 S[Z_c]}{\delta \hat{Z}^i(x) \delta \hat{Z}^j(y)} \right.$$
$$\left. + \int d^4z \, \hat{J}_k(z) \frac{\delta^2 \hat{A}^k[Z_c](z)}{\delta \hat{Z}^i(x) \delta \hat{Z}^j(y)} \right) \left(\hat{Z}^j(y) - \hat{Z}^j_c(y)\right) .$$

Loops with Z-field action vertcies

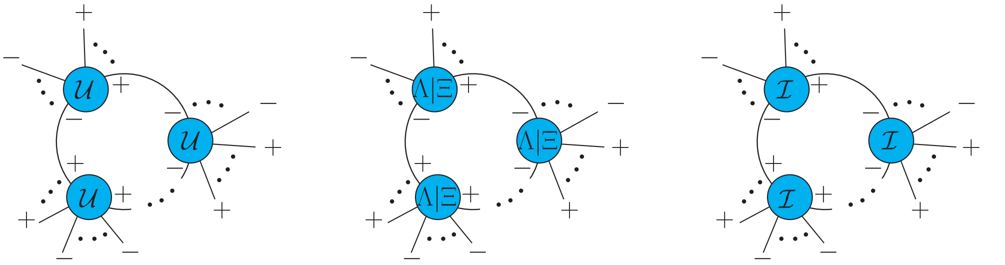
One-loop effective action:

$$\mathcal{Z}[J] \approx \exp\left\{ i \left(S[Z_c] + \int d^4x \, \mathrm{Tr}\, \hat{J}_l(x)\, \hat{A}^l[Z_c](x) \right) \right.$$
$$\left. - \frac{1}{2} \mathrm{Tr} \ln \left(\frac{\delta^2 S[Z_c]}{\delta \hat{Z}^i(x) \delta \hat{Z}^k(y)} + \int d^4z \, \hat{J}_l(z) \frac{\delta^2 \hat{A}^l[Z_c](z)}{\delta \hat{Z}^i(x) \delta \hat{Z}^k(y)} \right) \right\} .$$

Loops with Z-field action vertcies

Diagrams originating from the log term

$S[Z]$ vertices only! **$A^{I}[Z]$ kernels only!** **Mix of the two!**



$\mathcal{I} \in \{\mathcal{U}, \Lambda, \Xi\}$

Quantum corrections: The equivalence

Approach 1

$$\mathcal{Z}_{\rm YM}[J] \xrightarrow{\rm OLEA} \mathcal{Z}_{\rm YM}^{\rm one-loop}[A_c[J]] \xrightarrow{\hat{A}^{\bullet}[Z^{\bullet},Z^{\star}]\,,\hat{A}^{\star}[Z^{\bullet},Z^{\star}]} \mathcal{Z}^{\rm one-loop}[Z_c[J]]$$

Approach 2

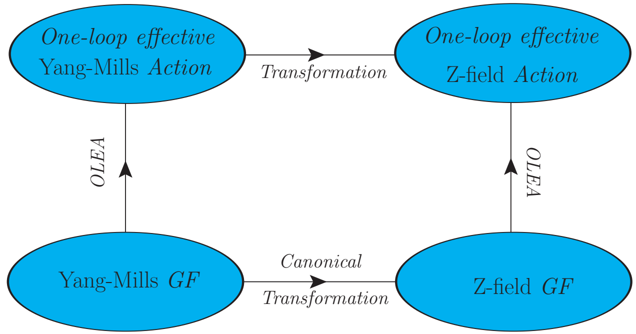
$$\mathcal{Z}_{\rm YM}[J] \xrightarrow{\hat{A}^{\bullet}[Z^{\bullet},Z^{\star}]\,,\hat{A}^{\star}[Z^{\bullet},Z^{\star}]} \mathcal{Z}[J] \xrightarrow{\rm OLEA} \mathcal{Z}^{\rm one-loop}[Z_c[J]]$$

The two approaches give equivalent actions.

$$\mathrm{Tr}\ln\left(\frac{\delta^2 S[Z_c]}{\delta\hat{A}^i(x)\delta\hat{A}^k(y)} + \int d^4z\,\hat{J}_l(z)\frac{\delta^2\hat{A}^l[Z_c](z)}{\delta\hat{A}^i(x)\delta\hat{A}^k(y)}\right) \longrightarrow \mathrm{Tr}\ln\left(\frac{\delta^2 S_{\rm YM}[A_c[Z_c]]}{\delta\hat{A}^i(x)\delta\hat{A}^j(y)}\right).$$

Quantum corrections: The equivalence

The equivalence:



Which approach is more efficient for computing pure gluonic amplitudes?

4. Summary

Summary

- Z-field action allows to efficiently compute pure gluonic amplitudes.
- There are no triple-gluon vertices.
- Vertices in Z-field action have an easy calculable form.
- No. of diagrams for split-helicity tree amplitudes follow Delannoy numbers.
- The Z-theory is geometrically rich and intriguing.
- Quantum corrections can be systematically developed using the One-loop effective action approach.

5. Outlook

Outlook

- Higher loops.
- Geometric exploartion of scattering amplitudes.
- Supersymmetric extension of the Z-action.

Thank You for your Time!

Hiren Kakkad
Krakow, January 12, 2023

kakkad@agh.edu.pl

Back-up

Color Decomposition

[F.A. Berends and W.T. Giele, 1987]; [M. Mangano, S. Parke and Z. Xu, 1988]; [M. Mangano, 1988]; [Z. Bern and D.A. Kosower, 1991]

- Technique to disentangle the color and kinematical degrees of freedom.
- Lie Algebra structure constants in terms of generators T^a.

$$i\sqrt{2}f^{abc} = \mathrm{Tr}\left(T^a T^b T^c\right) - \mathrm{Tr}\left(T^a T^c T^b\right); \qquad \mathrm{Tr}\left(T^a T^b\right) = \delta^{ab}$$

- Fierz Identity systematically combines them into a single trace.

$$\left(T^a\right)_{i_1}^{j_1}\left(T^a\right)_{i_2}^{j_2} = \delta_{i_1}^{j_2}\delta_{i_2}^{j_1} - \frac{1}{\mathrm{N}}\delta_{i_1}^{j_1}\delta_{i_2}^{j_2}$$

- n-gluon tree amplitudes:

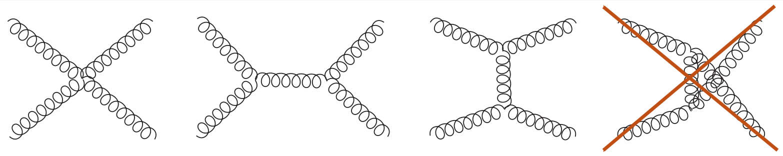
$$\mathcal{A}_n^{tree}(\{k_i, h_i, a_i\}) = \sum_{\sigma \in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}) A_n^{tree}(\sigma(1^{h_1}), \ldots, \sigma(n^{h_n}))$$

Color Decomposition

[F.A. Berends and W.T. Giele, 1987]; [M. Mangano, S. Parke and Z. Xu, 1988]; [M. Mangano, 1988]; [Z. Bern and D.A. Kosower, 1991]

Color ordered amplitude: Planar graphs with no leg-crossings allowed

$$A_4^{tree}(1^{h_1},\ldots,4^{h_4}) =$$



n	2	3	4	5	6	7	8
unordered	4	25	220	2485	34300	559405	10525900
ordered	3	10	38	154	654	2871	12925

$\sigma \in S_n/Z_n$

Spinor Helicity Formalism

[P. De Causmaecker et.al. 82]; [F. A. Berends et. al. 82]; [R. Kleiss et. al. 85]; [Z. Xu et. al. 87]; [R. Gastmans et. al. 90]

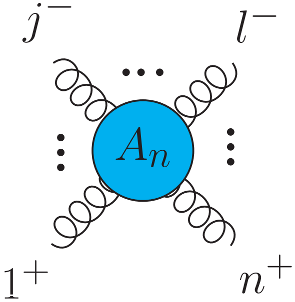
- Uniform description of the on-shell degrees of freedom (DOF).
- Spinors from massless Dirac equation.
- Kinematical DOF in terms of Spinors:
  - 4-Momentum $k_i^\mu \equiv (k_i^0, k_i^1, k_i^2, k_i^3)$ in terms of Spinors:

$$k_i^\mu (\sigma_\mu)_{\alpha\dot{\alpha}} = (k_i)_{\alpha\dot{\alpha}} = \begin{pmatrix} k_i^0 + k_i^3 & k_i^1 - ik_i^2 \\ k_i^1 + ik_i^2 & k_i^0 - k_i^3 \end{pmatrix} = \lambda_{i\,\alpha} \widetilde{\lambda}_{i\,\dot{\alpha}}.$$

  - Polarization vectors also in terms of Spinors.
- Renders the analytic expressions of amplitudes compact.
- In order to uniformize the description we take all particles as outgoing.

Helicity Amplitudes

Example: 2



MHV Amplitudes

Maximally Helicity Violating

$$A_n^{tree}(\dots, j^-, \dots, l^-, \dots) = \frac{\langle jl \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}.$$

[S.J.Parke, T.R Taylor, 1986]

Spinor product

$$\langle ij \rangle \equiv \langle \lambda_i \lambda_j \rangle = \epsilon_{\alpha\beta}\, \lambda_i^\alpha\, \lambda_j^\beta\,, \quad \text{where} \quad \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Helicity Amplitudes

Example: 2

$\overline{\text{MHV}}$ Amplitudes

$$A_n^{tree}(\ldots,j^+,\ldots,l^+,\ldots) = \frac{[jl]^4}{[12][23]\cdots[n1]}.$$

$$[ij] \equiv [\widetilde{\lambda}_i\widetilde{\lambda}_j] = -\epsilon_{\dot{\alpha}\dot{\beta}}\,\widetilde{\lambda}_i^{\dot{\alpha}}\,\widetilde{\lambda}_j^{\dot{\beta}}\,.$$

Our convention:

$\text{MHV} \equiv$ 2 gluons of $+$ helicity and rest minus.

$\overline{\text{MHV}} \equiv$ 2 gluons of $-$ helicity and rest plus.

Spinor product

$$\langle ij\rangle \equiv \langle\lambda_i\lambda_j\rangle = \epsilon_{\alpha\beta}\,\lambda_i^{\alpha}\,\lambda_j^{\beta}\,, \quad \text{where} \quad \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Cachazo-Svrcek-Witten (CSW) Method

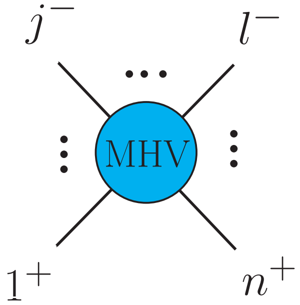
[F. Cachazo, P. Svrcek, E. Witten, 2004]

Basic idea

- Method truly motivated by the geometry.
- MHV amplitudes continued off-shell are used as interaction vertices.
- Any amplitude can be constructed by combining such vertices using scalar propagators.

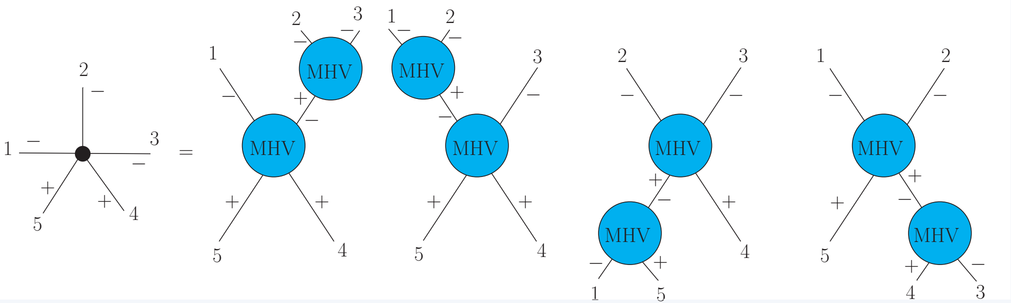
This technique gives a simple and systematic method of computing amplitudes of gluons.

Building blocks



Cachazo-Svrcek-Witten (CSW) Method

Example: 5 point $\overline{\mathrm{MHV}}$ $(- - - + +)$ in CSW method



Later, the so-called "MHV action" was derived whose Feynman rules for computing amplitudes correspond to the CSW rules.

[P. Mansfield, 2006]

Deriving the new action

[J. Scherk and J.H. Schwarz, 1975]

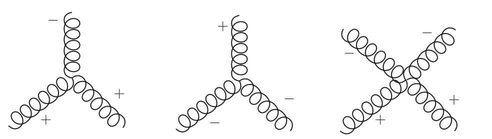
Yang-Mills action on the Light-cone

$$S_{\mathrm{YM}}\left[A^{\bullet},A^{\star}\right]=\int dx^{+}\left(\mathcal{L}_{+-}+\mathcal{L}_{++-}+\mathcal{L}_{+--}+\mathcal{L}_{++--}\right).$$

- Kinetic term:

$$\mathcal{L}_{+-}\left[A^{\bullet},A^{\star}\right]=-\int d^{3}\mathrm{x}\,\mathrm{Tr}\,\hat{A}^{\bullet}\Box\hat{A}^{\star}$$

$$\Box=\partial^{\mu}\partial_{\mu}=2(\partial_{+}\partial_{-}-\partial_{\bullet}\partial_{\star})\,,$$



Interaction vertices

Recap: Yang-Mills action on the Light-cone

$$S_{\mathrm{YM}} = -\frac{1}{4}\int d^4x\, \mathrm{Tr}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu} \quad \text{where} \quad \hat{F}^{\mu\nu} = \partial^\mu \hat{A}^\nu - \partial^\nu \hat{A}^\mu - ig\left[\hat{A}^\mu, \hat{A}^\nu\right] .$$

- Two light like four-vectors: $\eta = \frac{1}{\sqrt{2}}(1,0,0,-1) \quad \tilde{\eta} = \frac{1}{\sqrt{2}}(1,0,0,1)$
- Two complex transverse four-vectors: $\varepsilon_\perp^\pm = \frac{1}{\sqrt{2}}(0,1,\pm i,0)$
- The components of a four-vector v

$$v^+ = v\cdot\eta \quad v^- = v\cdot\tilde{\eta} \quad v^\bullet = v\cdot\varepsilon_\perp^+ \quad v^\star = v\cdot\varepsilon_\perp^-$$

- Light-cone gauge: $A\cdot\eta = A^+ = 0$.
- Action becomes quadratic in A^-, can be integrated out.

$$S_{\mathrm{YM}}\left[A^\bullet, A^\star\right] = \int dx^+ \left(\mathcal{L}_{+-} + \mathcal{L}_{++-} + \mathcal{L}_{+--} + \mathcal{L}_{++--}\right) .$$

Deriving the new action

[H. Kakkad, P. Kotko, A. Stasto, 2021]

Yang-Mills action on the Light-cone

$$S_{\mathrm{YM}}\left[A^{\bullet},A^{\star}\right]=\int dx^{+}\left(\mathcal{L}_{+-}+\mathcal{L}_{++-}+\mathcal{L}_{+--}+\mathcal{L}_{++--}\right)\,.$$

Transformation:

- Eliminates both the triple gluon vertices.

$$\left\{\hat{A}^{\bullet},\hat{A}^{\star}\right\}\rightarrow\left\{\hat{Z}^{\bullet}\left[A^{\bullet},A^{\star}\right],\hat{Z}^{\star}\left[A^{\bullet},A^{\star}\right]\right\},$$

- Generating functional:

$$\mathcal{G}[A^{\bullet},Z^{\star}](x^{+})=-\int d^{3}\mathbf{x}\;\mathrm{Tr}\,\hat{\mathcal{W}}^{-1}_{(-)}[Z](x)\;\partial_{-}\hat{\mathcal{W}}_{(+)}[A](x)\,.$$

Deriving the new action

[H. Kakkad, P. Kotko, A. Stasto, 2021]

Wilson Line

$$\mathcal{W}[A]\,(x,y) = \mathbb{P}\exp\left[ig\int_C dz_\mu\,\hat{A}^\mu\,(z)\right]$$

[P. Kotko, 2014], [P. Kotko, A. Stasto, 2017]

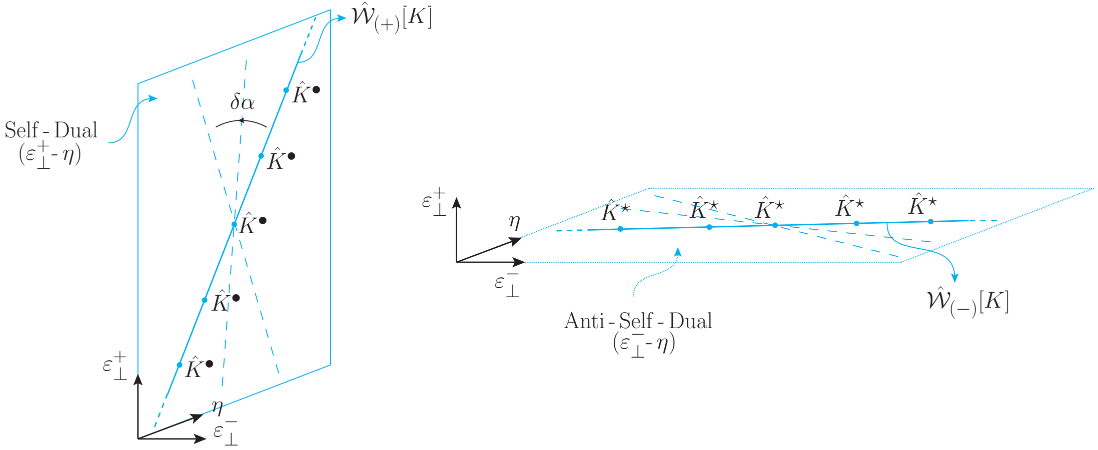
$$\mathcal{W}^a_{(\pm)}[K](x) = \int_{-\infty}^{\infty} d\alpha\,\mathrm{Tr}\left\{\frac{1}{2\pi g}t^a\partial_-\,\mathbb{P}\exp\left[ig\int_{-\infty}^{\infty} ds\,\varepsilon^{\pm}_{\alpha}\cdot\hat{K}\left(x+s\varepsilon^{\pm}_{\alpha}\right)\right]\right\}\,.$$

$$\varepsilon^{\pm\,\mu}_{\alpha} = \varepsilon^{\pm\,\mu}_{\perp} - \alpha\eta^\mu\,.$$

$$\mathcal{W}[\mathcal{W}^{-1}[K]] = K\,.$$

Deriving the new action

Geometric Representation of $\mathcal{W}^{a}_{(\pm)}[K](x)$:



Deriving the MHV action

[P. Mansfield, 2006]

Basic Idea

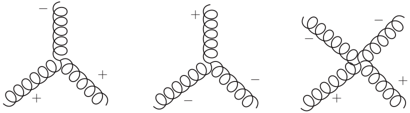
$$S_{\mathrm{YM}}\left[A^{\bullet}, A^{\star}\right] = \int dx^{+} \left(\mathcal{L}_{+-} + \mathcal{L}_{++-} + \mathcal{L}_{+--} + \mathcal{L}_{++--}\right) .$$

Transformation:

$$\{A^{\bullet}, A^{\star}\} \rightarrow \{B^{\bullet}, B^{\star}\}$$

$$\mathcal{L}_{+-} + \mathcal{L}_{++-} \longrightarrow \mathcal{L}_{+-}$$

Interaction vertices



MHV action: Action with MHV vertices

$$S_{\mathrm{YM}}\left[B^{\bullet}, B^{\star}\right] = \int dx^{+} \left(\mathcal{L}_{+-} + \mathcal{L}_{--+} + \cdots + \mathcal{L}_{--+\cdots+} + \dots\right)$$

Tree Amplitudes: Delannoy Numbers

No. of diagrams

$A_{n,m}$	2	3	4	5	...
2	1	1	1	1	MHV
3	1	3	5	7	NMHV
4	1	5	13	25	NNMHV
5	1	7	25	63	NNNMHV

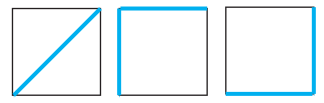
Delannoy Numbers

(n,m)	0	1	2	3
0	1	1	1	1
1	1	3	5	7
1	1	5	13	25
3	1	7	25	63

Delannoy Numbers

Definition:
The Delannoy numbers D(a,b) are the number of lattice paths from (0,0) to (b,a) in which only east (1, 0), north (0, 1), and northeast (1, 1) steps are allowed.

Example:



The 3 paths to go from $(0,0)$ to $(1,1)$ using just three moves: east $\rightarrow$, north $\uparrow$, and north-east $\nearrow$.

Tree Amplitudes: Delannoy Numbers

No. of diagrams

$A_{n,m}$	2	3	4	5	...
2	1	1	1	1	MHV
3	1	3	5	7	NMHV
4	1	5	13	25	NNMHV
5	1	7	25	63	NNNMHV

Delannoy Numbers

(n,m)	0	1	2	3
0	1	1	1	1
1	1	3	5	7
1	1	5	13	25
3	1	7	25	63

The correspondence

$$\# \, A^{(n+m-4\,\mathrm{Tree})}_{\underbrace{-\cdots-}_{m-2}\underbrace{+\cdots+}_{n-2}} = D(n,m) = \sum_{i=0}^{\min(n,m)} \binom{m}{i}\binom{n+m-i}{m} = \sum_{i=0}^{\min(n,m)} 2^i \binom{m}{i}\binom{n}{i}.$$

Yang-Mills action on the light cone

[J. Scherk and J.H. Schwarz, 1975]

$$S_{\rm YM}\left[A^{\bullet},A^{\star}\right]=\int dx^{+}\left(\mathcal{L}_{+-}+\mathcal{L}_{++-}+\mathcal{L}_{+--}+\mathcal{L}_{++--}\right)$$

$$\mathcal{L}_{+-}\left[A^{\bullet},A^{\star}\right]=-\int d^{3}\mathbf{x}\,\mathrm{Tr}\,\hat{A}^{\bullet}\Box\hat{A}^{\star}$$

$$\mathcal{L}_{++-}\left[A^{\bullet},A^{\star}\right]=-2ig'\int d^{3}\mathbf{x}\,\mathrm{Tr}\,\gamma_{\mathbf{x}}\hat{A}^{\bullet}\left[\partial_{-}\hat{A}^{\star},\hat{A}^{\bullet}\right]$$

$$\mathcal{L}_{--+}\left[A^{\bullet},A^{\star}\right]=-2ig'\int d^{3}\mathbf{x}\,\mathrm{Tr}\,\overline{\gamma}_{\mathbf{x}}\hat{A}^{\star}\left[\partial_{-}\hat{A}^{\bullet},\hat{A}^{\star}\right]$$

$$\mathcal{L}_{++--}\left[A^{\bullet},A^{\star}\right]=-g^{2}\int d^{3}\mathbf{x}\,\mathrm{Tr}\,\left[\partial_{-}\hat{A}^{\bullet},\hat{A}^{\star}\right]\partial_{-}^{-2}\left[\partial_{-}\hat{A}^{\star},\hat{A}^{\bullet}\right]$$

$$\gamma_{\mathbf{x}}=\partial_{-}^{-1}\partial_{\bullet},\quad \overline{\gamma}_{\mathbf{x}}=\partial_{-}^{-1}\partial_{\star},\qquad g'=\frac{g}{\sqrt{2}}$$

B - Fields

B - Fields as Wilson lines

[P. Kotko, 2014], [P. Kotko, A. Stasto, 2017]

$$B^{\bullet}_a[A](x) = \int_{-\infty}^{\infty} d\alpha\, \mathrm{Tr}\left\{\frac{1}{2\pi g} t^a \partial_- \,\mathbb{P}\exp\left[ig\int_{-\infty}^{\infty} ds\, \varepsilon_\alpha^+ \cdot \hat{A}\left(x + s\varepsilon_\alpha^+\right)\right]\right\}$$

$$\varepsilon_\alpha^+ = \varepsilon_\perp^+ - \alpha\eta, \quad \hat{A} = A_a t^a$$

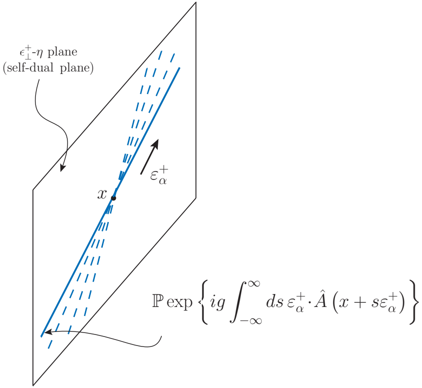
[H. Kakkad, P. Kotko, A. Stasto, 2020]

$$B^{\star}_a(x) = \int d^3\mathbf{y}\left[\frac{\partial_-^2(y)}{\partial_-^2(x)}\frac{\delta B^{\bullet}_a(x^+;\mathbf{x})}{\delta A^{\bullet}_c(x^+;\mathbf{y})}\right] A^{\star}_c(x^+;\mathbf{y})$$

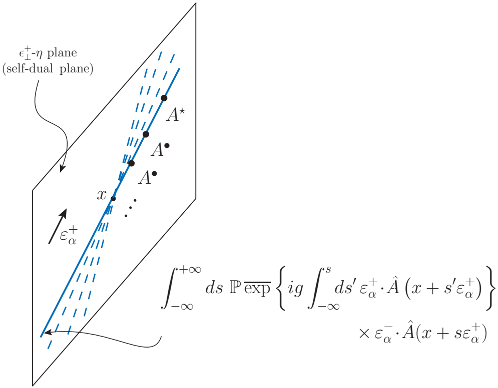
B Fields

Geometrical Representation.

$B^\bullet[A^\bullet]$



$B^\star[A^\bullet, A^\star]$



[P. Kotko, 2014], [P. Kotko, A. Stasto, 2017], [H. Kakkad, P. Kotko, A. Stasto, 2020]

Wilson line kernels

$$\widetilde{B}^{\bullet}_a(x^+;\mathbf{P}) = \sum_{n=1}^{\infty}\int d^3\mathbf{p}_1\ldots d^3\mathbf{p}_n\,\widetilde{\Gamma}_n^{a\{b_1\ldots b_n\}}(\mathbf{P};\{\mathbf{p}_1,\ldots,\mathbf{p}_n\})\prod_{i=1}^{n}\widetilde{A}^{\bullet}_{b_i}(x^+;\mathbf{p}_i)$$

$$\widetilde{B}^{\star}_a(x^+;\mathbf{P}) = \sum_{n=1}^{\infty}\int d^3\mathbf{p}_1\ldots d^3\mathbf{p}_n\,\widetilde{\Upsilon}_n^{ab_1\{b_2\ldots b_n\}}(\mathbf{P};\mathbf{p}_1,\{\mathbf{p_2},\ldots,\mathbf{p_n}\})\widetilde{A}^{\star}_{b_1}(x^+;\mathbf{p}_1)\prod_{i=2}^{n}\widetilde{A}^{\bullet}_{b_i}(x^+;\mathbf{p}_i)$$

where

$$\widetilde{\Gamma}_n^{a\{b_1\ldots b_n\}}(\mathbf{P};\{\mathbf{p}_1,\ldots,\mathbf{p}_n\}) = (-g)^{n-1}\frac{\delta^3\left(\mathbf{p}_1+\cdots+\mathbf{p}_n-\mathbf{P}\right)\,\mathrm{Tr}\left(t^a t^{b_1}\ldots t^{b_n}\right)}{\widetilde{v}^*_{1(1\cdots n)}\widetilde{v}^*_{(12)(1\cdots n)}\cdots\widetilde{v}^*_{(1\cdots n-1)(1\cdots n)}}$$

$$\widetilde{\Upsilon}_n^{ab_1\{b_2\cdots b_n\}}(\mathbf{P};\mathbf{p}_1,\{\mathbf{p_2},\ldots,\mathbf{p_n}\}) = n\left(\frac{p_1^+}{p_{1\cdots n}^+}\right)^2\widetilde{\Gamma}_n^{ab_1\ldots b_n}(\mathbf{P};\mathbf{p}_1,\ldots,\mathbf{p}_n)$$

Inverse Wilson line kernels

$$\widetilde{A}^{\bullet}_{a}(x^+;\mathbf{P}) = \sum_{n=1}^{\infty}\int d^3\mathbf{p}_1\dots d^3\mathbf{p}_n\,\widetilde{\Psi}_n^{a\{b_1\dots b_n\}}(\mathbf{P};\{\mathbf{p}_1,\dots,\mathbf{p}_n\})\prod_{i=1}^{n}\widetilde{B}^{\bullet}_{b_i}(x^+;\mathbf{p}_i)$$

$$\widetilde{A}^{\star}_{a}(x^+;\mathbf{P}) = \sum_{n=1}^{\infty}\int d^3\mathbf{p}_1\dots d^3\mathbf{p}_n\,\widetilde{\Omega}_n^{ab_1\{b_2\cdots b_n\}}(\mathbf{P};\mathbf{p}_1,\{\mathbf{p_2},\dots,\mathbf{p_n}\})\widetilde{B}^{\star}_{b_1}(x^+;\mathbf{p}_1)\prod_{i=2}^{n}\widetilde{B}^{\bullet}_{b_i}(x^+;\mathbf{p}_i)$$

where the kernels are

$$\widetilde{\Psi}_n^{a\{b_1\cdots b_n\}}(\mathbf{P};\{\mathbf{p}_1,\dots,\mathbf{p}_n\}) = -(-g)^{n-1}\,\frac{\widetilde{v}^{\star}_{(1\cdots n)1}}{\widetilde{v}^{\star}_{1(1\cdots n)}}\,\frac{\delta^3(\mathbf{p}_1+\dots+\mathbf{p}_n-\mathbf{P})\ \mathrm{Tr}(t^at^{b_1}\cdots t^{b_n})}{\widetilde{v}^{\star}_{21}\widetilde{v}^{\star}_{32}\cdots\widetilde{v}^{\star}_{n(n-1)}}$$

$$\widetilde{\Omega}_n^{ab_1\{b_2\cdots b_n\}}(\mathbf{P};\mathbf{p}_1,\{\mathbf{p}_2,\dots,\mathbf{p}_n\}) = n\left(\frac{p_1^+}{p_{1\cdots n}^+}\right)^2\widetilde{\Psi}_n^{ab_1\cdots b_n}(\mathbf{P};\mathbf{p}_1,\dots,\mathbf{p}_n)$$

Z-field Action

Important features

- There are MHV vertices, $(- - + \cdots +)$, corresponding to MHV amplitudes in the on-shell limit.

$$\mathcal{A}\left(1^-, 2^-, 3^+, \ldots, n^+\right) \equiv \left(\frac{p_1^+}{p_2^+}\right)^2 \frac{\widetilde{v}_{21}^{*4}}{\widetilde{v}_{1n}^{*}\widetilde{v}_{n(n-1)}^{*}\widetilde{v}_{(n-1)(n-2)}^{*}\ldots\widetilde{v}_{21}^{*}}$$

- There are $\overline{\mathrm{MHV}}$ vertices, $(- \cdots - ++)$, corresponding to $\overline{\mathrm{MHV}}$ amplitudes in the on-shell limit.

$$\mathcal{A}\left(1^-, \ldots, n-2^-, n-1^+, n^+\right) \equiv \left(\frac{p_{n-1}^+}{p_n^+}\right)^2 \frac{\widetilde{v}_{n(n-1)}^{4}}{\widetilde{v}_{1n}\widetilde{v}_{n(n-1)}\widetilde{v}_{(n-1)(n-2)}\ldots\widetilde{v}_{21}}$$

Z field Wilson Line

Z - Fields as Wilson Line functionals

$$Z_a^{\star}[B^{\star}](x) = \int_{-\infty}^{\infty} d\alpha \, \mathrm{Tr}\left\{ \frac{1}{2\pi g} t^a \partial_- \, \mathbb{P}\exp\left[ig \int_{-\infty}^{\infty} ds \, \varepsilon_\alpha^- \cdot \hat{B}\left(x + s\varepsilon_\alpha^-\right)\right]\right\}$$

$$\varepsilon_\alpha^- = \varepsilon_\perp^- - \alpha\eta, \quad \hat{B} = B_a t^a$$

$$Z_a^{\bullet}(x) = \int d^3\mathbf{y} \left[\frac{\partial_-^2(y)}{\partial_-^2(x)} \frac{\delta Z_a^{\star}(x^+;\mathbf{x})}{\delta B_c^{\star}(x^+;\mathbf{y})} \right] B_c^{\bullet}(x^+;\mathbf{y})$$

Inverse Wilson line kernels

$$\widetilde{B}^{\star}_{a}(x^+;\mathbf{P}) = \sum_{n=1}^{\infty}\int d^3\mathbf{p}_1\ldots d^3\mathbf{p}_n\,\overline{\widetilde{\Psi}}\,{}_{n}^{a\{b_1\ldots b_n\}}(\mathbf{P};\{\mathbf{p}_1,\ldots,\mathbf{p}_n\})\prod_{i=1}^{n}\widetilde{Z}^{\star}_{b_i}(x^+;\mathbf{p}_i)$$

$$\widetilde{B}^{\bullet}_{a}(x^+;\mathbf{P}) = \sum_{n=1}^{\infty}\int d^3\mathbf{p}_1\ldots d^3\mathbf{p}_n\,\overline{\widetilde{\Omega}}\,{}_{n}^{ab_1\{b_2\cdots b_n\}}(\mathbf{P};\mathbf{p}_1,\{\mathbf{p}_2,\ldots,\mathbf{p}_n\})\widetilde{Z}^{\bullet}_{b_1}(x^+;\mathbf{p}_1)\prod_{i=2}^{n}\widetilde{Z}^{\star}_{b_i}(x^+;\mathbf{p}_i)$$

with

$$\overline{\widetilde{\Psi}}\,{}_{n}^{a\{b_1\cdots b_n\}}(\mathbf{P};\{\mathbf{p}_1,\ldots,\mathbf{p}_n\}) = -(-g)^{n-1}\,\frac{\widetilde{v}_{(1\cdots n)1}}{\widetilde{v}_{1(1\cdots n)}}\,\frac{\delta^3(\mathbf{p}_1+\cdots+\mathbf{p}_n-\mathbf{P})\ \mathrm{Tr}(t^a t^{b_1}\cdots t^{b_n})}{\widetilde{v}_{21}\widetilde{v}_{32}\cdots\widetilde{v}_{n(n-1)}}$$

$$\overline{\widetilde{\Omega}}\,{}_{n}^{ab_1\{b_2\cdots b_n\}}(\mathbf{P};\mathbf{p}_1,\{\mathbf{p}_2,\ldots,\mathbf{p}_n\}) = n\left(\frac{p_1^+}{p_{1\cdots n}^+}\right)^2\overline{\widetilde{\Psi}}\,{}_{n}^{ab_1\cdots b_n}(\mathbf{P};\mathbf{p}_1,\ldots,\mathbf{p}_n)$$

V-symbols

$$\widetilde{v}_{ij} = p_i^+ \left(\frac{p_j^\star}{p_j^+} - \frac{p_i^\star}{p_i^+} \right), \qquad \widetilde{v}^*_{ij} = p_i^+ \left(\frac{p_j^\bullet}{p_j^+} - \frac{p_i^\bullet}{p_i^+} \right)$$

THE END!