KRAKOW SCHOOL OF INTERDYSCIPLINARY PHD STUDIES

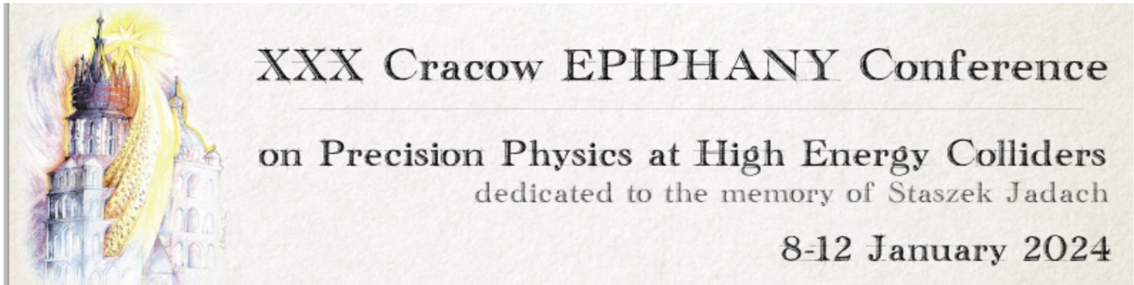
Markov Chain Monte Carlo Methods: An Alternative Approach for Nuclear PDF Determination

Nasim Derakhshanian

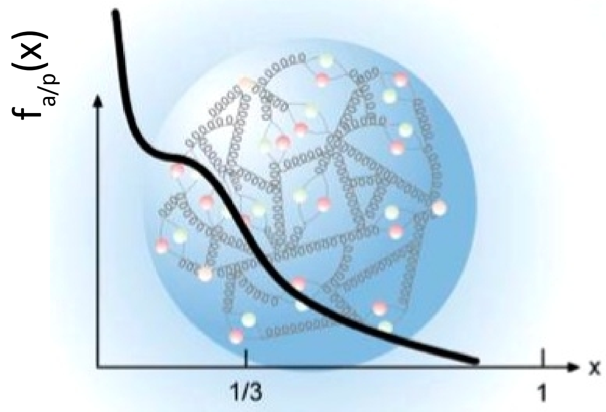
Dr. Aleksander Kusina

IFJ PAN

Department of Theoretical Particle Physics



nCTEQ
nuclear parton distribution functions



Parton Distribution Function (PDF):

The probability $\mathbf{f_{a/p}(x,\mu)}$ that a parton **a** carries fraction **x** of the proton's momentum

μ: Factorization scale

x: momentum fraction

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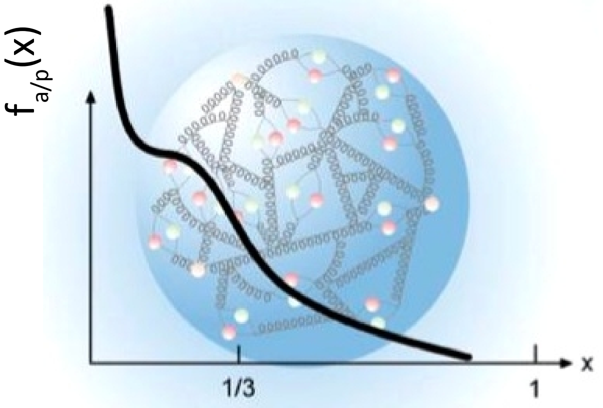

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Factorization Theorem:

$$\sigma_{P\gamma\rightarrow c} = f_{P\rightarrow a} \otimes \hat{\sigma}_{a\gamma\rightarrow c}$$

Parton densities (long-distance)

Parton interaction (short-distance)

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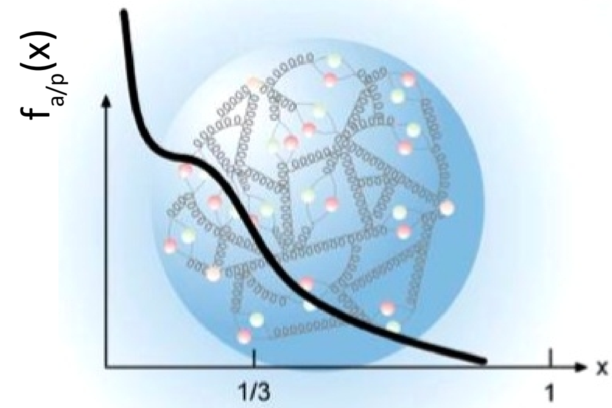

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Parton densities (long-distance)

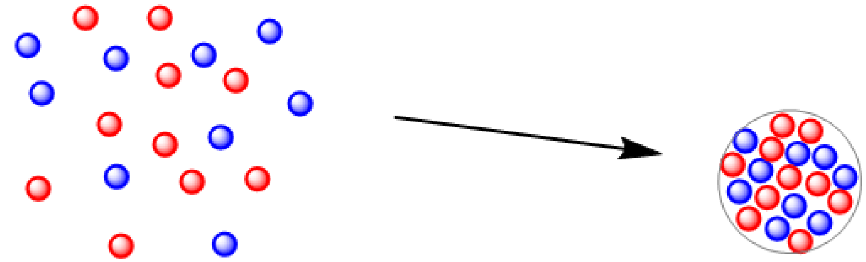
Parton interaction (short-distance)

PDF properties:

- Universal (independent of the process)
- Constrained through momentum and number sum rules
- μ^2-dependence governed by DGLAP evolution equations
- **Non-perturbative**: x-dependence of PDF is NOT calculable in pQCD

→ **extracting from a fit to experimental data**

Nuclear PDFs(nPDFs):

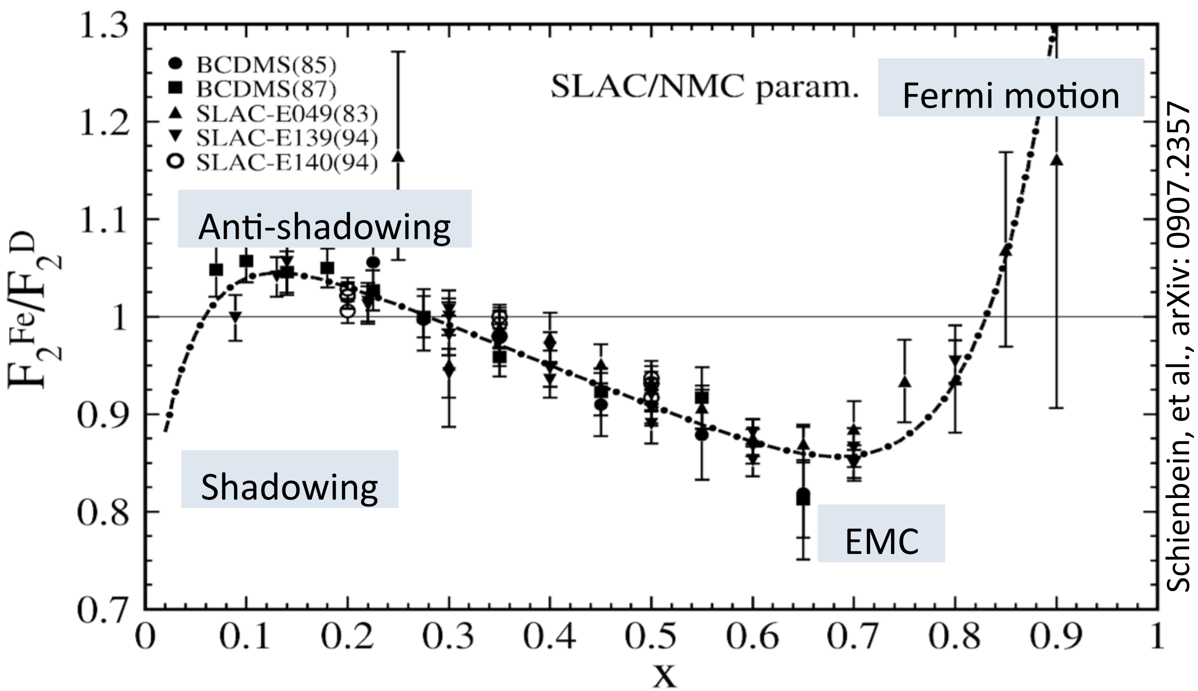


Free nucleons

Bound nucleons within a nucleus

$$F_2^A(x) \neq ZF_2^p(x) + NF_2^n(x)$$

Nuclear correction ratio:



Nuclear PDFs(nPDFs):

Motivations:

- Interpreting heavy-ion collision data
- Understanding Nuclear Structure

The accuracy of PDF(nPDF) determination plays an important role in high-energy physics

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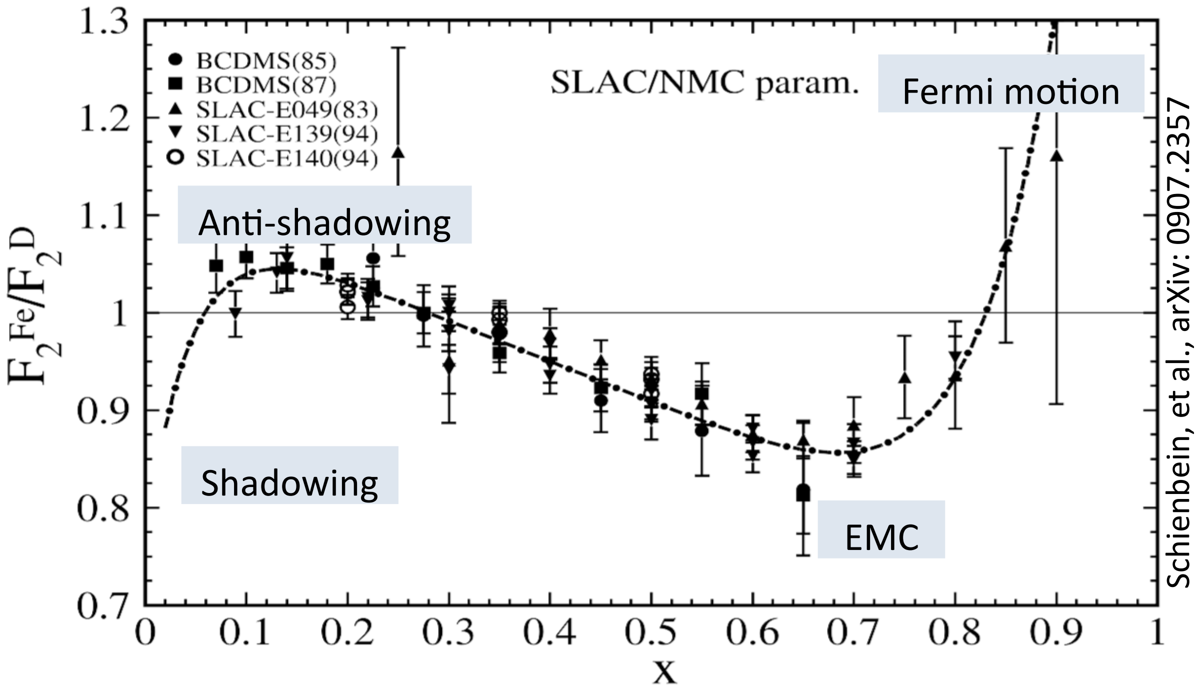
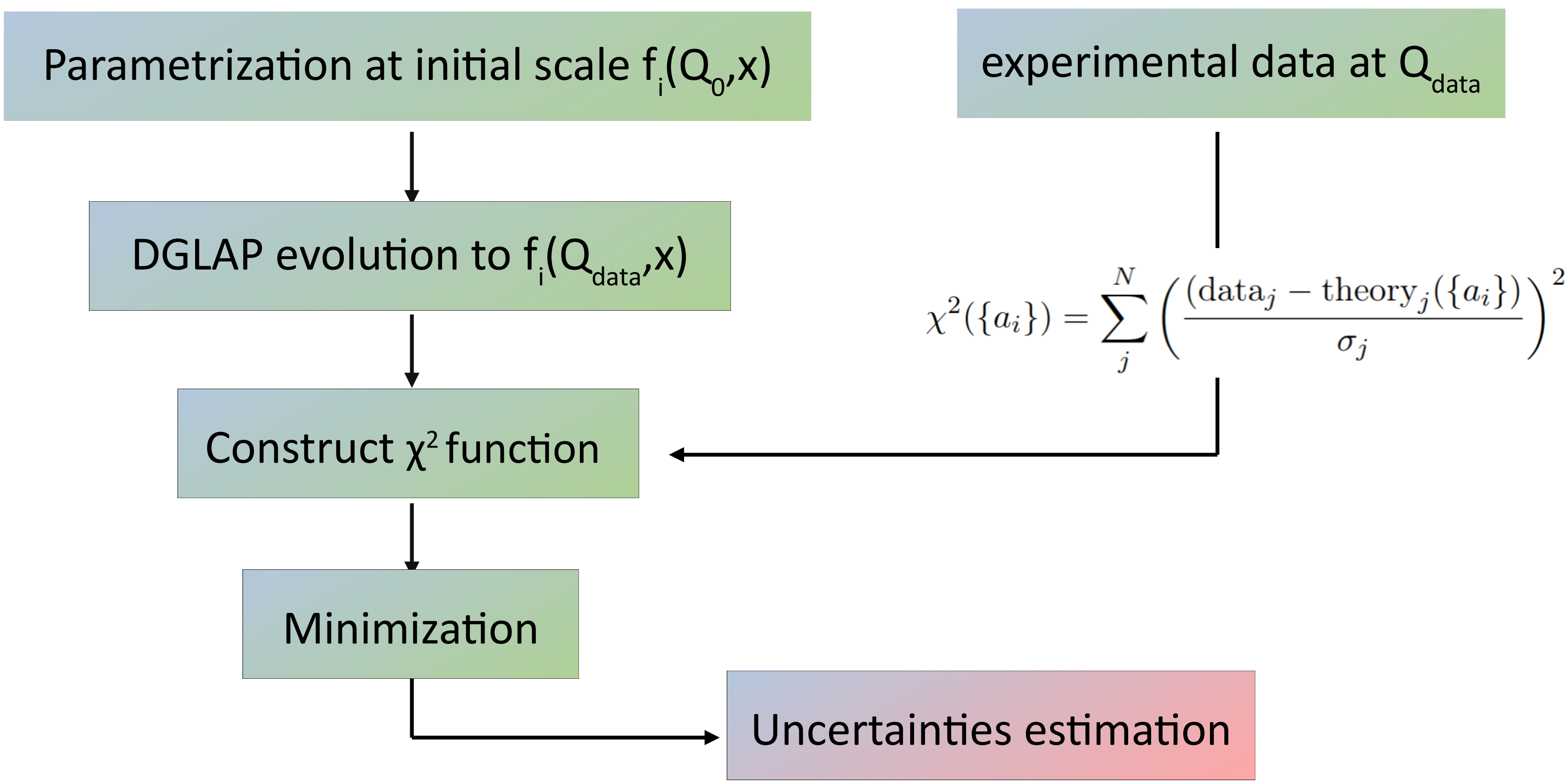


QCD Global analysis:



nCTEQ15 framework for nuclear PDF:

Kovarik et al., arXiv:1509.00792

Functional form for bound proton at Q_0:

$$x f_i^{p/A}(x, Q_0) = c_0\, x^{c_1} (1-x)^{c_2} e^{c_3 x} (1 + e^{c_4} x)^{c_5}$$

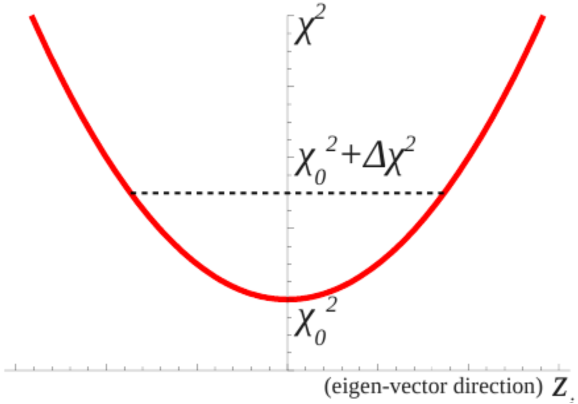
$$i = u_v, d_v, g, \bar{u} + \bar{d}, s, \bar{s}$$

Atomic number dependence is characterized in the c_k coefficients as

$$c_k \to c_k(A) \equiv p_k + a_k(1 - A^{-b_k}), \qquad k = \{1, \ldots, 5\}.$$

PDF of a nucleus (A – mass, Z – charge):

$$f_i^{(A,Z)}(x, Q) = \frac{Z}{A} f_i^{p/A}(x, Q) + \frac{A-Z}{A} f_i^{n/A}(x, Q)$$



PDF uncertainties estimation:

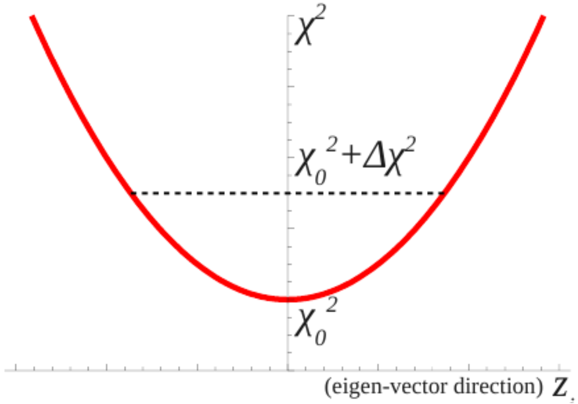
Hessian method: Common method for estimating uncertainties in PDFs.

↓

relying on the Gaussian approximation of $\Delta\chi^2$

Shortcomings:

- Non-gaussian errors
- Global minima judgment
- Choice of $\Delta\chi^2$



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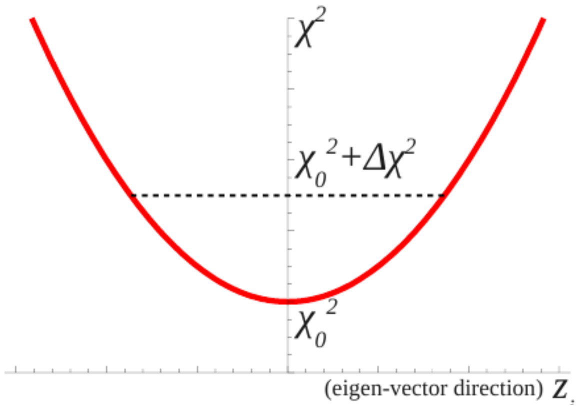
nPDF difficulties
- Lacking data (need low-x & precise data, for several nuclei)
- Complexity and nature of nuclear effects



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Markov Chain Monte Carlo method

advanced statistical method as an alternative for Hessian

Markov Chain Monte Carlo (MCMC)

A sequence of random variables where the current value is dependent on the value of the prior variable (Memory-less property)

A technique for randomly sampling a probability distribution and approximating a desired quantity.

Bayes theorem:

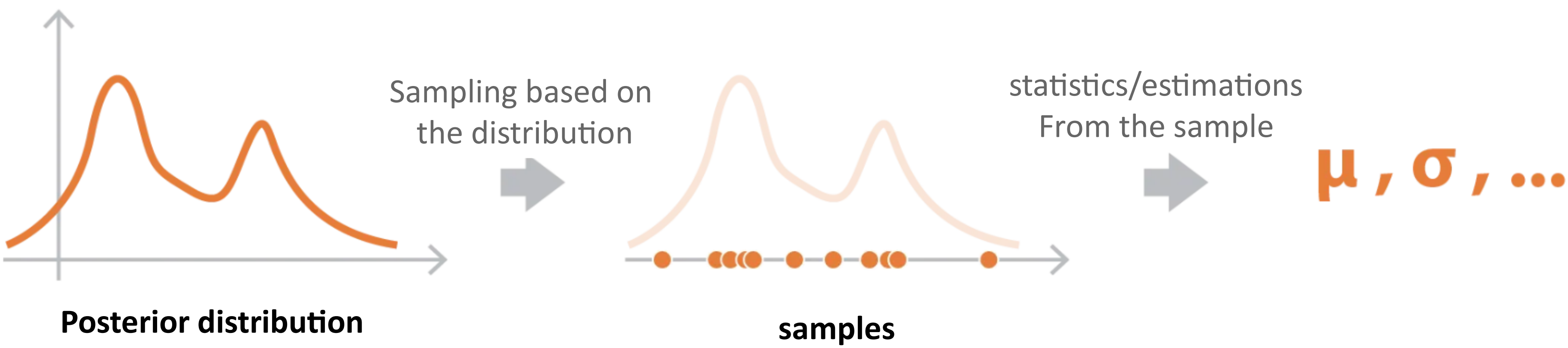
Posterior, Likelihood, Prior, Normalization

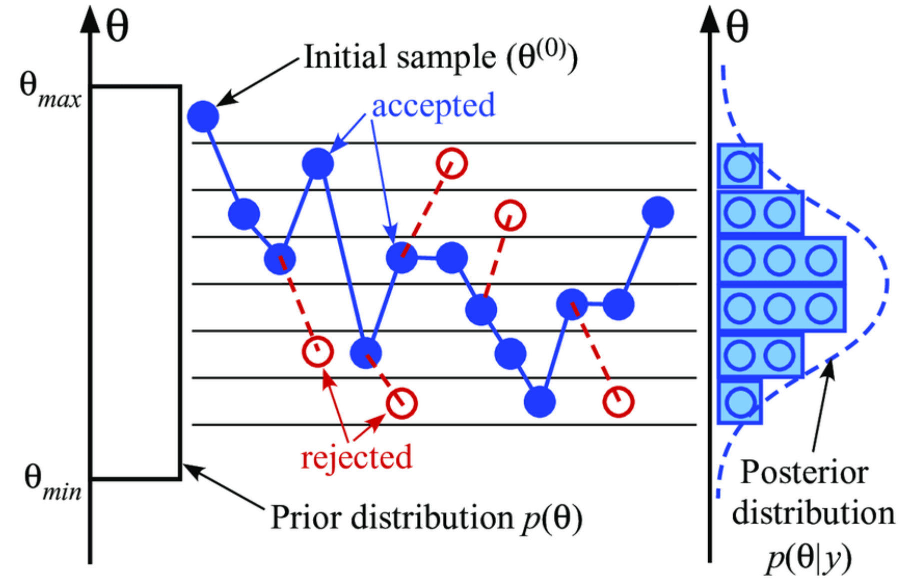
$$\underbrace{p(\theta \mid \text{data})}_{\text{Posterior}} = \frac{\overbrace{p(\text{data} \mid \theta)}^{\text{Likelihood}} \cdot \overbrace{p(\theta)}^{\text{Prior}}}{\underbrace{p(\text{data})}_{\text{Normalization}}}$$

Prior: initial belief about the parameter before considering the data.
Likelihood: probability of observing the data given a specific value of the parameter.
Posterior: updated belief about the parameter given the data.

We aim to find the set of nPDF parameters that maximizes the posterior probability distribution given the experimental data.

Bayesian inference → **MCMC algorithms**





Metropolis algorithm:

- Initialize parameters
- for i=1 to i=N:

Propose a new sample

Generate proposed parameters via proposition function: θ * ~ q(θ * | θ i)

Sample from uniform distribution: u ~ U(0,1)

Compute acceptance ratio: α = p(θ * | D) / p(θ i | D)

If u < min(1, α) then x_{i+1} =x *

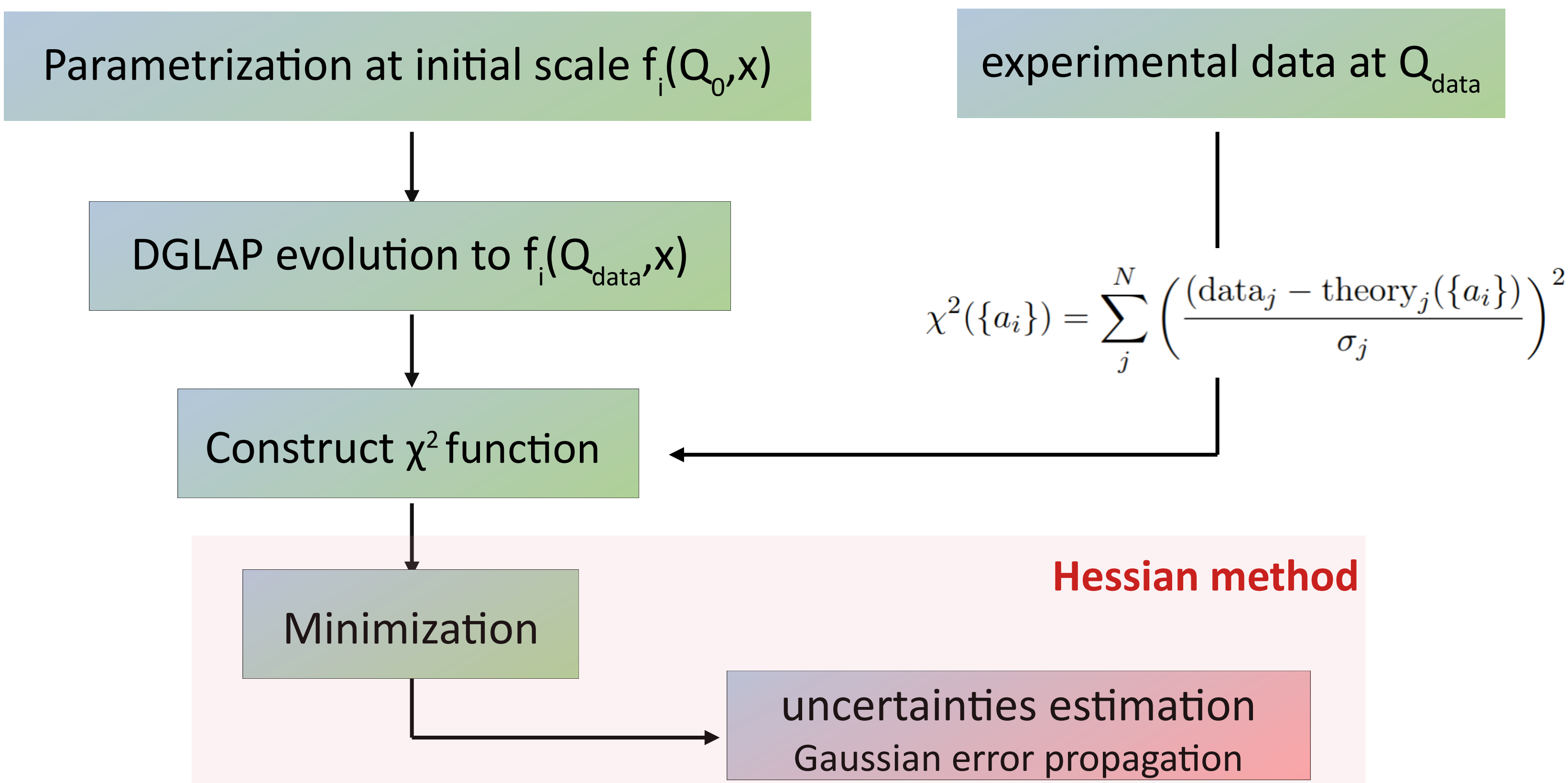
Judgment of proposed sample

- Else x_{i+1} =x i

Metropolis-Hasting: $x_{t+1} = \mathcal{N}(x_t, C_0)$

Adaptive Metropolis-Hasting: $x_{t+1} = \beta\mathcal{N}(x_t, C_0) + (1-\beta)\mathcal{N}(x_t, \hat{C}_n)$

PDF Global analysis:



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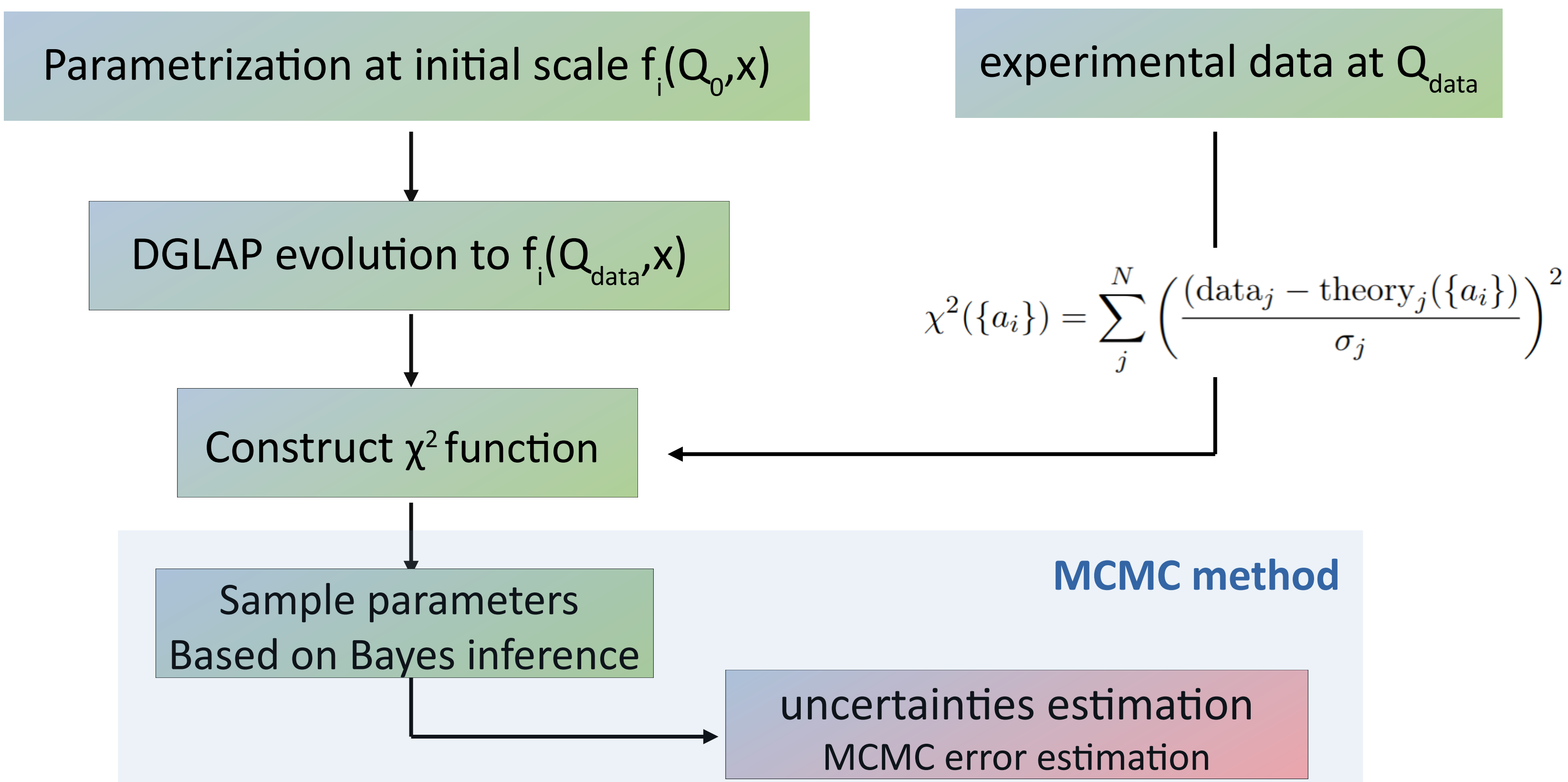


Parametrization: $x f_i^{p/A}(x, Q_0) = c_0 \, x^{c_1} (1-x)^{c_2} e^{c_3 x} (1 + e^{c_4} x)^{c_5}$

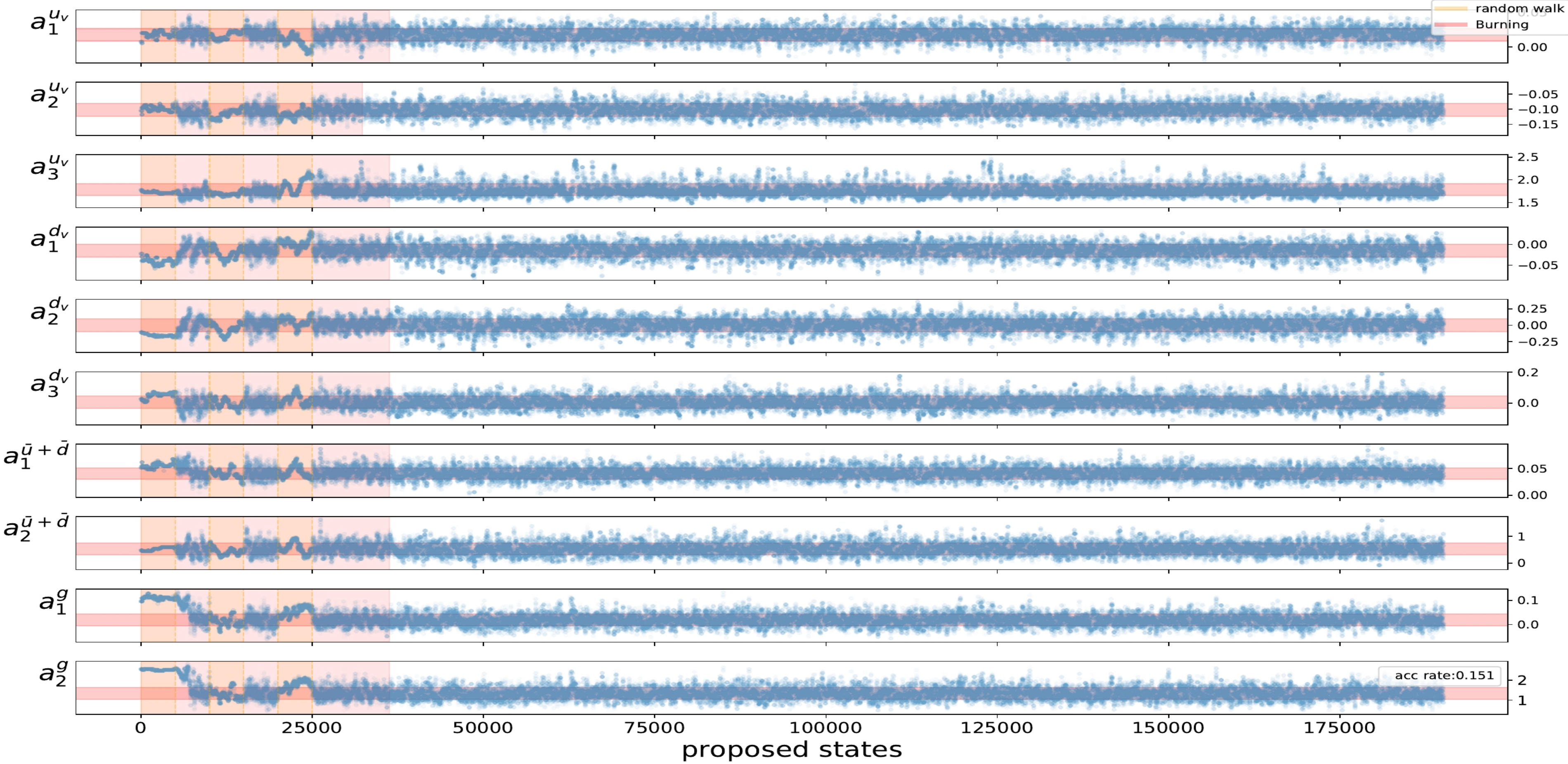
$$c_k \to c_k(A) \equiv p_k + a_k(1 - A^{-b_k}),$$

Generating the Markov Chain of nPDF parameters:

Each point of the chain is representing a set of nPDF parameters

(parameter a_k for 6 valances, 2 sea quarks and 2 gluon)

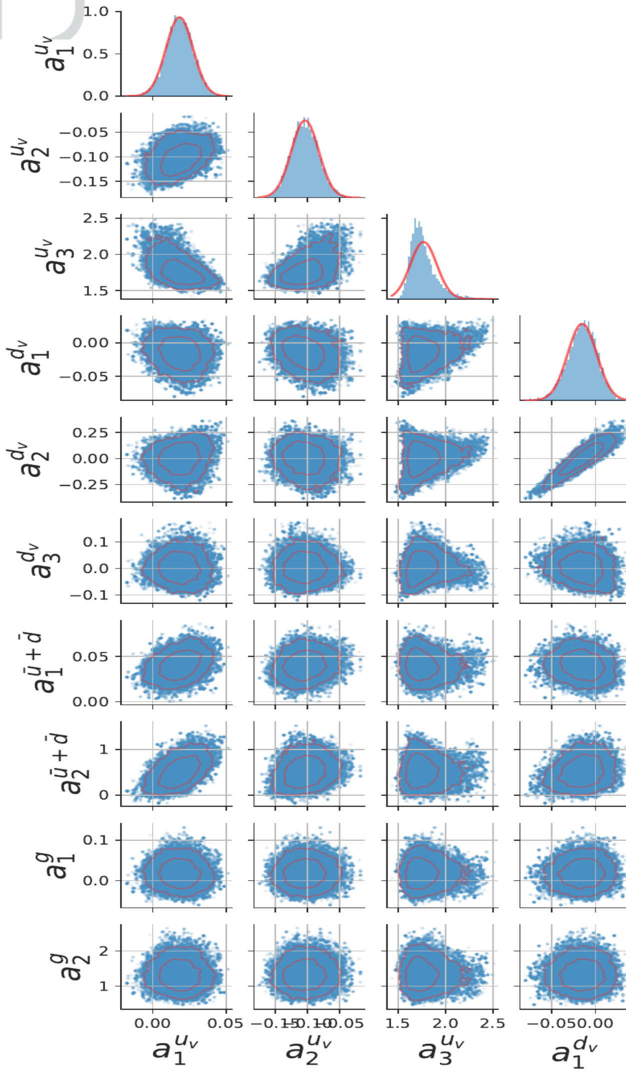
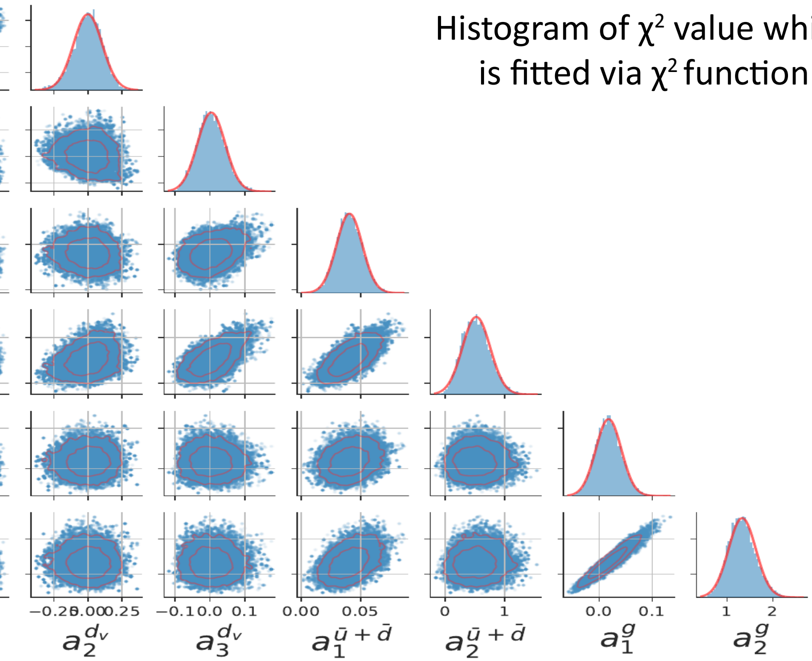
DIS and W/Z boson data:
436 data points

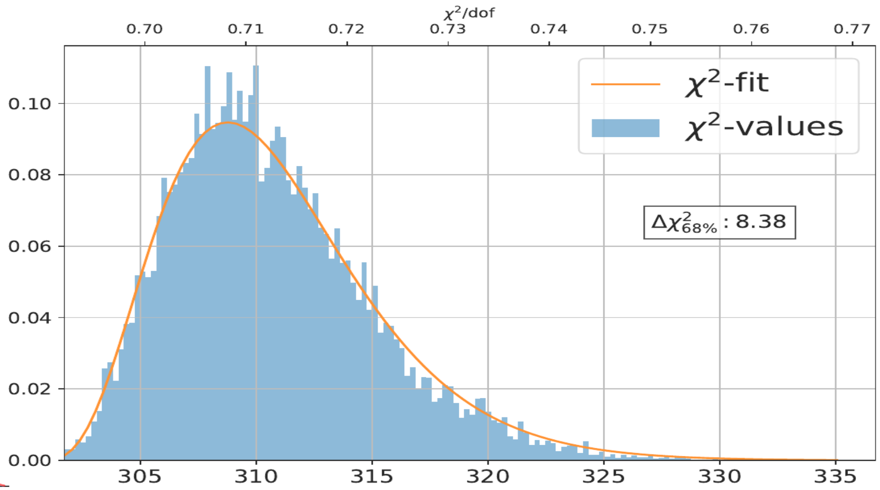


MCMC can reveal non-Gaussian features of the underlying distribution

Pairwise plot

diagonal: histogram of each parameter
off-diagonal: 2D correlation plots between parameters



Histogram of χ^2 value which is fitted via χ^2 function

Error estimation:

Monte Carlo error estimation (uncorrelated)

$$\sigma^2_{MC} = \frac{1}{n-1}\sum_{t=1}^{n}(X_t - \hat{\mu})^2$$

MCMC error estimation (correlated)

$$\sigma^2_{MCMC} = 2\,\tau_{int}\,\sigma^2_{MC}$$

Autocorrelation function (ACF)

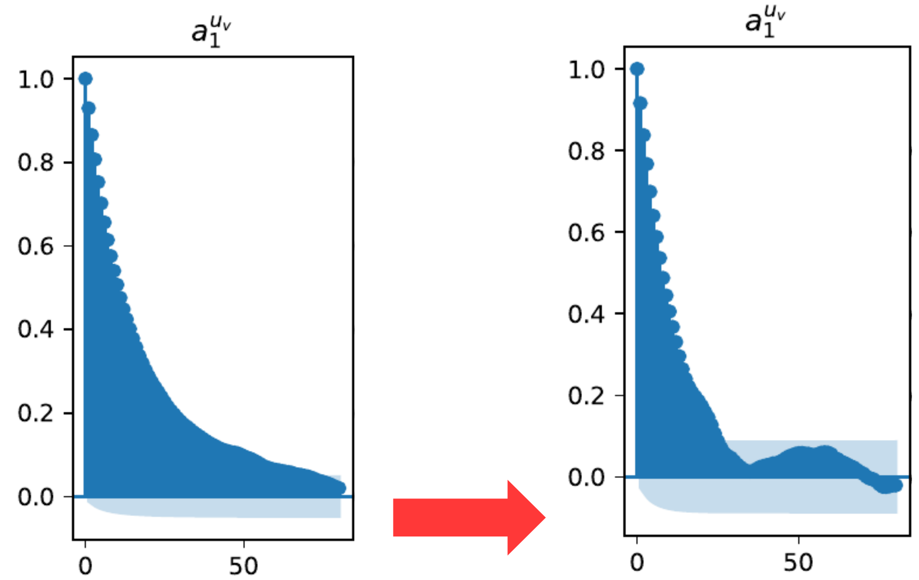
$$\rho(k) = \frac{\mathrm{Cov}(k)}{\mathrm{Cov}(0)}$$

$$\mathrm{Cov}(k) = \frac{1}{n}\sum_{t=1}^{n-k}(x_{t+k} - \bar{x})(x_t - \bar{x}),$$

Autocorrelation time

$$\tau_{int} = \frac{1}{2}\sum_{-\infty}^{+\infty}\rho(k)$$

Autocorrelation function versus time interval



Thinning by rate 40

Thinning method:

keep only every k-th sample in the Markov chain and discard the rest

Why Thinning?

- It provides an **uncorrelated** chain so we can use Monte-Carlo error estimation:

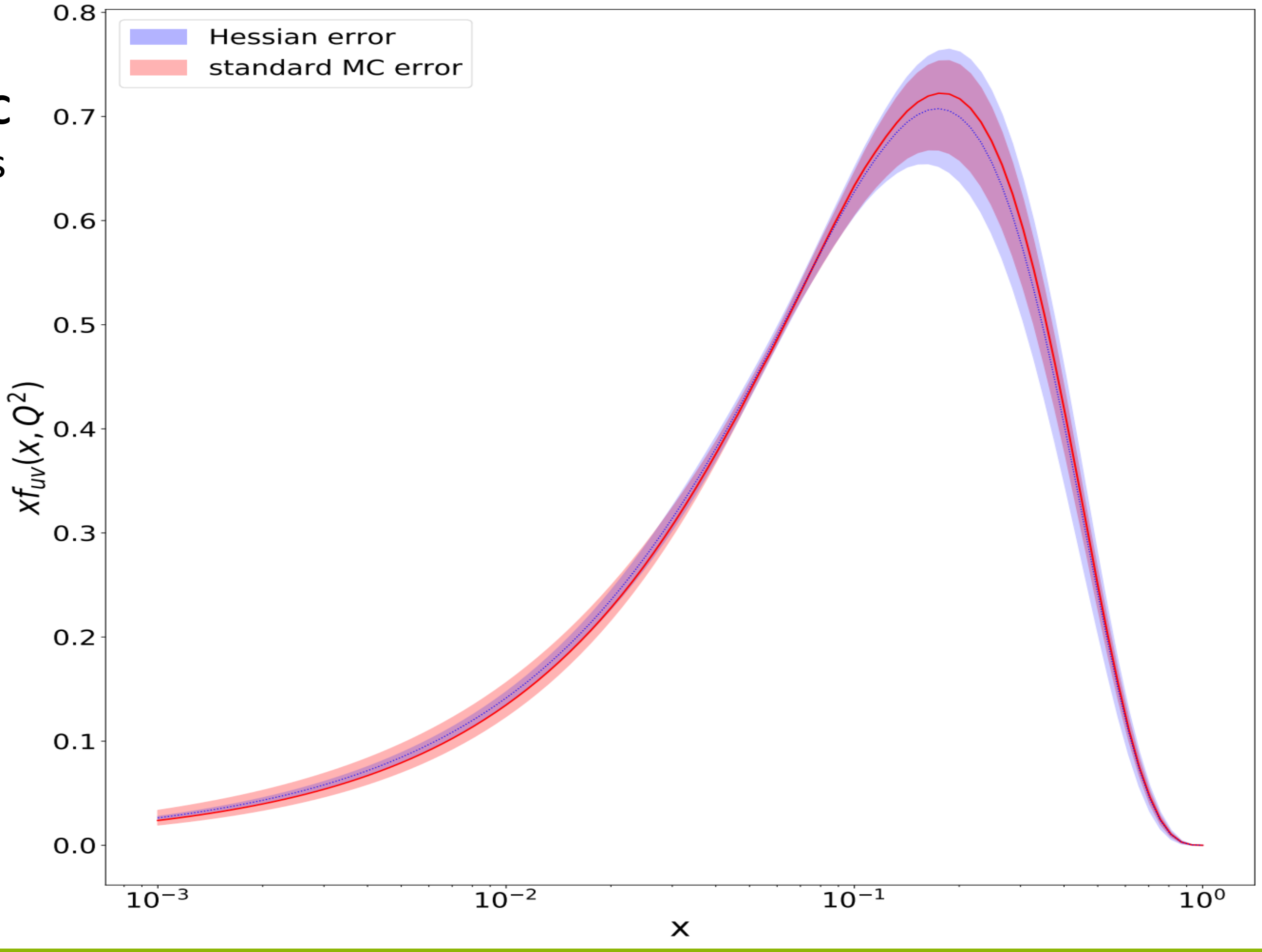
$$\sigma^2_{MCMC} = 2\,\tau_{int}\,\sigma^2_{MC} \quad \longrightarrow \quad \sigma^2_{MC} = \frac{1}{n-1}\sum_{t=1}^{n}(X_t - \hat{\mu})^2$$

- We aim to generate a set of PDF grids corresponding chain's units. Thinning the chain makes it more applicable.

MCMC approach:

- Generating the Markov Chain
- Thinning the chain
- Construct nPDF corresponding to each unit of the thinned chain and perform standard MC error estimation (Saving them in the standard LHAPDF format so that anyone can use such nPDFs)

Pb^{208} PDF(u-valence) resulting from **MCMC** and **Hessian** methods



Preliminary results

Conclusion:

- Despite the MCMC challenges (mainly computational cost), this method has become a powerful tool for determining nPDFs.

- The implementation of this alternative method hasn't finished yet, but so far we have obtained promising results (comparing with Hessian).

new nCTEQ global nPDF release: CJ15

Accardi et al., arXiv:1602.03154

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