applied Quantum algorithms
Leiden

Topics in Quantum Topological Data Analysis

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Quantum 6, 855 (2022), arXiv:2209.13581

w/ Casper Gyurik

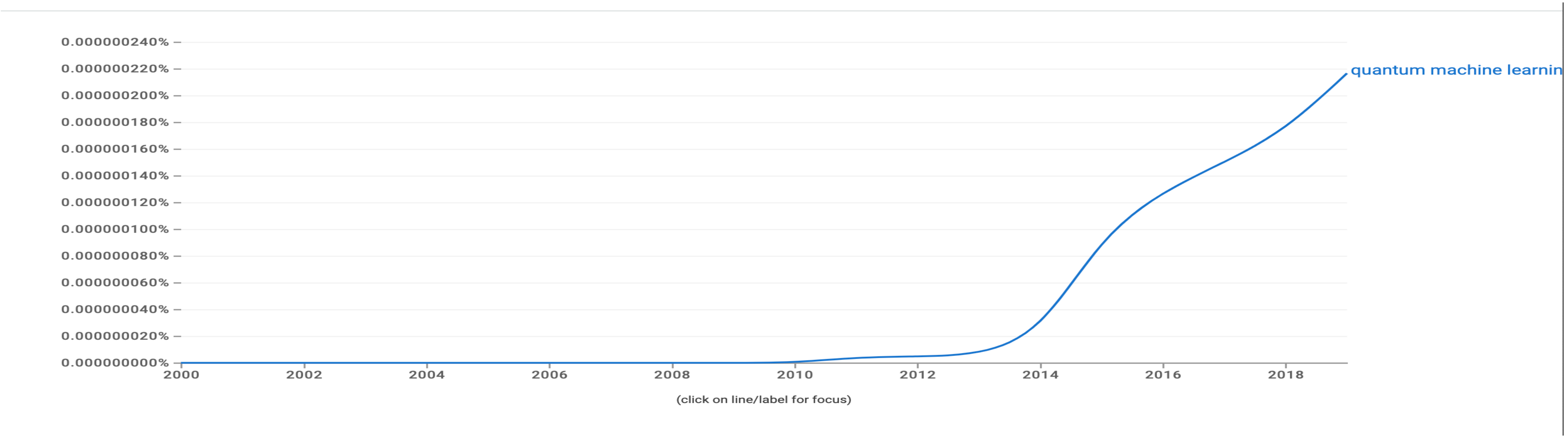
and: Cade, Crichigno, Berry, Su, King, Basso, Del Toro Barba, Rajput, Wiebe, Babbush

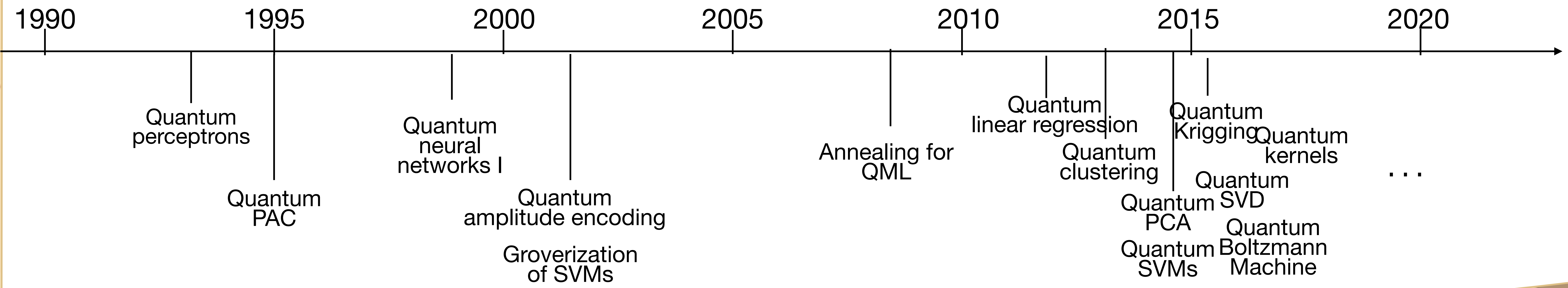
Quantum Delta NL

Outline:

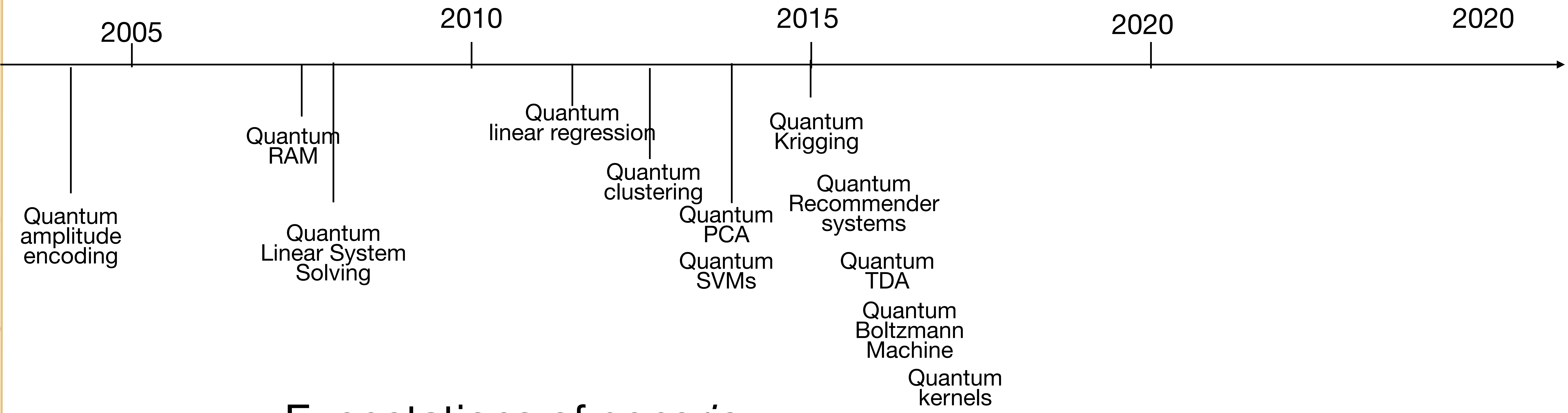
1) the motivation
2) the TDA problem and the algorithm(s)
3) TDA, bounds and complexity theory
4) QTDA versus current algorithms
5) Other open questions

Timeline of QML



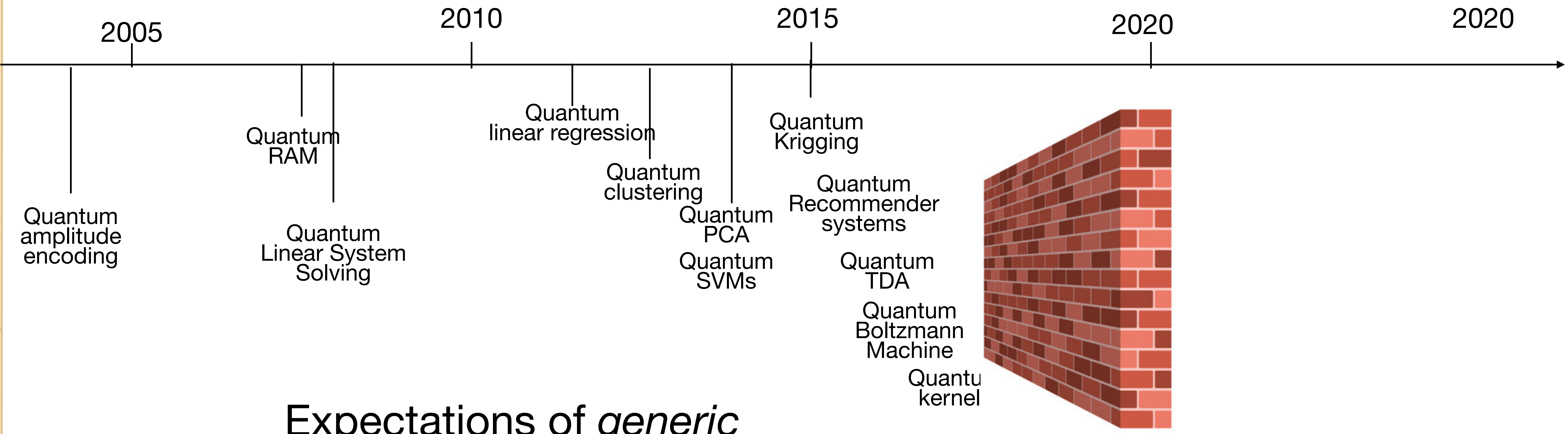


Timeline of QML



Expectations of *generic* exponential speed-ups

Timeline of QML

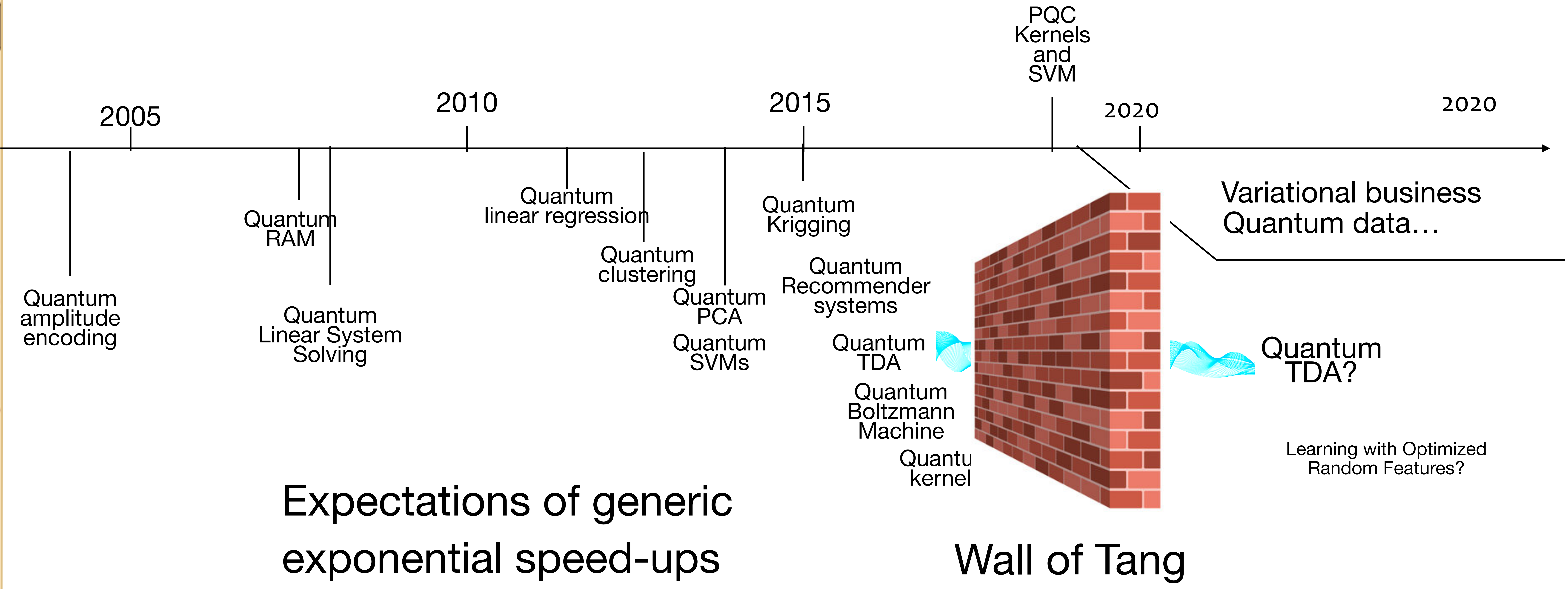


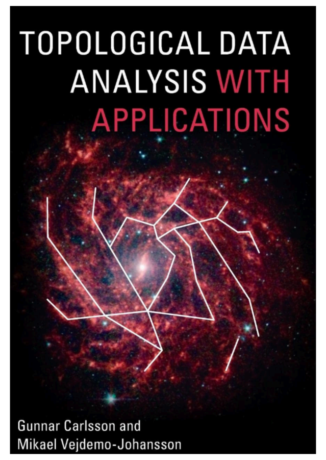
Expectations of *generic* exponential speed-ups

Wall of Tang

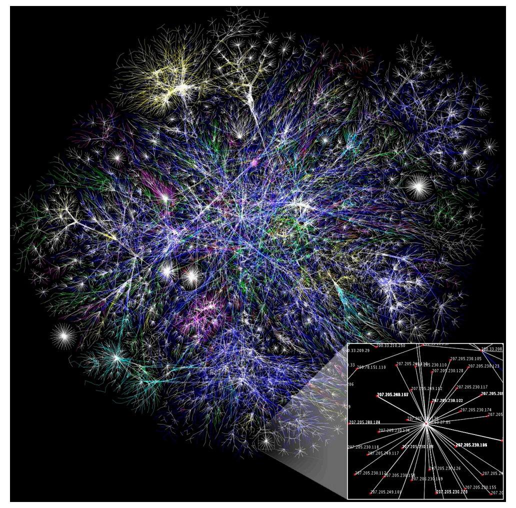
Ewin Tang
Quantum Recommendation Systems
arXiv:1807.04271
Ewin Tang
arXiv:1811.00414
Quantum PCA
Ewin Tang
arXiv:1811.04909
Low rank HHL

Timeline of QML



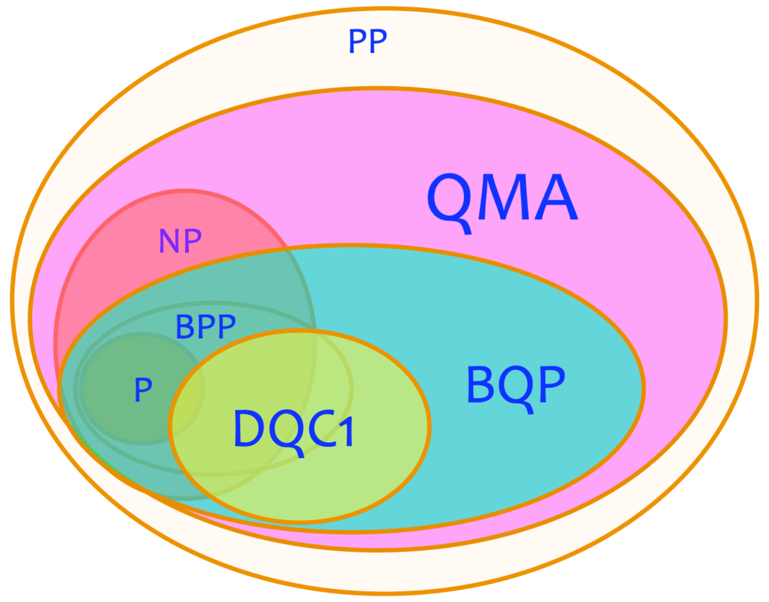


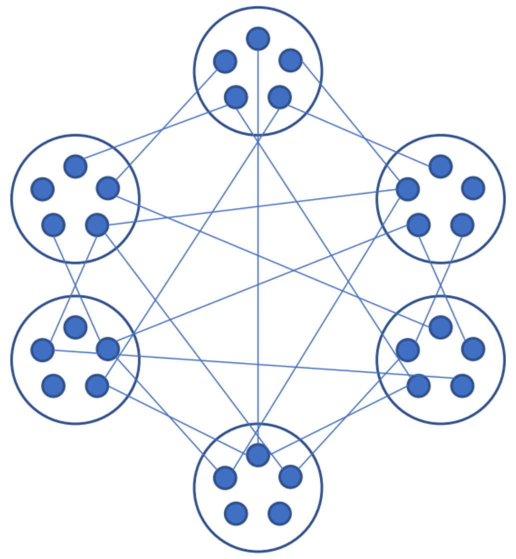
medicine, materials,
entanglement, time-series…



Complex network
analysis

(Q)TDA



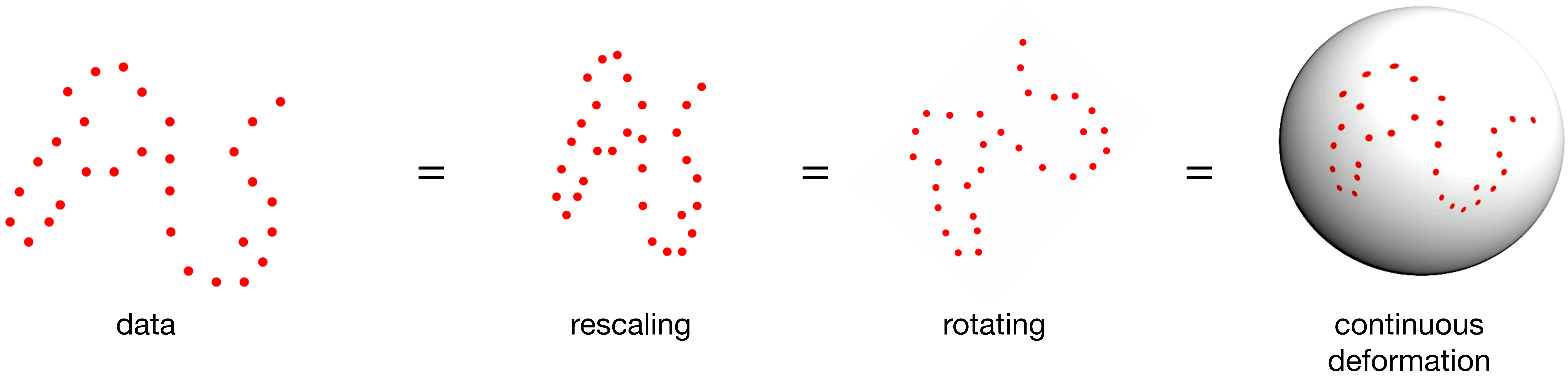


80 billion Toffolis in TDA…

*Image credit: Scientific American, JSTOR

Topological Data Analysis

Machine learning: about “robust” properties of data



Topological Data Analysis

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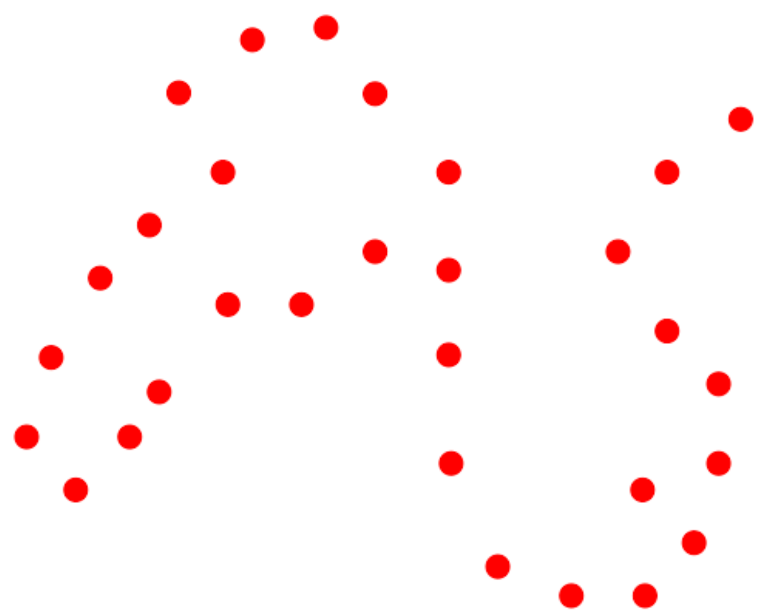
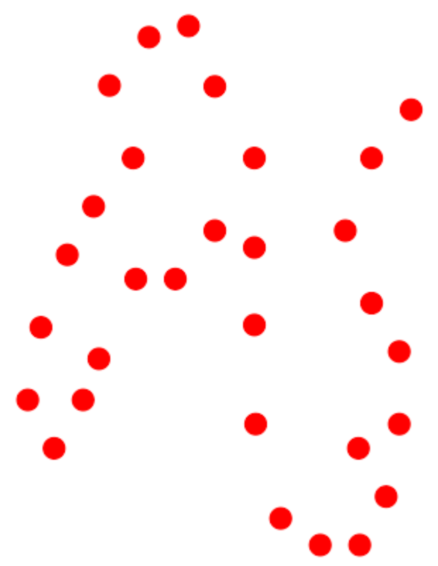


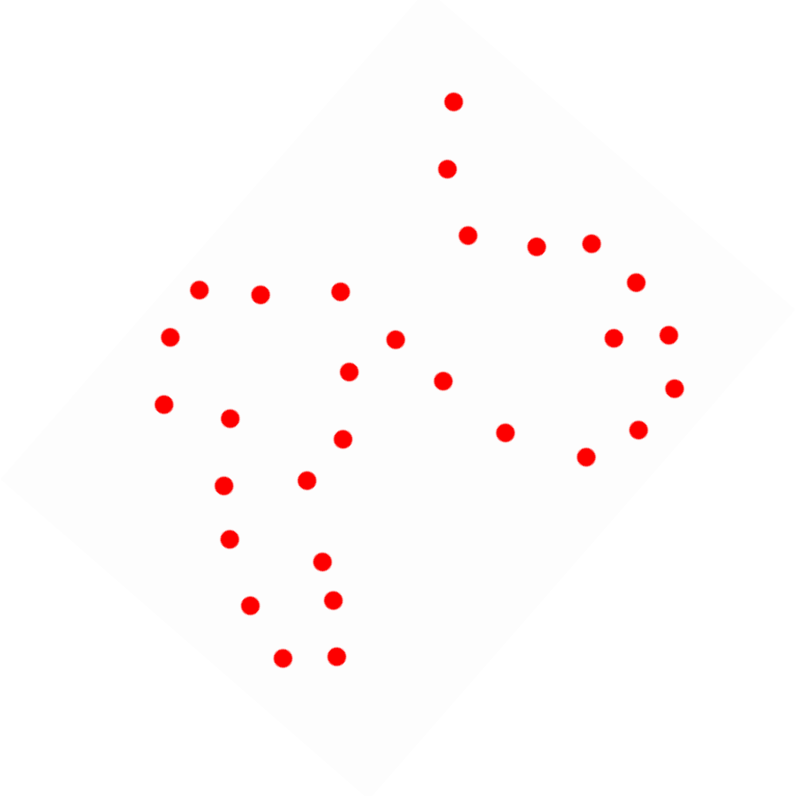
data

=



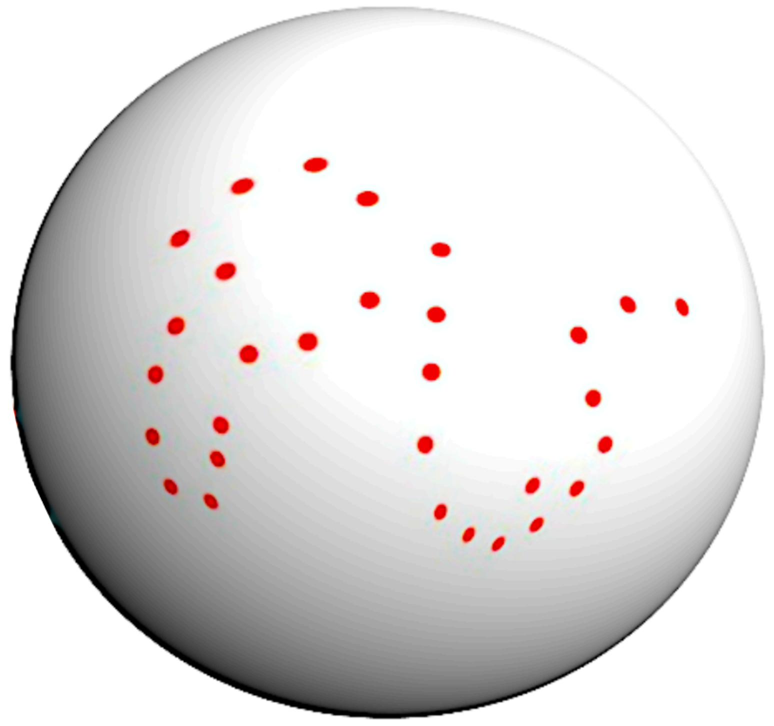
rescaling

=



rotating

=



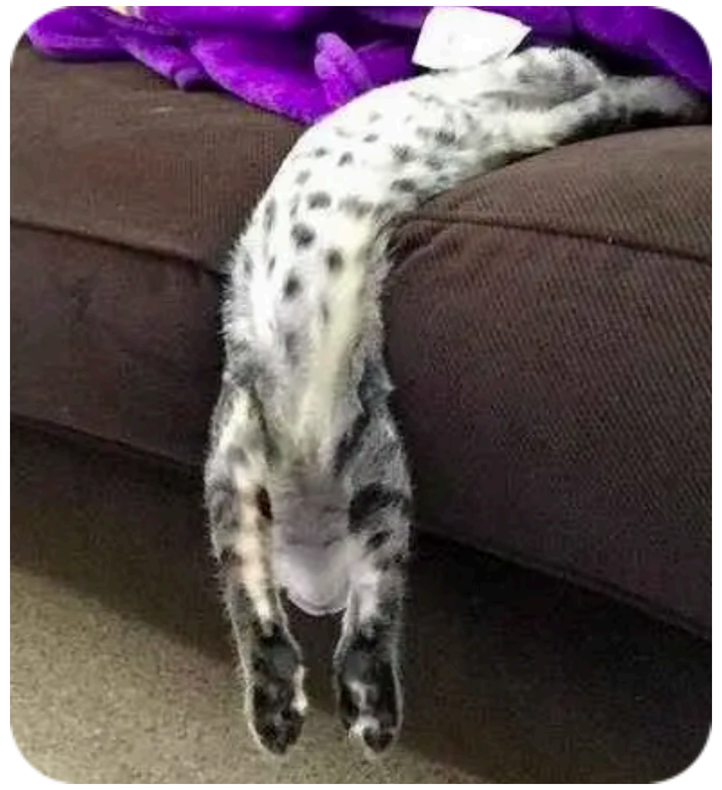
continuous
deformation

A

is a

is a

is a



Topological Data Analysis

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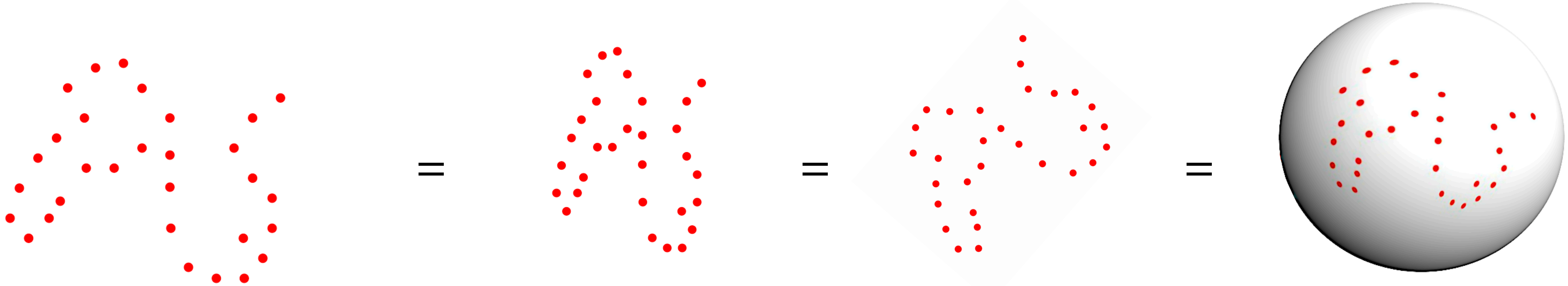
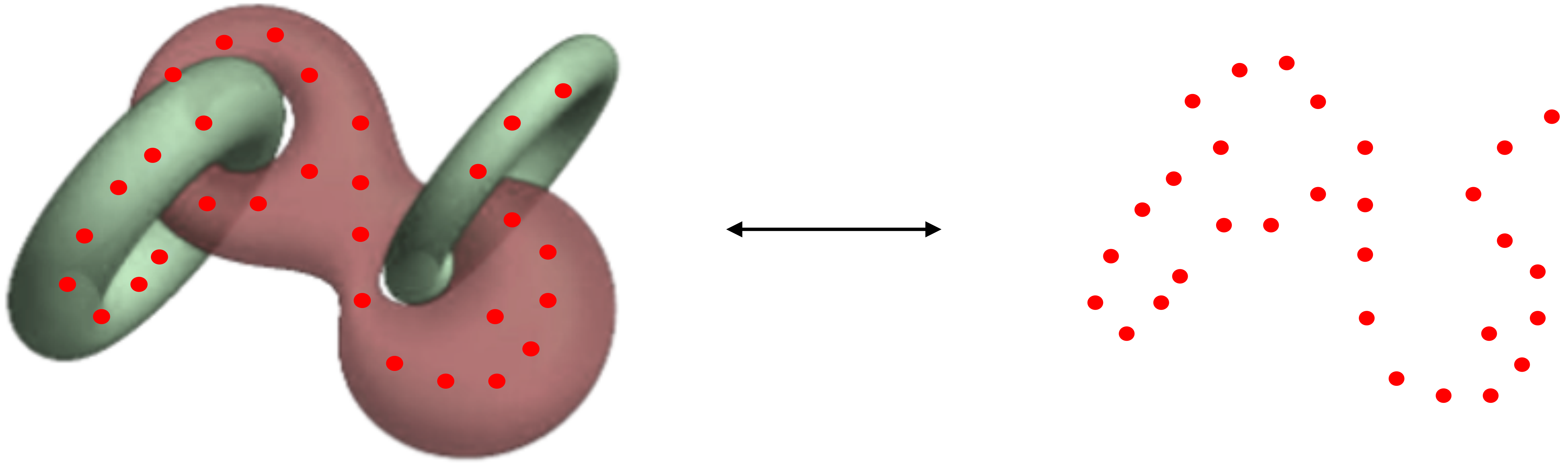


data = rescaling = rotating = continuous deformation

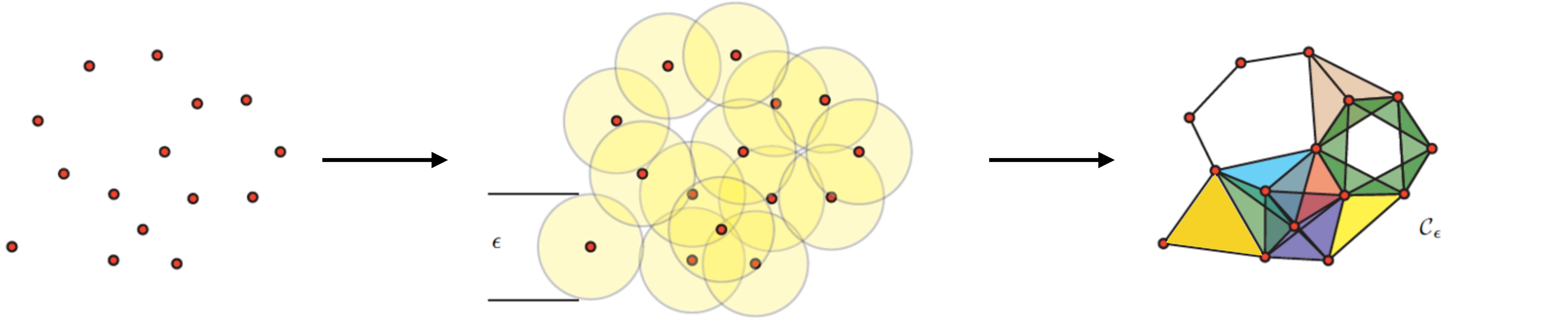
Remains:



Topology of *data*

Topological Data Analysis

part of pipeline



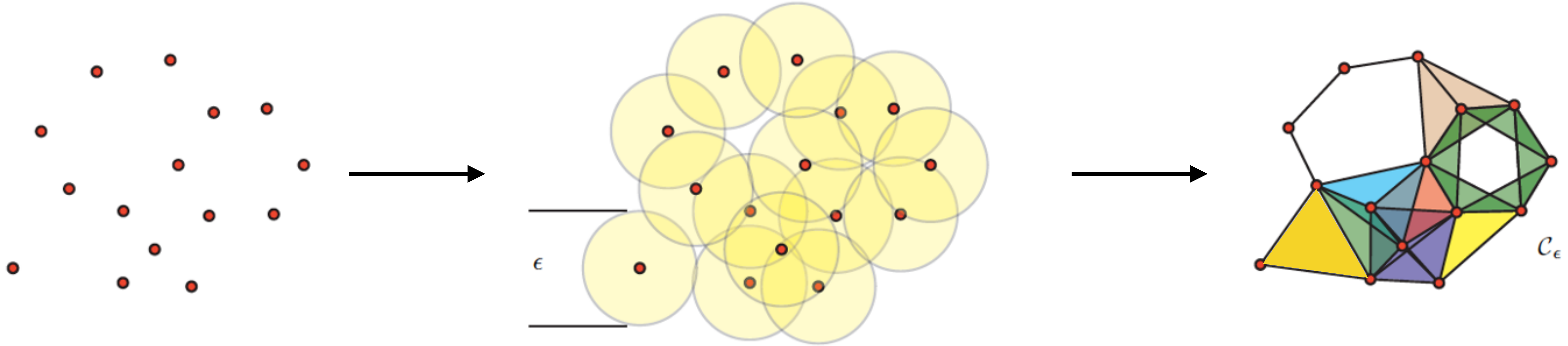
“connect if close”

clique complex (graph)

“simplicial complex”

Topological Data Analysis

part of pipeline



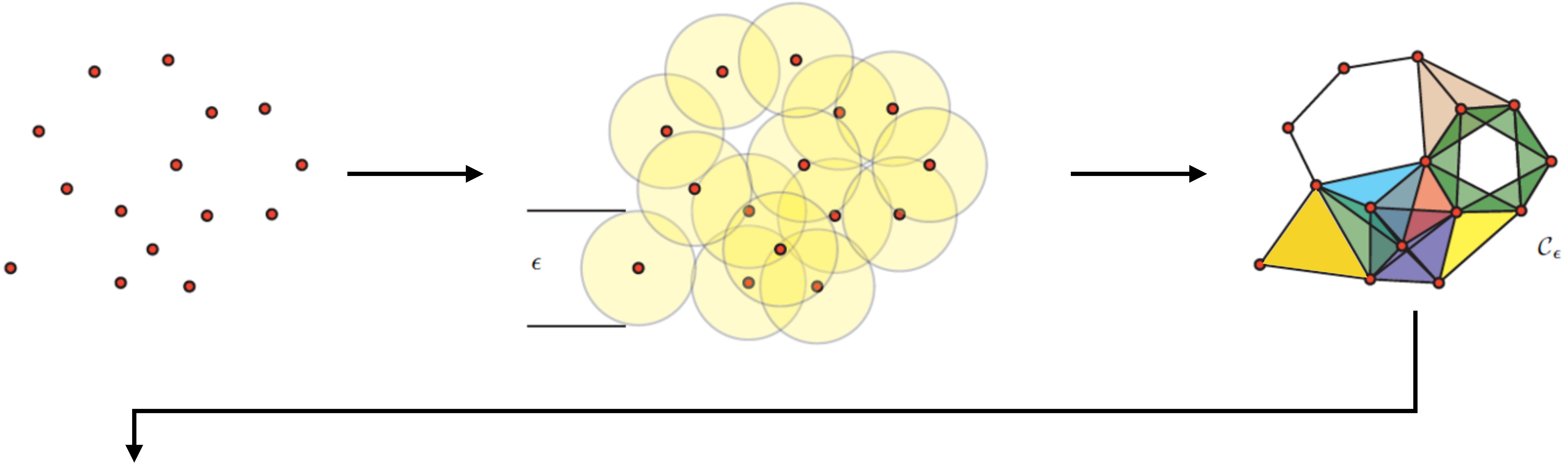
features = # *k*-dimensional *holes*

= $(\beta_k)_k$ *(Betti numbers)*

*(persistent homology, barcodes, consider all ϵ)

Topological Data Analysis

part of pipeline



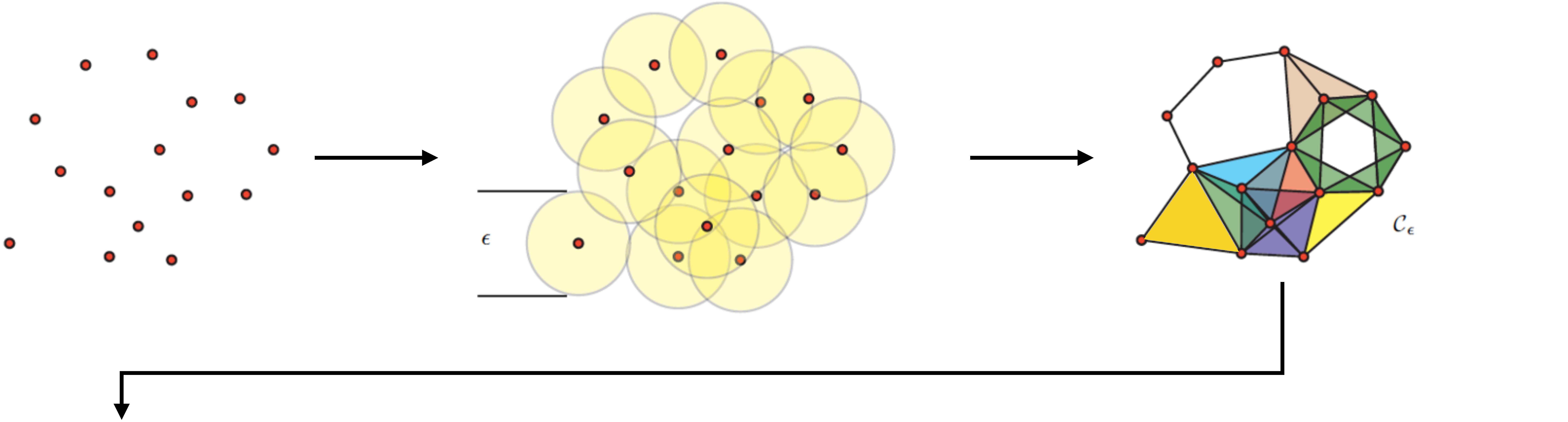
$$\Delta_k^G$$

"connectivity" of $(k+1)$-cliques

combinatorial Laplacian

Topological Data Analysis

part of pipeline



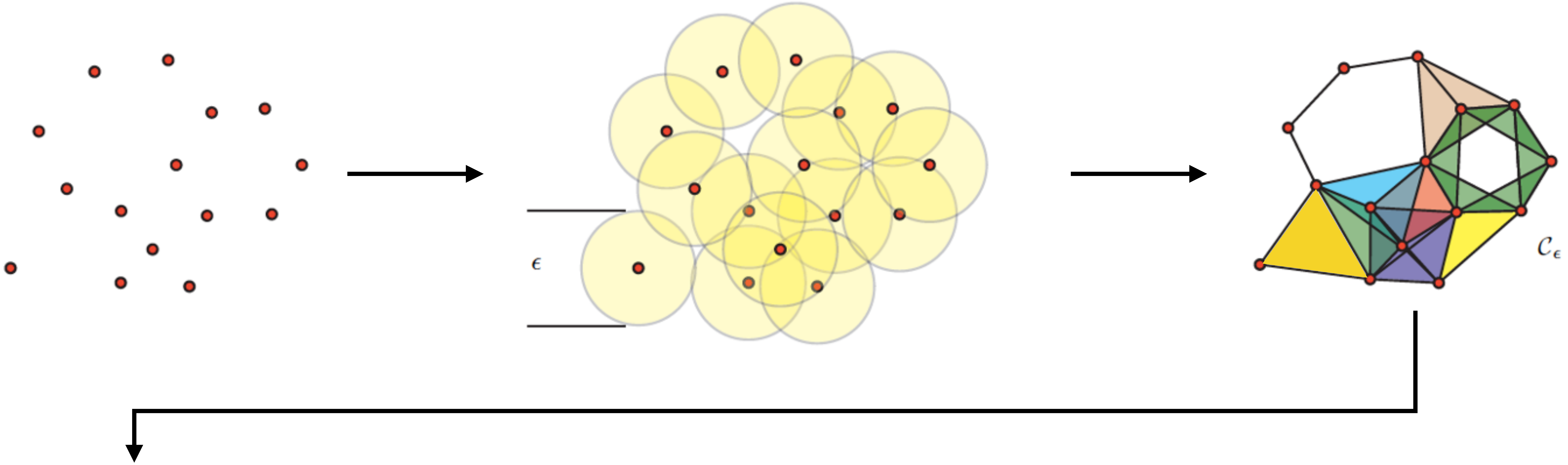
$$\Delta_k^G$$

“connectivity” of $(k+1)$-cliques for β_k $\qquad \beta_k = dim(Ker(\Delta_k))$

combinatorial Laplacian

Topological Data Analysis

part of pipeline



$$\Delta_k^G$$

combinatorial Laplacian

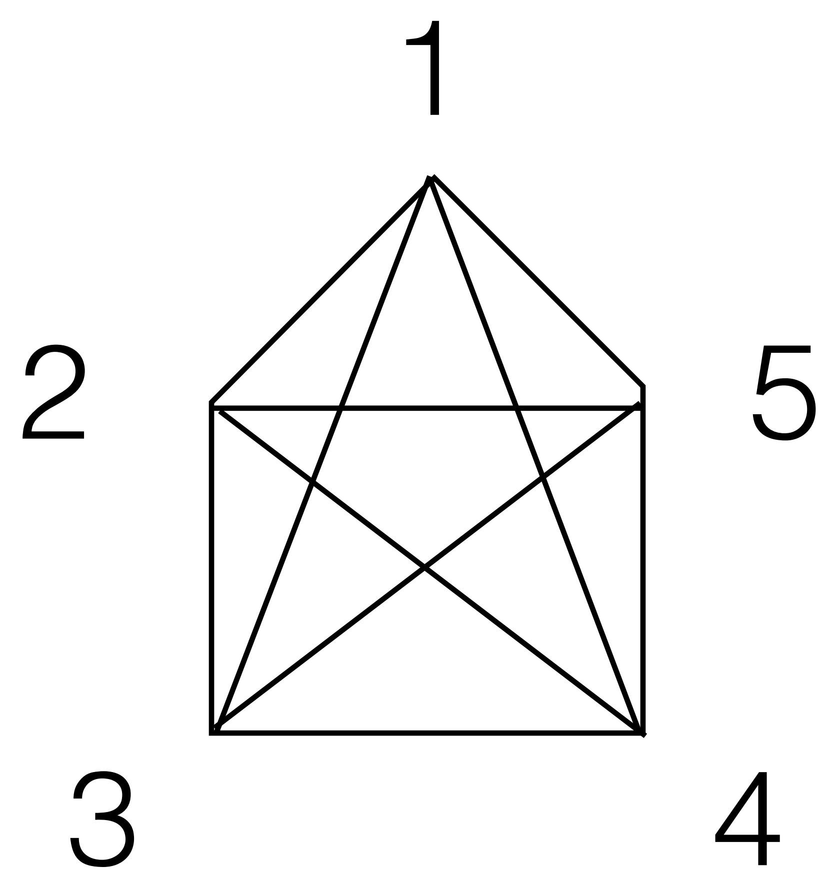
$\binom{n}{k+1}$ – dimensional…

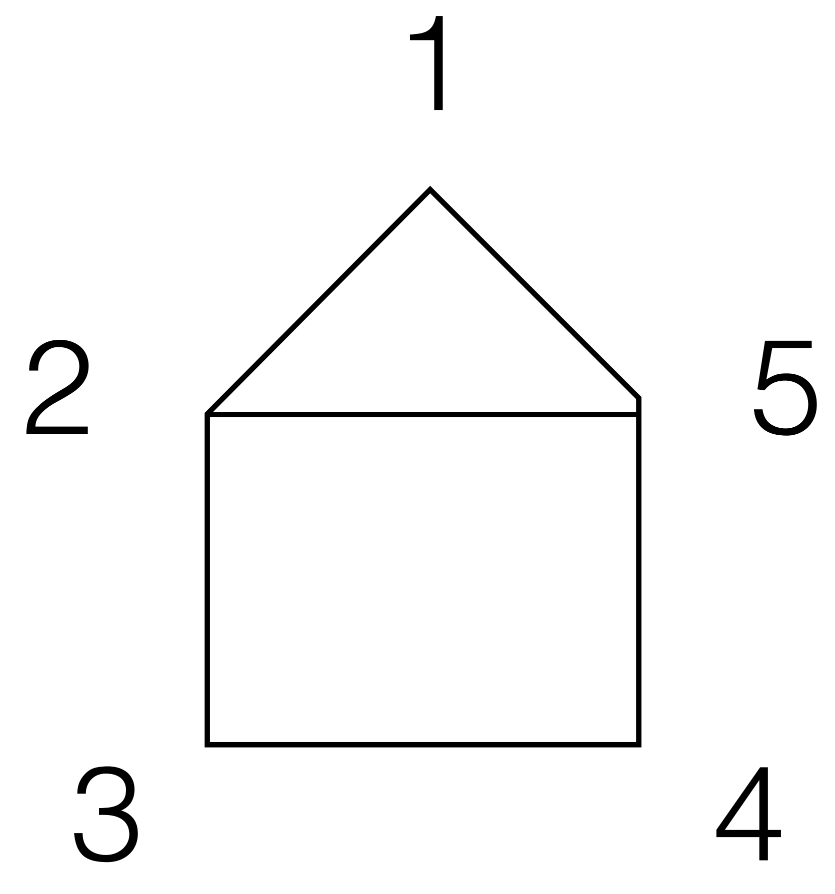
Hermitian… sparse access…

Hint-hint

$$\beta_k = dim(Ker(\Delta_k))$$

Input: Graph. Vertices = qubits.

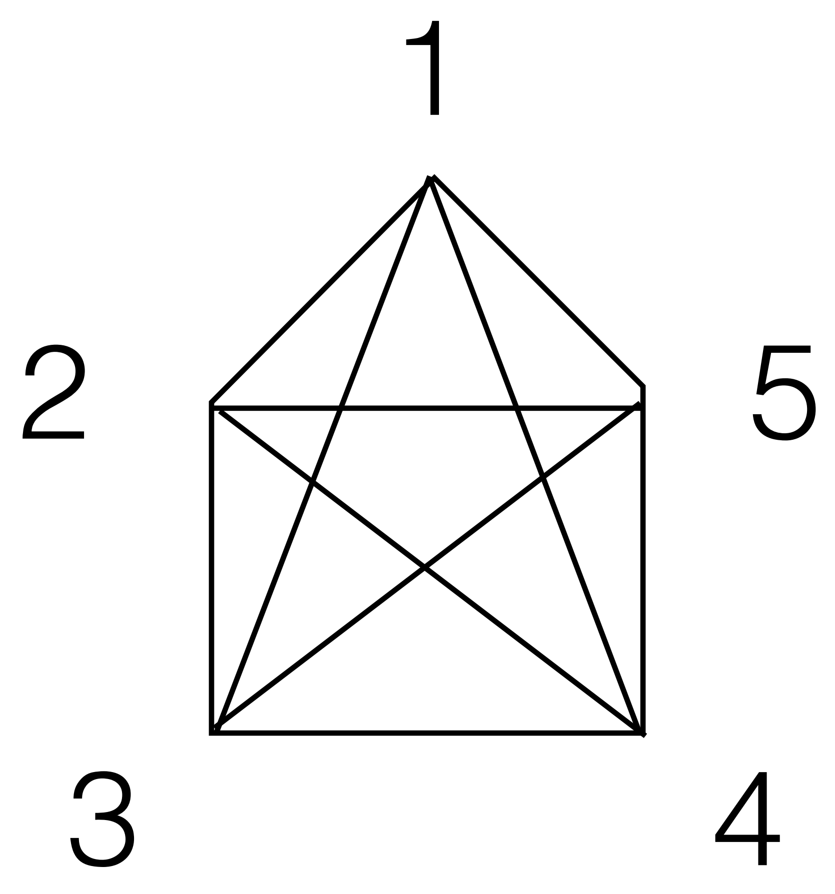


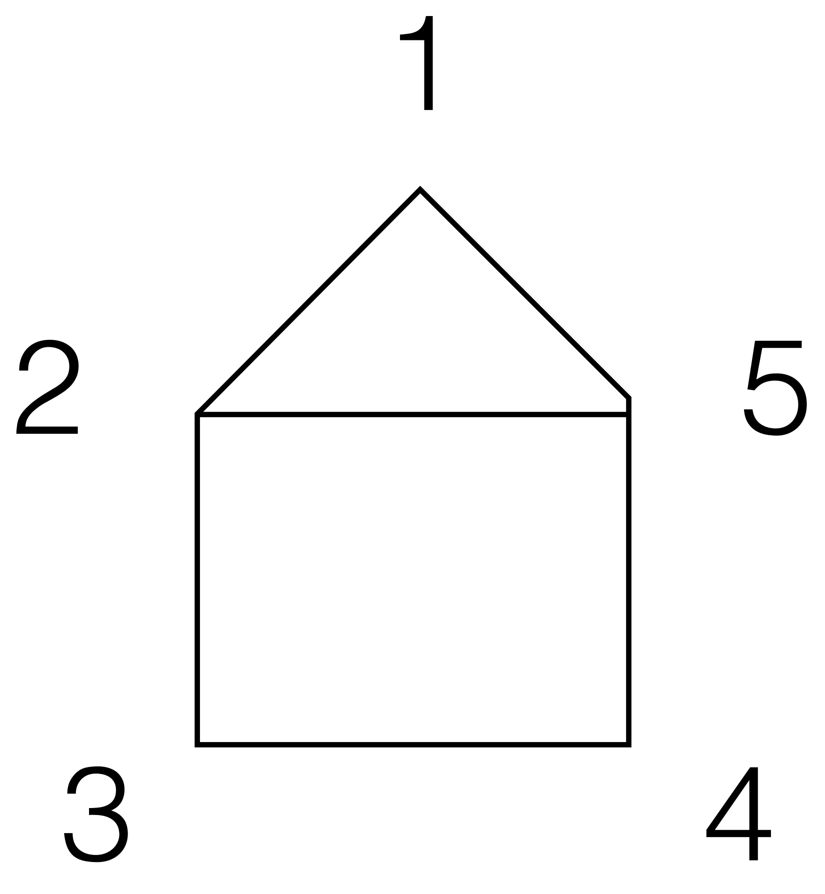


$|11100\rangle$ - 3 clique

$|11100\rangle$ - not a 3 clique

Boundary map ($\partial_k^G : \mathscr{H}_{k+1}^G \to \mathscr{H}_k^G$)





$$\partial_k |x\rangle = \sum_{j=0}^{k} (-1)^j |x\backslash(j)\rangle$$

set j-th to zero

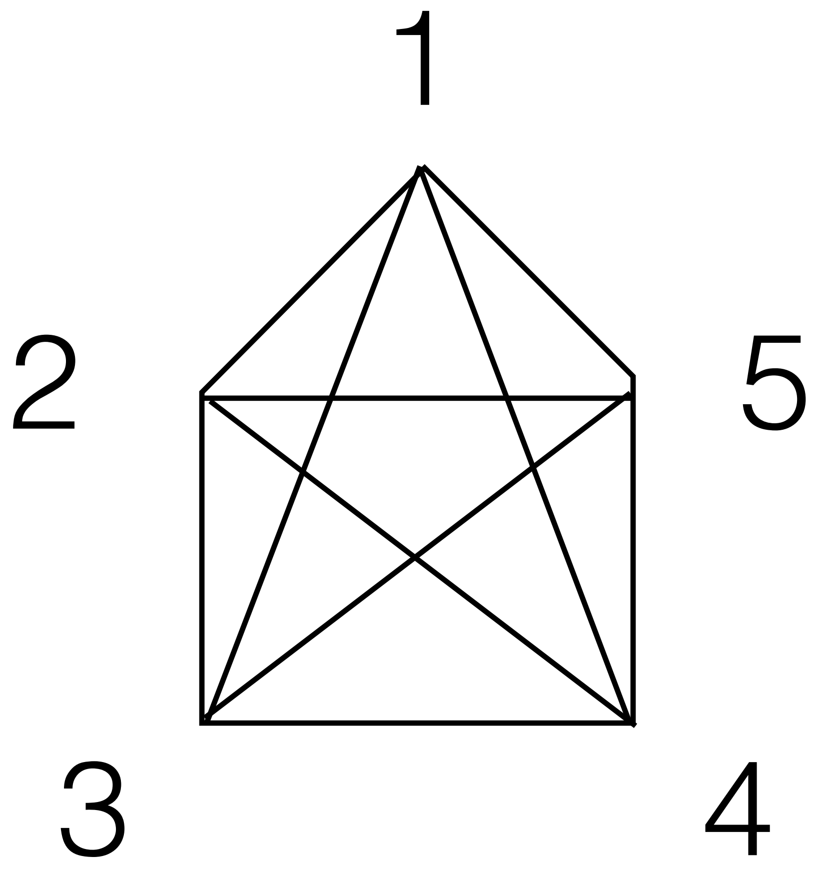
$$|11100\rangle \to |01100\rangle - |10100\rangle + |11000\rangle$$

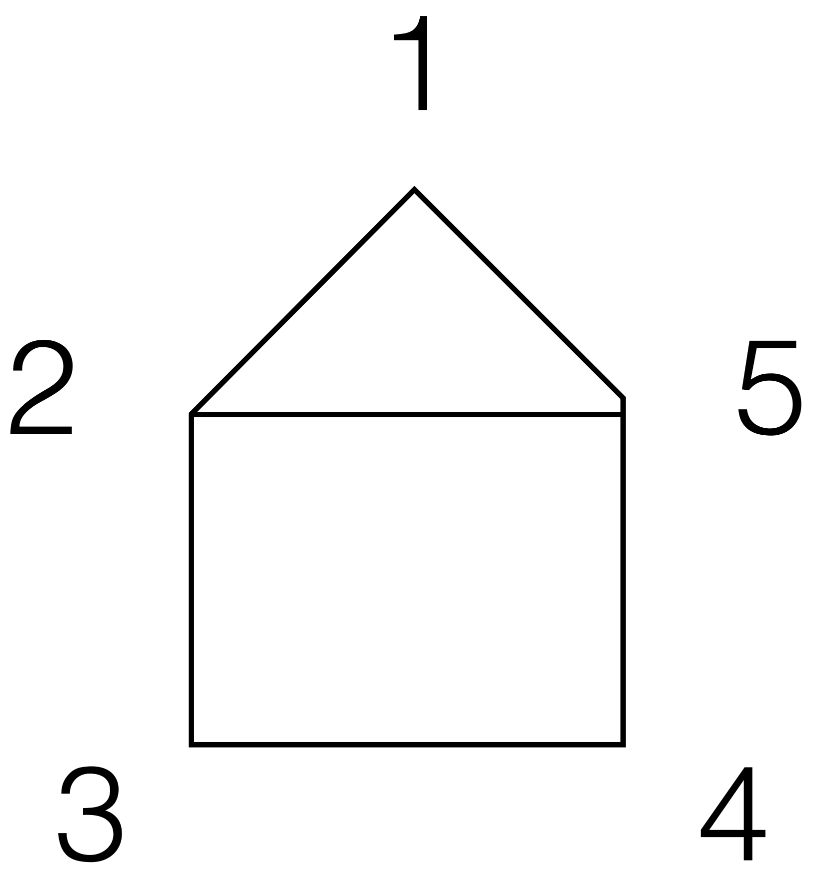
$$\partial_k^G = P_k^G(\partial_k)P_{k+1}^G$$

$$P_k^G = \sum_{c \in Cl_k(G)} |c\rangle\langle c|$$

Restriction to *G will be vital*

Boundary map and combinatorial Laplacian



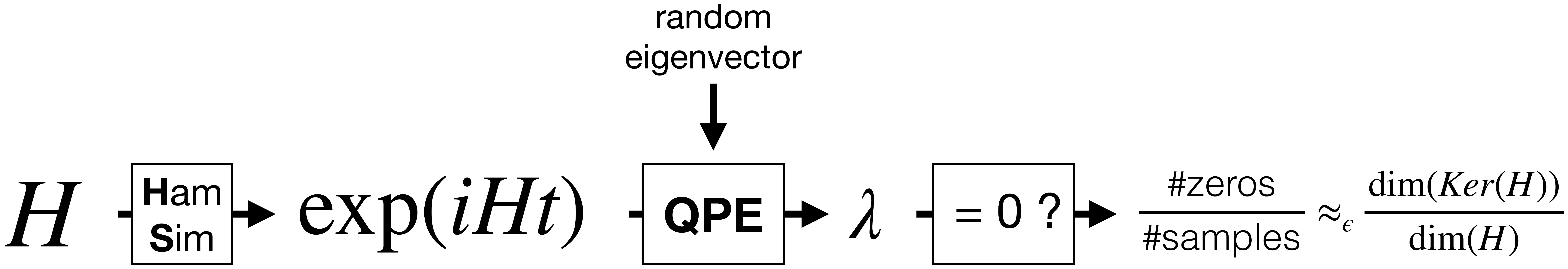


$$\Delta_k^G = \partial^{G\dagger}_k \partial^G_k + \partial^G_{k+1} \partial^{G\dagger}_{k+1}$$

$$dim(Ker(\Delta_k)) = \beta_k$$

compute on a QC!

Lloyd, Garnerone, Zanardi (LGZ)* ideas



*S. Lloyd, S. Garnerone & P. Zanardi, Nat Commun. Vol. 7, Article no.: 10138 (2016)

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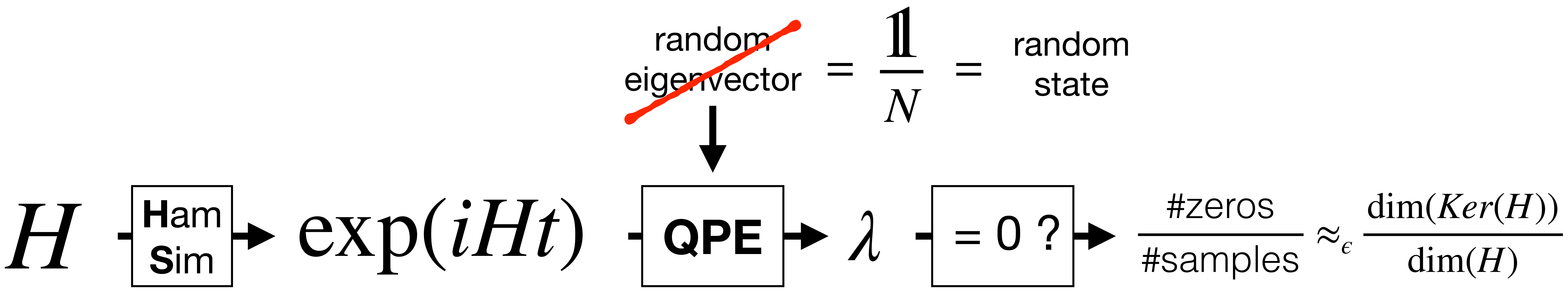

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Lloyd, Garnerone, Zanardi (LGZ)* ideas

~~random eigenvector~~ $= \frac{\mathbb{1}}{N} =$ random state

H → [Ham Sim] → $\exp(iHt)$ → [QPE] → λ → [= 0 ?] → $\frac{\text{\#zeros}}{\text{\#samples}} \approx_{\epsilon} \frac{\dim(Ker(H))}{\dim(H)}$

$$H \to \Delta^G \qquad \partial_k^G = P_k^G(\partial_k)P_{k+1}^G$$

must only operate on *valid cliques*

*S. Lloyd, S. Garnerone & P. Zanardi, Nat Commun. Vol. 7, Article no.: 10138 (2016)

Lloyd, Garnerone, Zanardi (LGZ)* ideas

~~random eigenvector~~ $= \dfrac{\sum_{\vec{b} \in Cl_k(G)} |\vec{b}\rangle\langle\vec{b}|}{|Cl_k(G)|} =$ random clique in graph

↓

H_G → **Ham Sim** → $\exp(iHt)$ → **QPE** → λ → $= 0$? → $\dfrac{\#\text{zeros}}{\#\text{samples}} \approx_\epsilon \dfrac{\dim(Ker(H))}{\dim(H)}$

+ some projections needed

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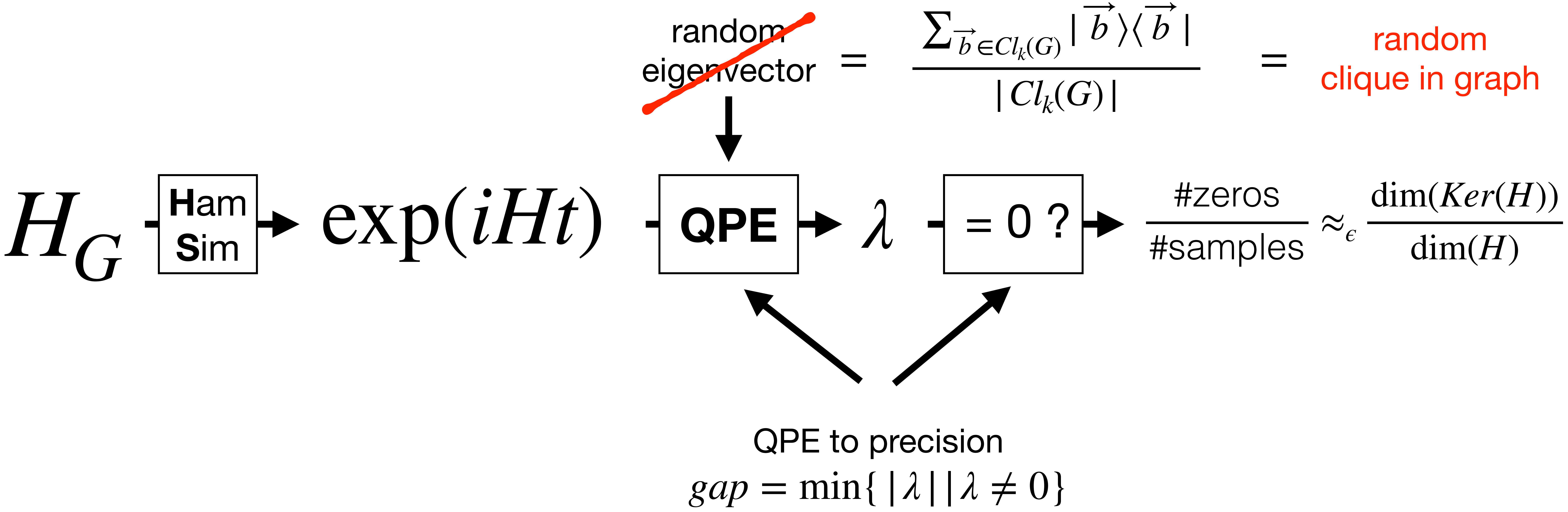

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H_G → [Ham Sim] → $\exp(iHt)$ → [**QPE**] → λ → [$= 0$?] → $\dfrac{\#\text{zeros}}{\#\text{samples}} \approx_\epsilon \dfrac{\dim(Ker(H))}{\dim(H)}$

Quantum costs:

- Ham. sim. = cheap *(low-deg poly n)*
- QPE to prec. *gap = could be cheap*
- *random clique sampling*

Classical (vanilla) costs:

- $O(n^k)$
- note if $k \sim n \;\rightarrow\; O(\exp(n))$

*S. Lloyd, S. Garnerone & P. Zanardi, "Nat Commun. Vol. 7, Article no.: 10138 (2016)

details matter

Lloyd, Garnerone, Zanardi (LGZ)* ideas

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$$H_G \text{ — [Ham Sim]} \rightarrow \exp(iHt) \text{ — [QPE]} \rightarrow \lambda \text{ — [} = 0\text{ ?]} \rightarrow \frac{\#\text{zeros}}{\#\text{samples}} \approx_\epsilon \frac{\dim(Ker(H))}{\dim(H)}$$

Quantum costs:

- Ham. sim. = cheap *(low-deg poly n)*
- QPE to prec. *gap* = *could be cheap*
- *random clique sampling* = ***NP-hard***

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$$H_G \xrightarrow{\text{Ham Sim}} \exp(iHt) \rightarrow \text{QPE} \rightarrow \lambda \rightarrow [= 0\,?] \rightarrow \frac{\#\text{zeros}}{\#\text{samples}} \approx_\epsilon \frac{\dim(Ker(H))}{\dim(H)}$$

Quantum

Efficient if clique sampling efficient

Dense graphs.

Classical (vanilla) costs:

Vanilla algorithm still exponential

But this is certainly a *special case*

*S. Lloyd, S. Garnerone & P. Zanardi, "Nat Commun. Vol. 7, Article no.: 10138 (2016)

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H_G → [Ham Sim] → $\exp(iHt)$ → [QPE] → λ → [= 0 ?] → $\frac{\#\text{zeros}}{\#\text{samples}} \approx_\epsilon \frac{\dim(Ker(H))}{\dim(H)}$

Quantum

Estimates normalized

Betti numbers $= \frac{\beta_k}{\#k\text{-cliques}}$

Classical (vanilla) costs:

Gets exact Betti!

*S. Lloyd, S. Garnerone & P. Zanardi, "Nat Commun. Vol. 7, Article no.: 10138 (2016)

Lloyd, Garnerone, Zanardi (LGZ)* ideas

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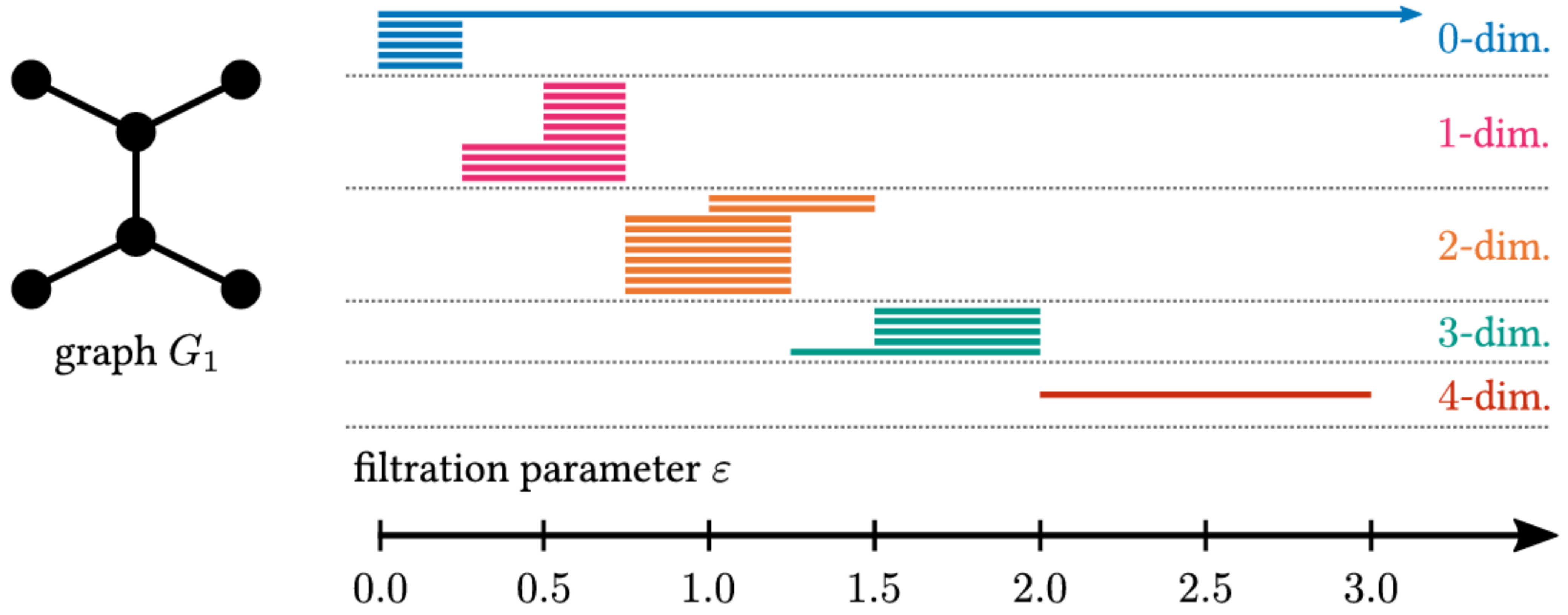
H_G → **Ham Sim** → $\exp(iHt)$ → **QPE** → λ → **= 0 ?** → $\dfrac{\#\text{zeros}}{\#\text{samples}} \approx_\epsilon \dfrac{\dim(Ker(H))}{\dim(H)}$

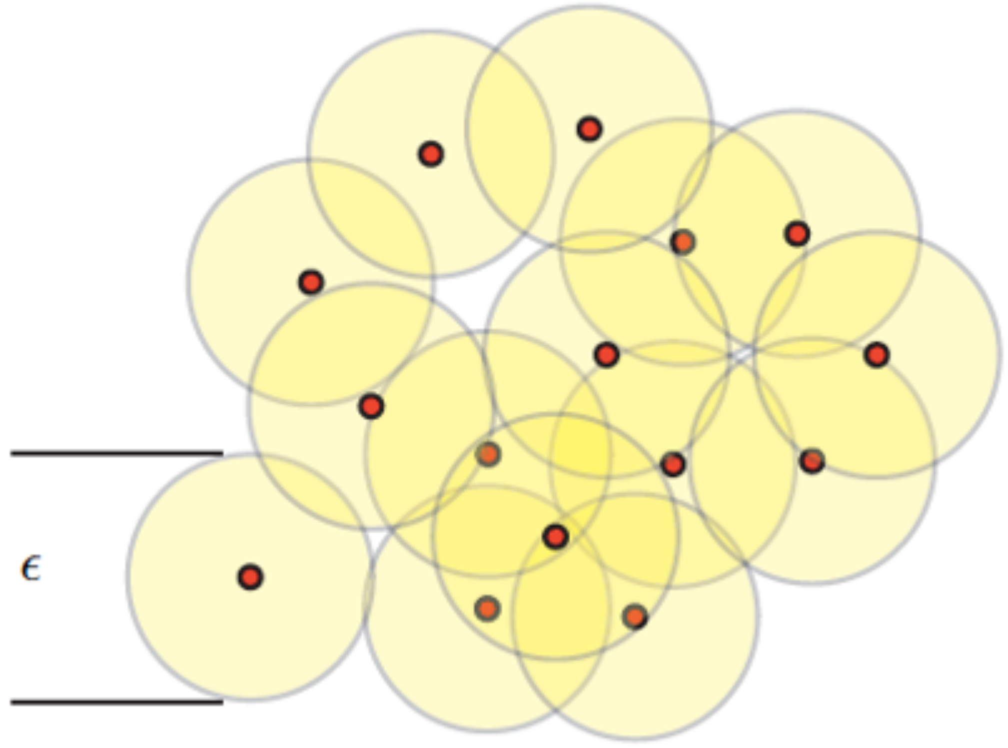
Quantum computers enable

efficient (*additive* error) estimation of *normalized approximate Betti numbers in regimes where clique sampling is efficient*

*S. Lloyd, S. Garnerone & P. Zanardi, "Nat Commun. Vol. 7, Article no.: 10138 (2016)

What is the context! TDA v.s. QTDA





G. A. Hamilton, F. Leditzky, arXiv:2307.07492 (2023)

Image source: https://indico.cern.ch/event/958074/contributions/4133637/attachments/2163528/3652970/Shiu-sd2020.pdf

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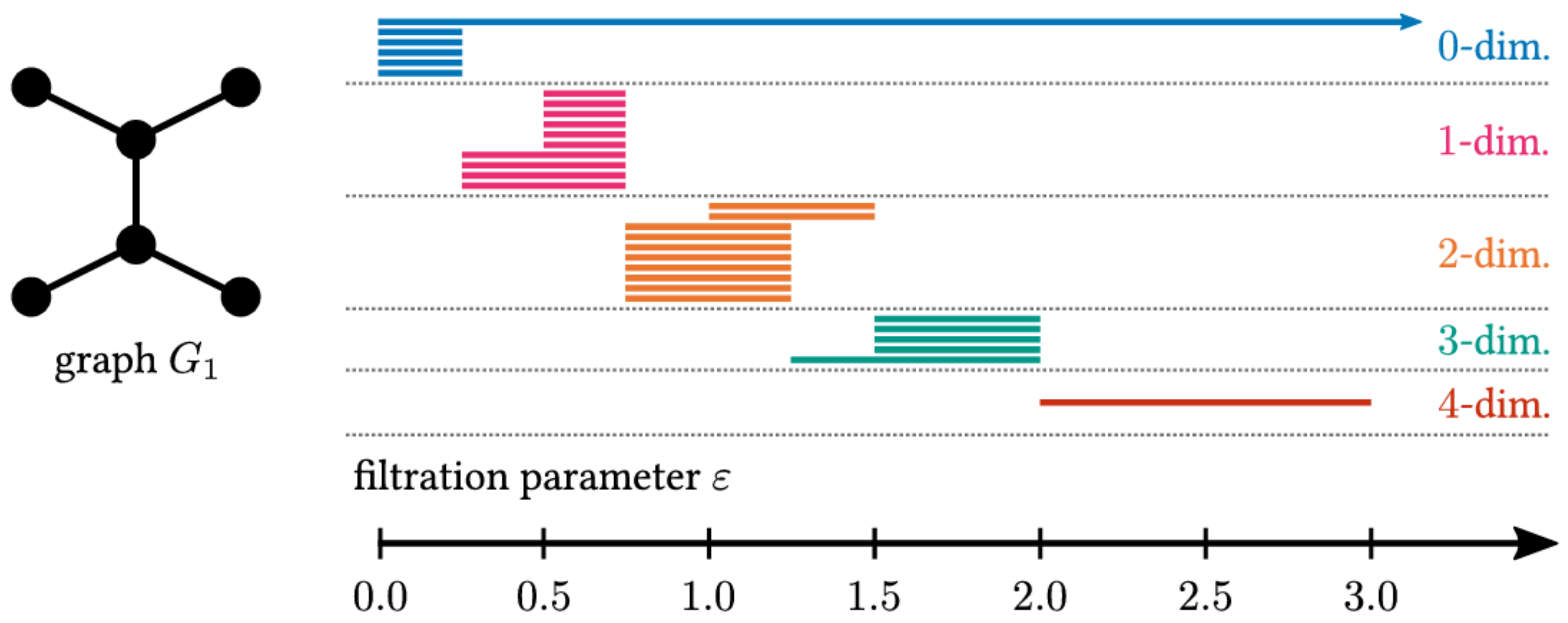

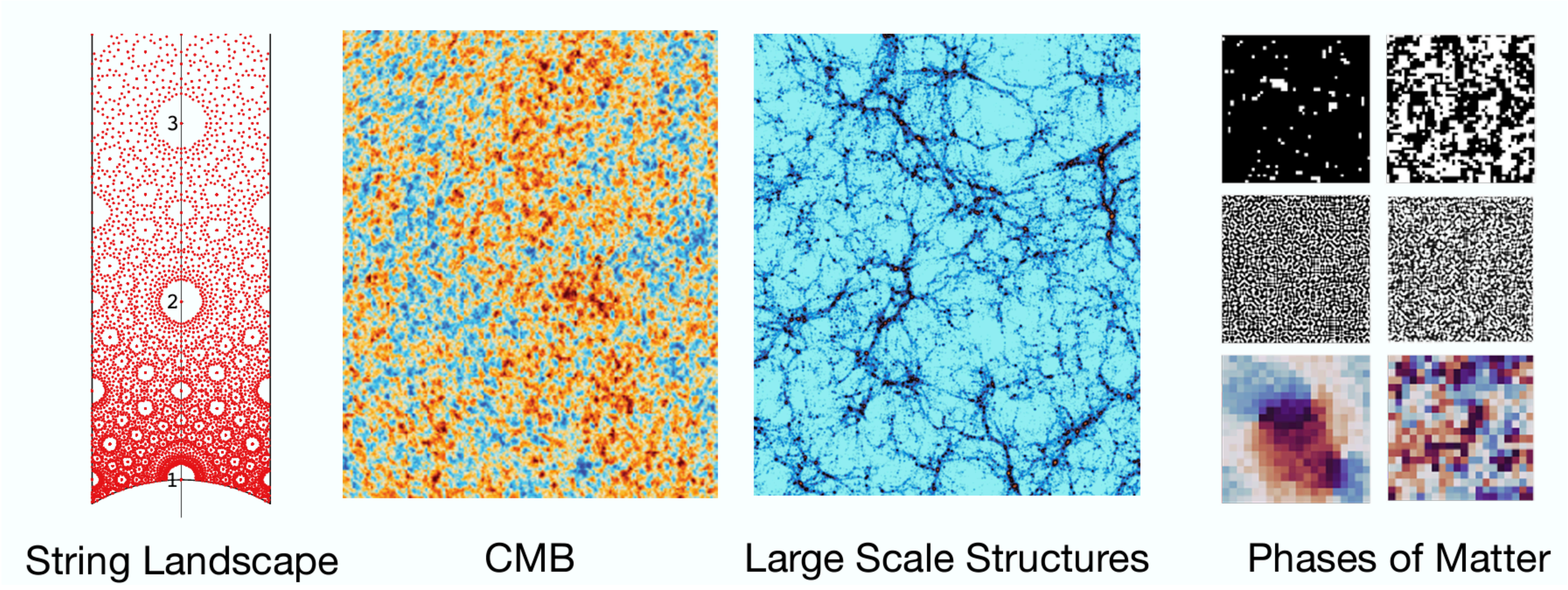

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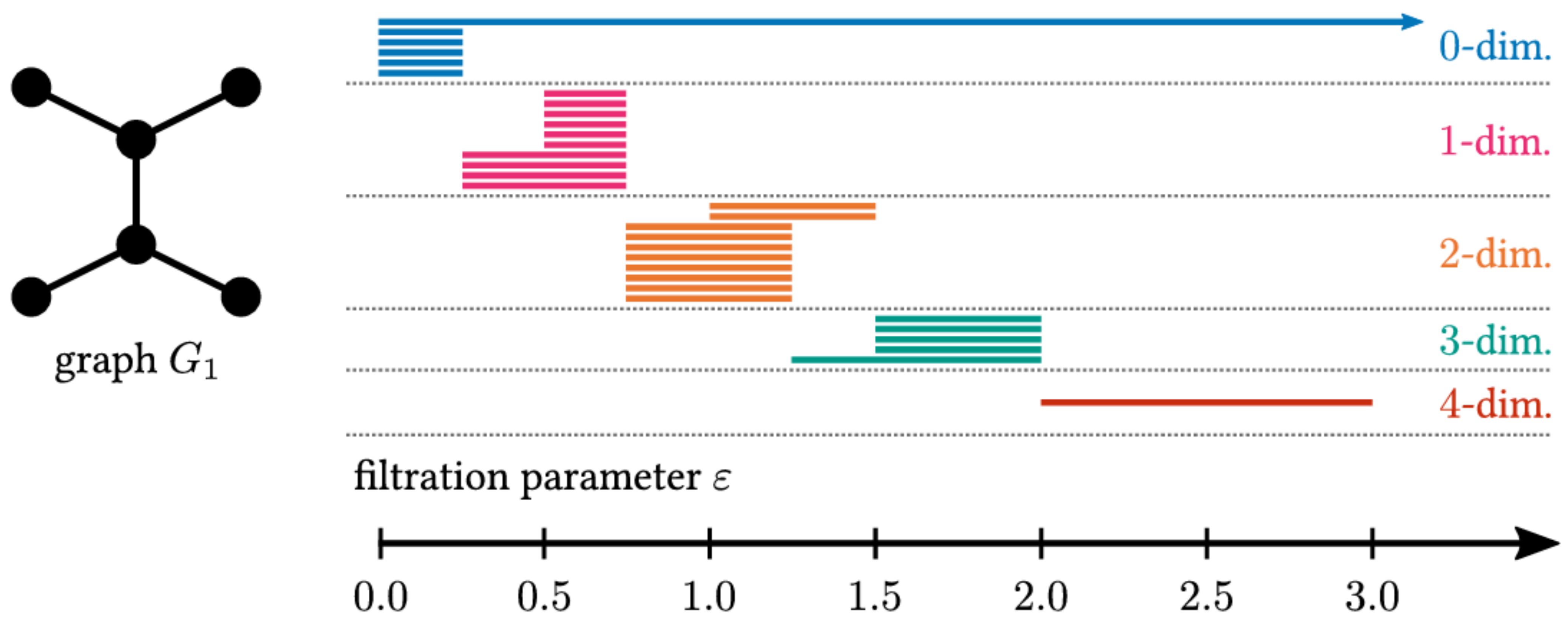

Persistent Homology of $\mathbb{Z}_2$ Gauge Theories

Dan Sehayek and Roger G. Melko
Department of Physics and Astronomy, University of Waterloo, Ontario, N2L 3G1, Canada and Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada

(Dated: September 16, 2022)

Probing center vortices and deconfinement in SU(2) lattice gauge theory with persistent homology

Nicholas Sale* and Biagio Lucini†
Department of Mathematics, Swansea University, Bay Campus, SA1 8EN, Swansea, Wales, UK

Jeffrey Giansiracusa
Department of Mathematical Sciences, Durham University, Upper Mountjoy Campus, Durham, DH1 3LE, UK

(Dated: January 16, 2023)

Article | Open access | Published: 15 August 2022

Precision dynamical mapping using topological data analysis reveals a hub-like transition state at rest

Manish Saggar, James M. Shine, Raphaël Liégeois, Nico U. F. Dosenbach & Damien Fair

Nature Communications **13**, Article number: 4791 (2022) | Cite this article

Article | Open access | Published: 11 April 2018

Towards a new approach to reveal dynamical organization of the brain using topological data analysis

Manish Saggar, Olaf Sporns, Javier Gonzalez-Castillo, Peter A. Bandettini, Gunnar Carlsson, Gary Glover & Allan L. Reiss

Nature Communications **9**, Article number: 1399 (2018) | Cite this article

Article | Published: 21 July 2015

Topological data analysis of contagion maps for examining spreading processes on networks

Dane Taylor, Florian Klimm, Heather A. Harrington, Miroslav Kramár, Konstantin Mischaikow, Mason A. Porter & Peter J. Mucha

Nature Communications **6**, Article number: 7723 (2015) | Cite this article

Topological Data Analysis of Financial Time Series: Landscapes of Crashes

Marian Gidea
Yeshiva University, Department of Mathematical Sciences, New York, NY 10016, USA
Yuri Katz
S&P Global Market Intelligence, 55 Water Str., New York, NY 10040, USA

G. A. Hamilton, F. Leditzky, arXiv:2307.07492 (2023)

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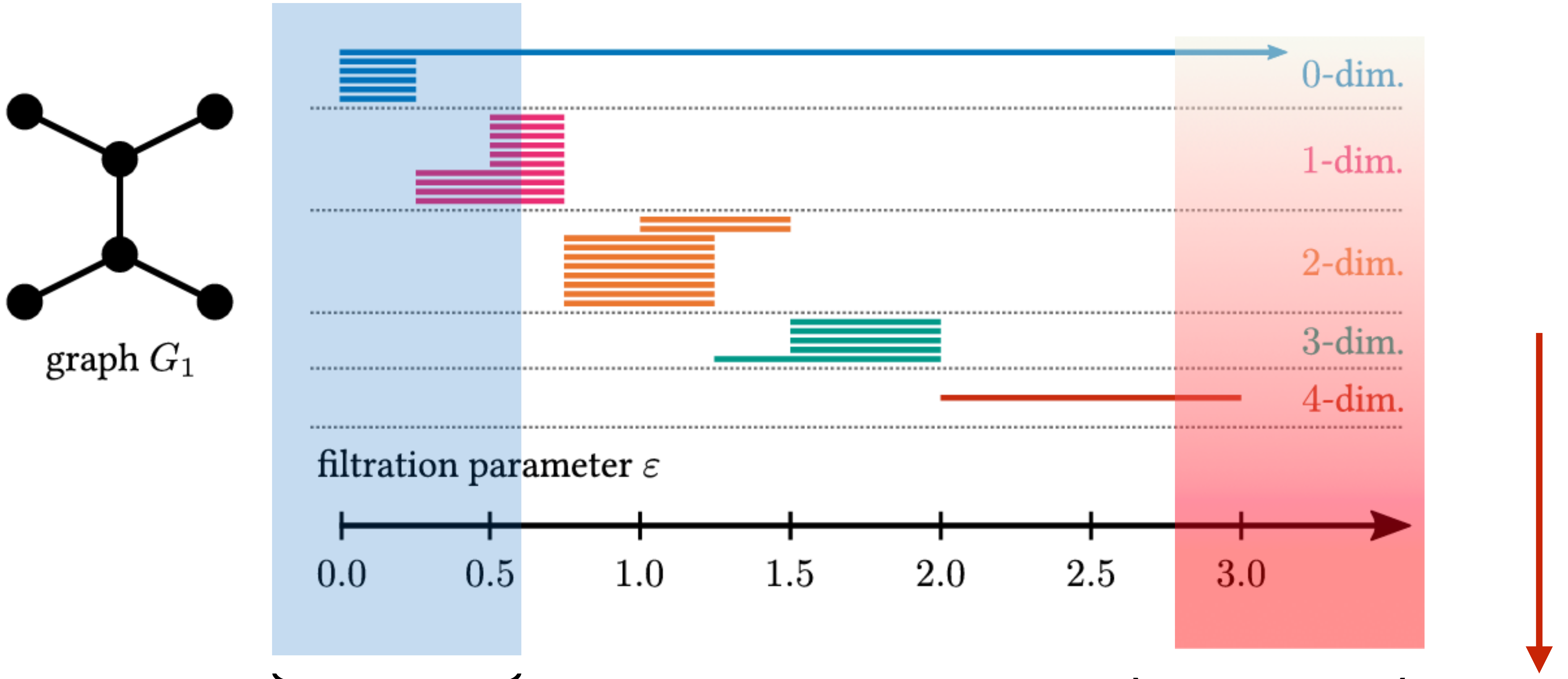
What is the context! TDA v.s. QTDA



need to need
high Bettis
for Q. advantage

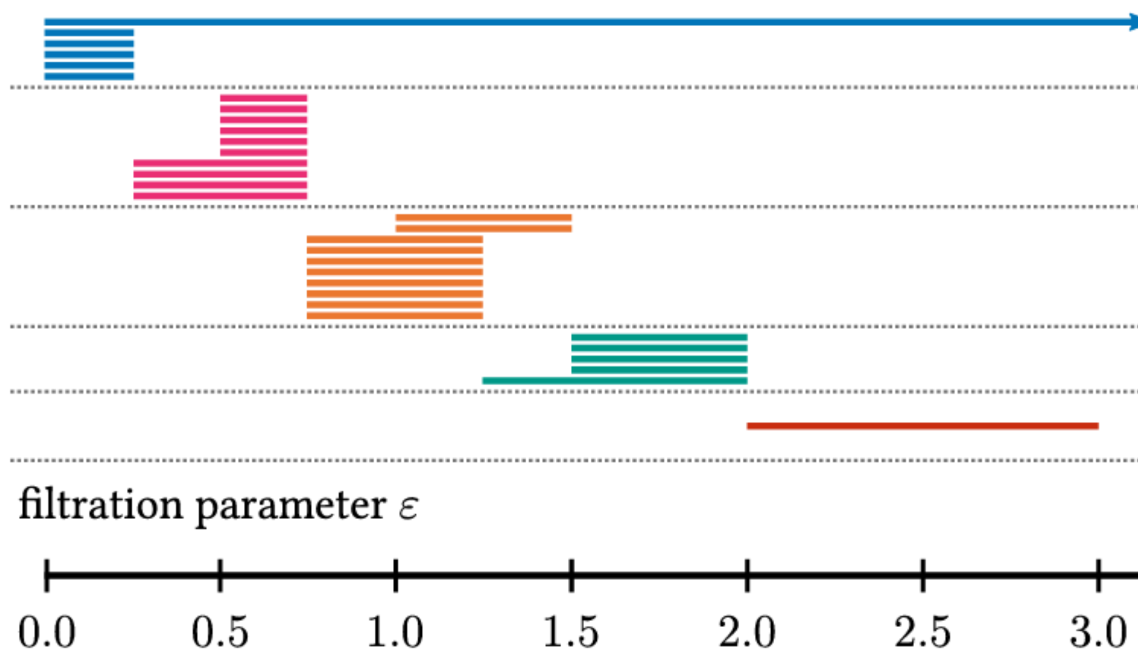
classical and
quantum efficient

quantum
efficient for
normalized Betti

classical
unclear

G. A. Hamilton, F. Leditzky, arXiv:2307.07492 (2023)

Questions



1) **can we do *significantly better* on a QC?**
   - *better precision?*
   - *full range of densities?*

2) **is there a *guaranteed* quantum advantage *for what we do have*, and can it be relevant?**
   - *hardness of TDA?*
   - *when is the QC algorithm truly faster against vanilla…*
   - *…and new classical algorithms?*

3) **Beyond basic TDA applications, or**
   - are there better applications we are missing

Can we do *significantly better* on a QC?

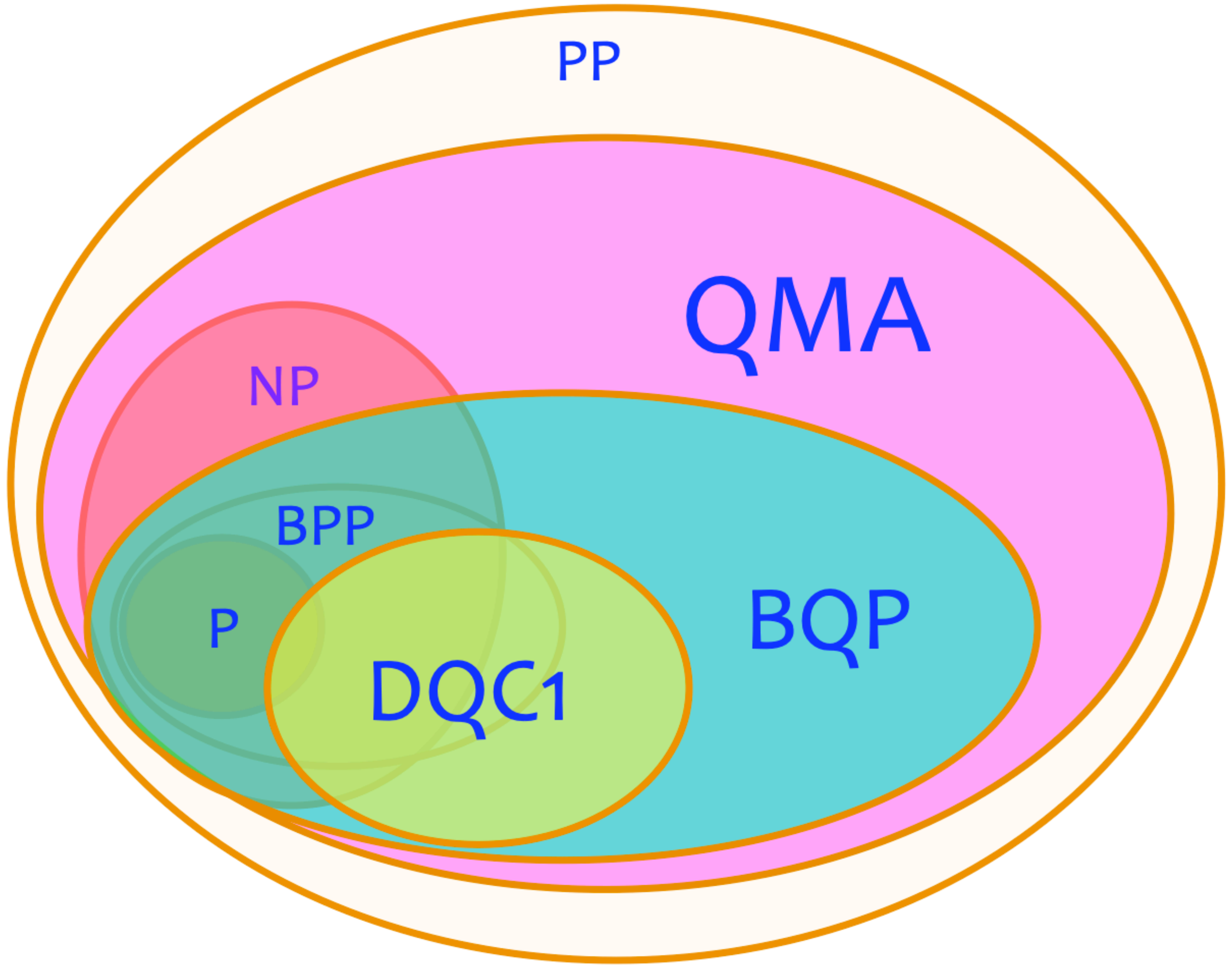
- *better precision?*
- *full range of densities?*

No.

No.

Multiplicative estimation of Betti numbers is QMA1-hard (2022)

New (King,Kohler, ’23-ish):
under *promised eigenvalue gap,*
QMA1-hard and *in QMA*



M. Crichigno, T. Kohler, Clique Homology is QMA1-hard (2022)
R. King,T. Kohler, ??? (in 1 week?)

No.

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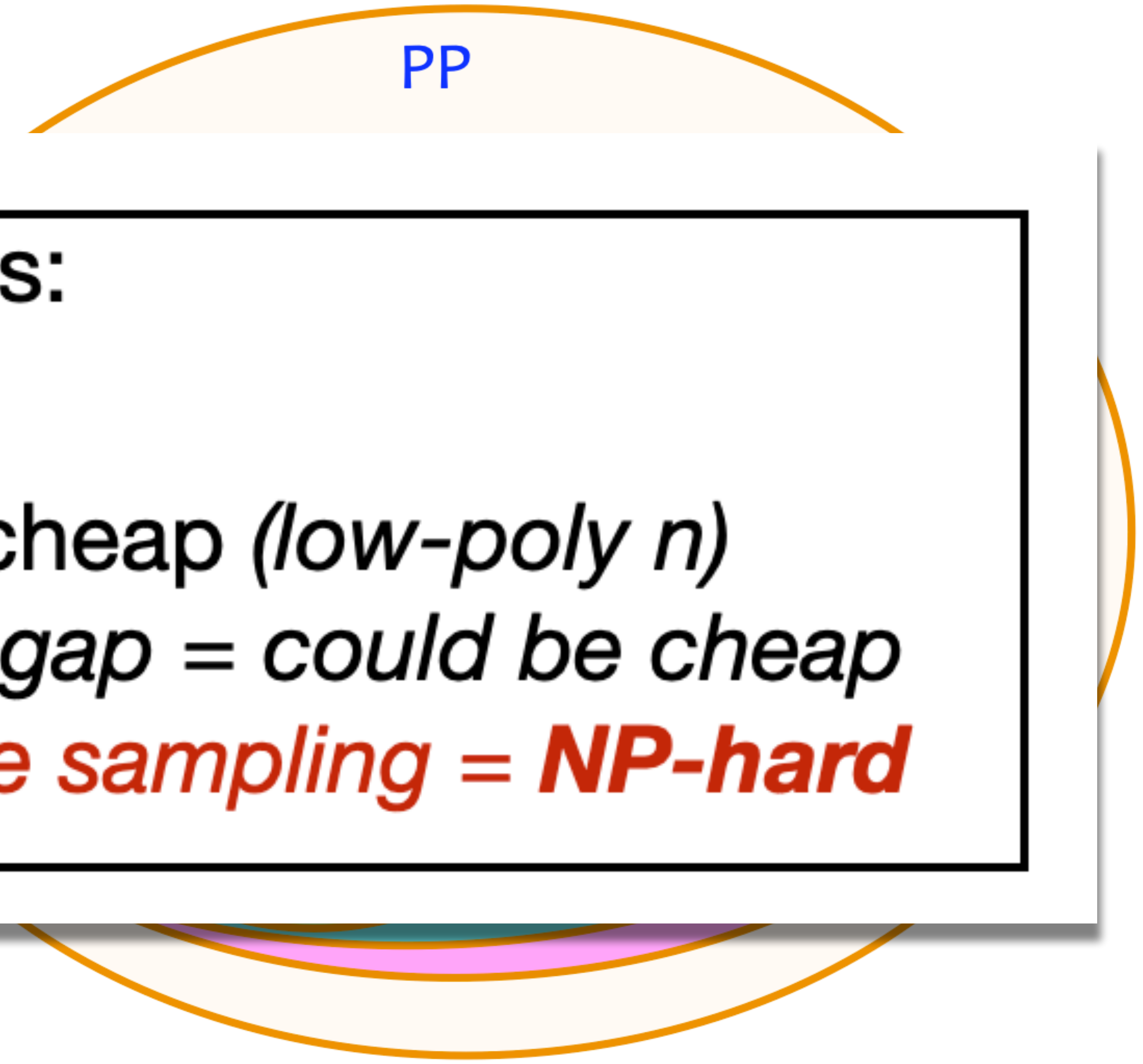

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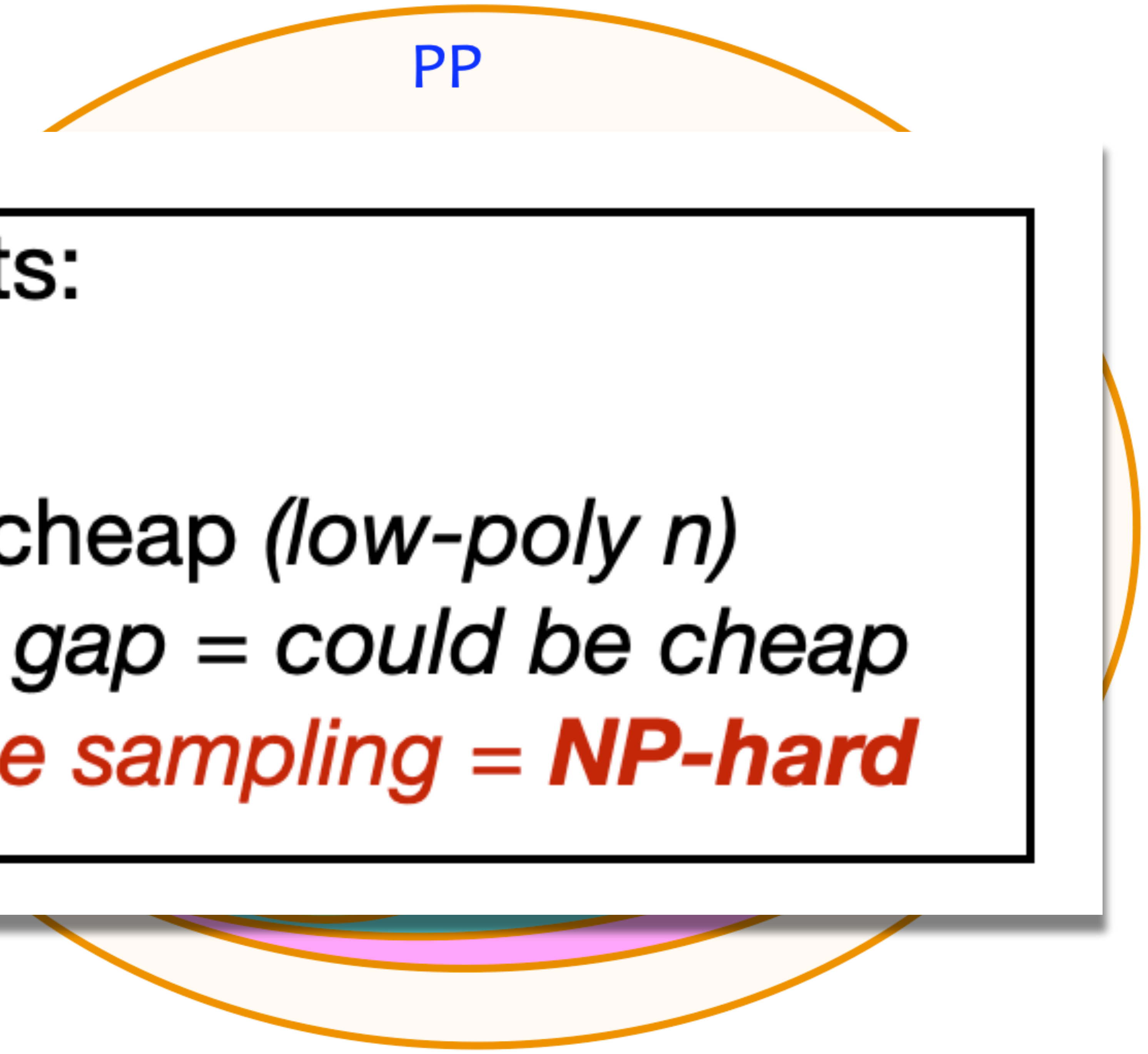
R. King,T. Kohler, ??? (in 1 week?)

No.

Multiplicative estimation of Betti numbers is QMA1-hard (2022)

New (King,Kohler, ’23-ish): under *promised eigenvalue gap*, *QMA1-hard* and *in QMA*



Quantum costs:

- Ham. sim. = cheap *(low-poly n)*
- QPE to prec. *gap = could be cheap*
- *random clique sampling = **NP-hard***

*Lemma: it is also hard when the graph is **clique-dense***

➡*Hardness is NOT in clique sampling*

➡*“Homology is quantum”*

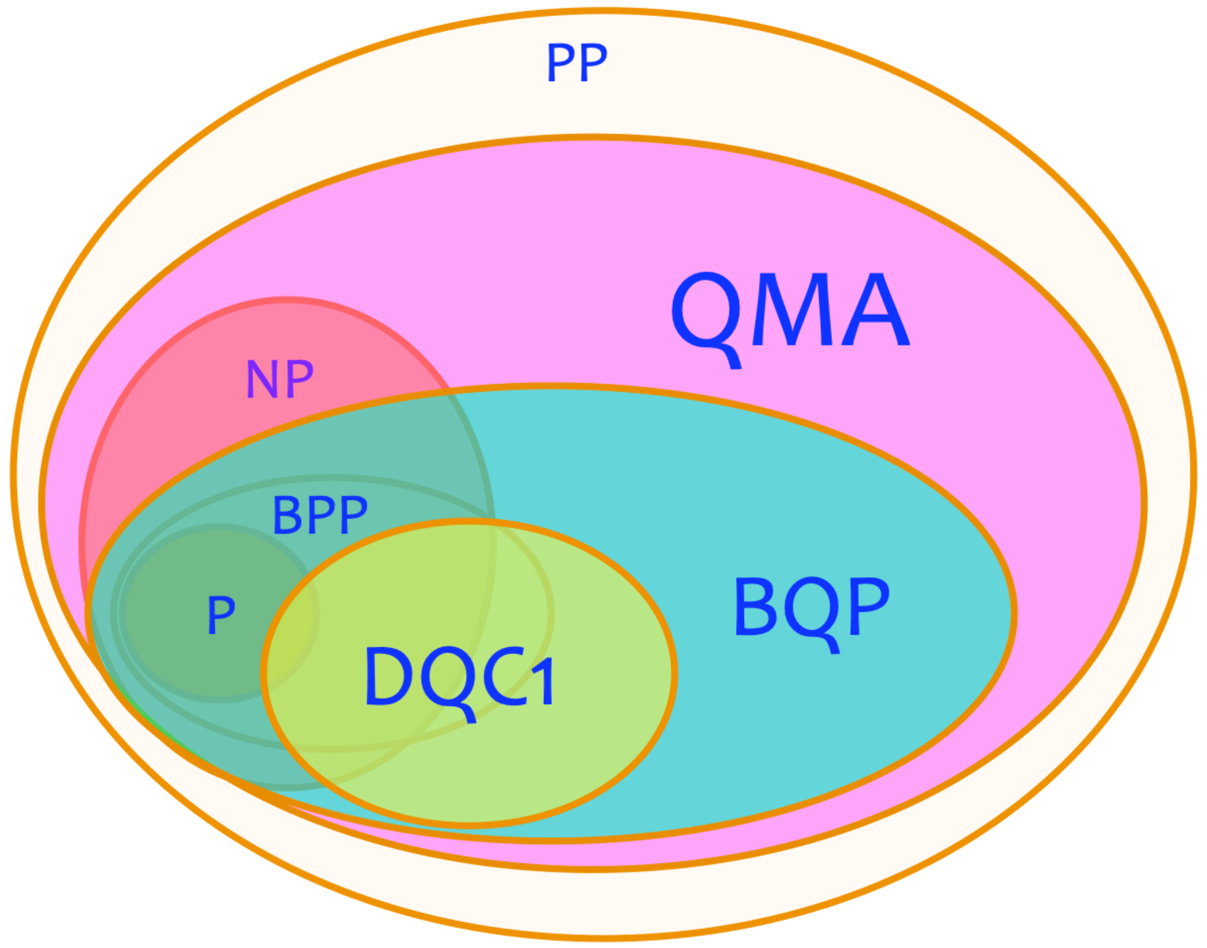
M. Crichigno, T. Kohler, Clique Homology is QMA1-hard (2022)

R. King,T. Kohler, ??? (in 1 week?)

Is there a *guaranteed* quantum advantage
***for what we do have*, and can it be relevant?**

- ***hardness of TDA?***
- *when is the QC algorithm truly faster against “vanilla”…*
- *…and new classical algorithms?*

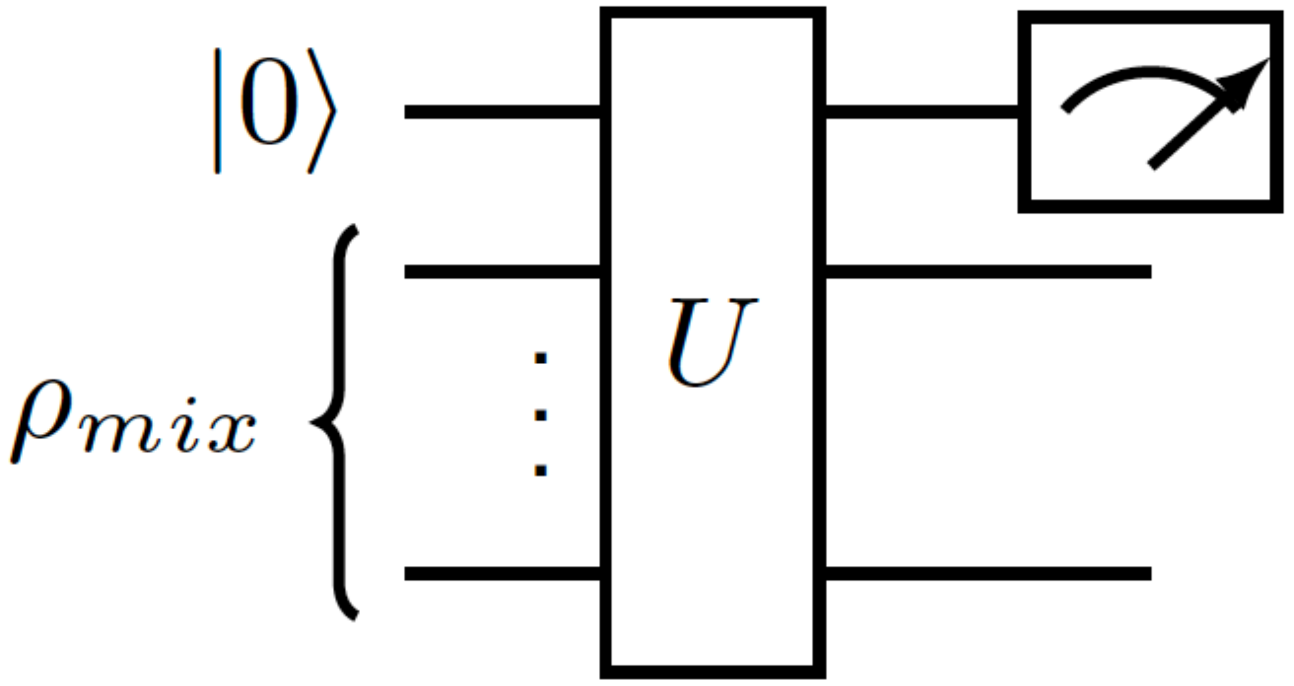
Generalizations of normalized-Betti-estimation are *DQC1-hard*



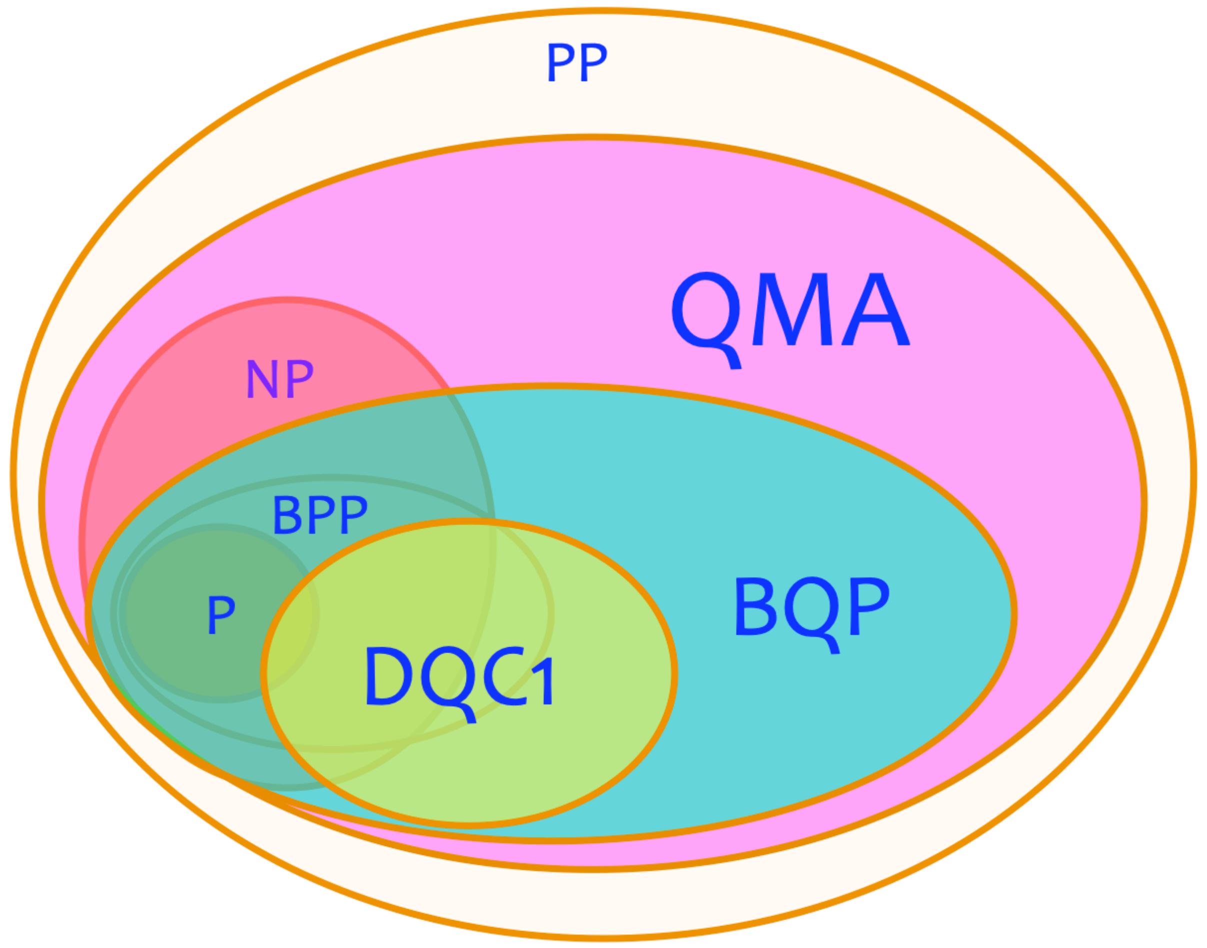
C. Gyurik, C. Cade, VD, Towards quantum advantage for topological data analysis (2020)
C. Cade, M. Crichigno,Complexity of Supersymmetric Systems (2021)

Generalizations of normalized-Betti-estimation are *DQC1-hard*

DQC1 model of computation:



$$\begin{cases} p_0 < 1/2 - \text{gap} \\ p_0 > 1/2 + \text{gap} \end{cases}$$



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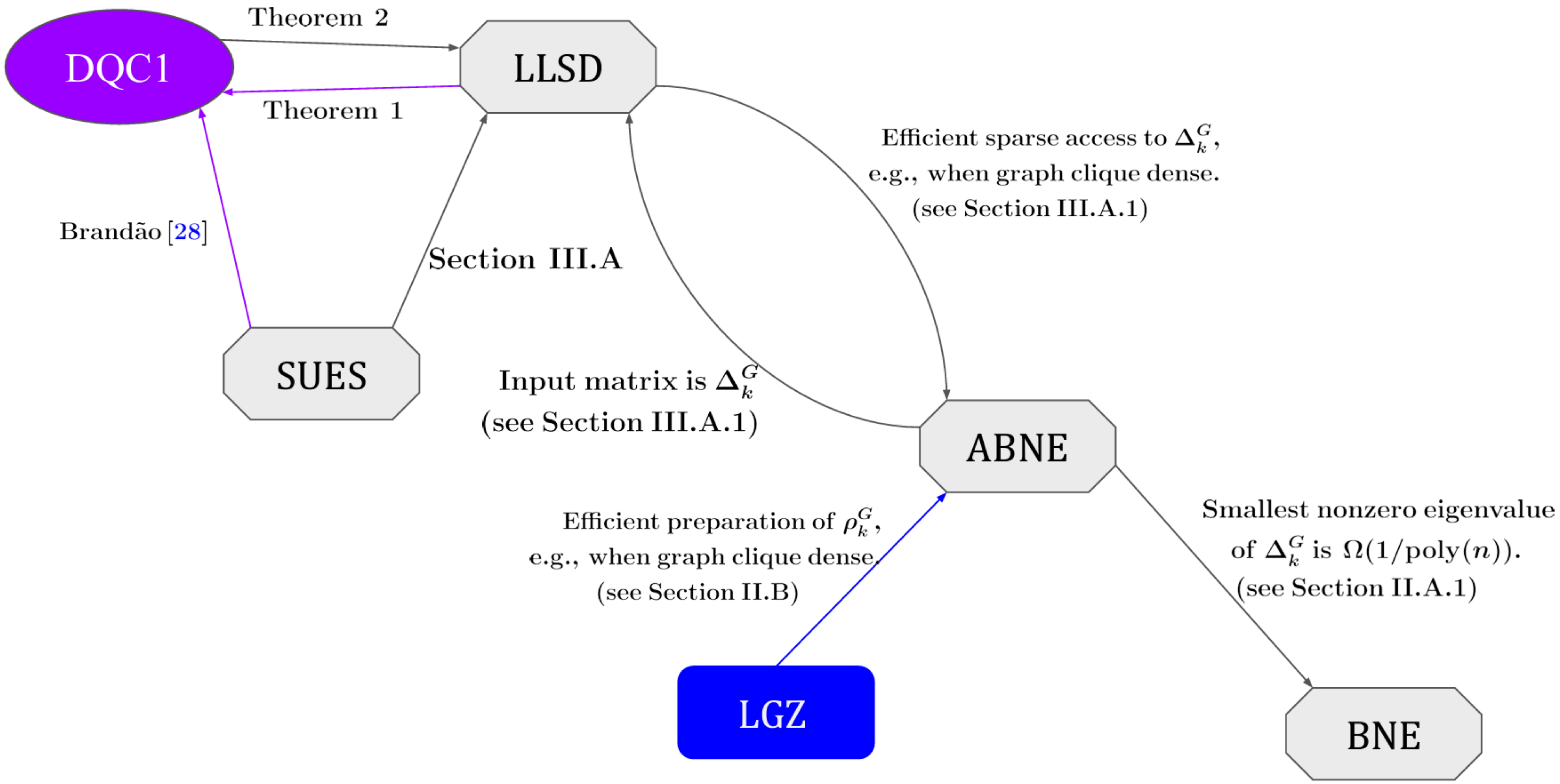

Marcos and Chris different (better) generalization

C. Gyurik, C. Cade, VD, Towards quantum advantage for topological data analysis (2020)
C. Cade, M. Crichigno,Complexity of Supersymmetric Systems (2021)

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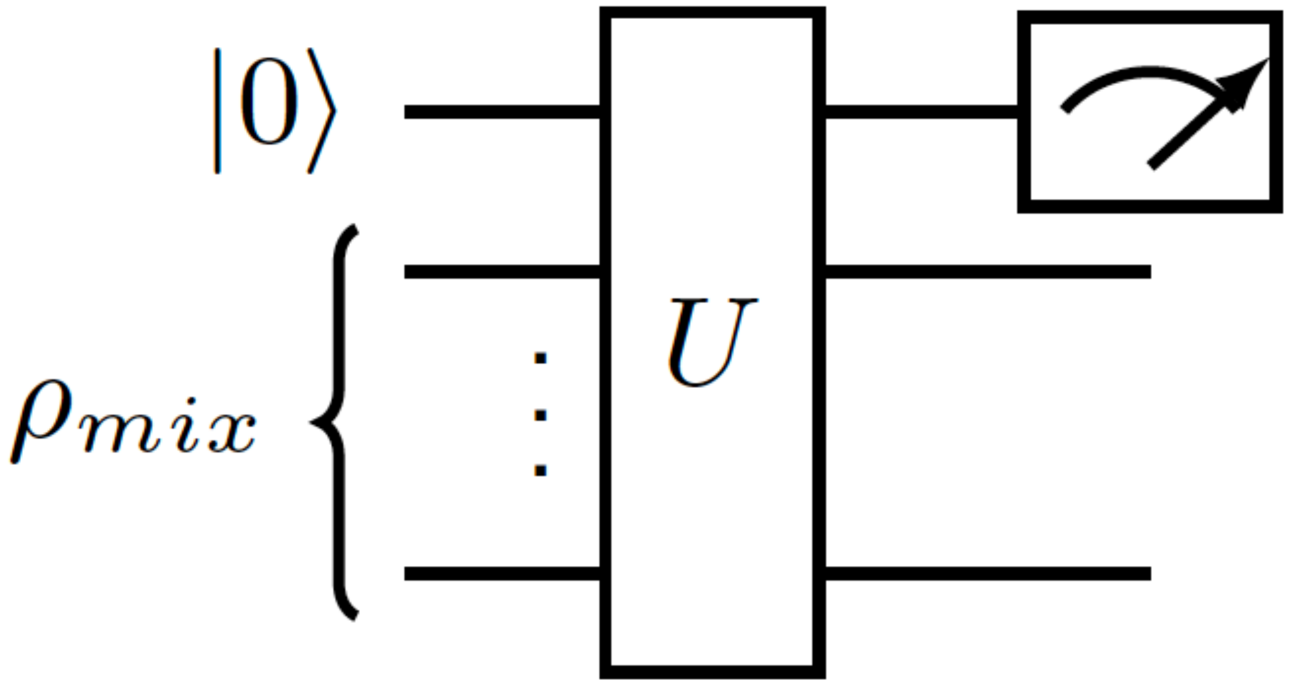

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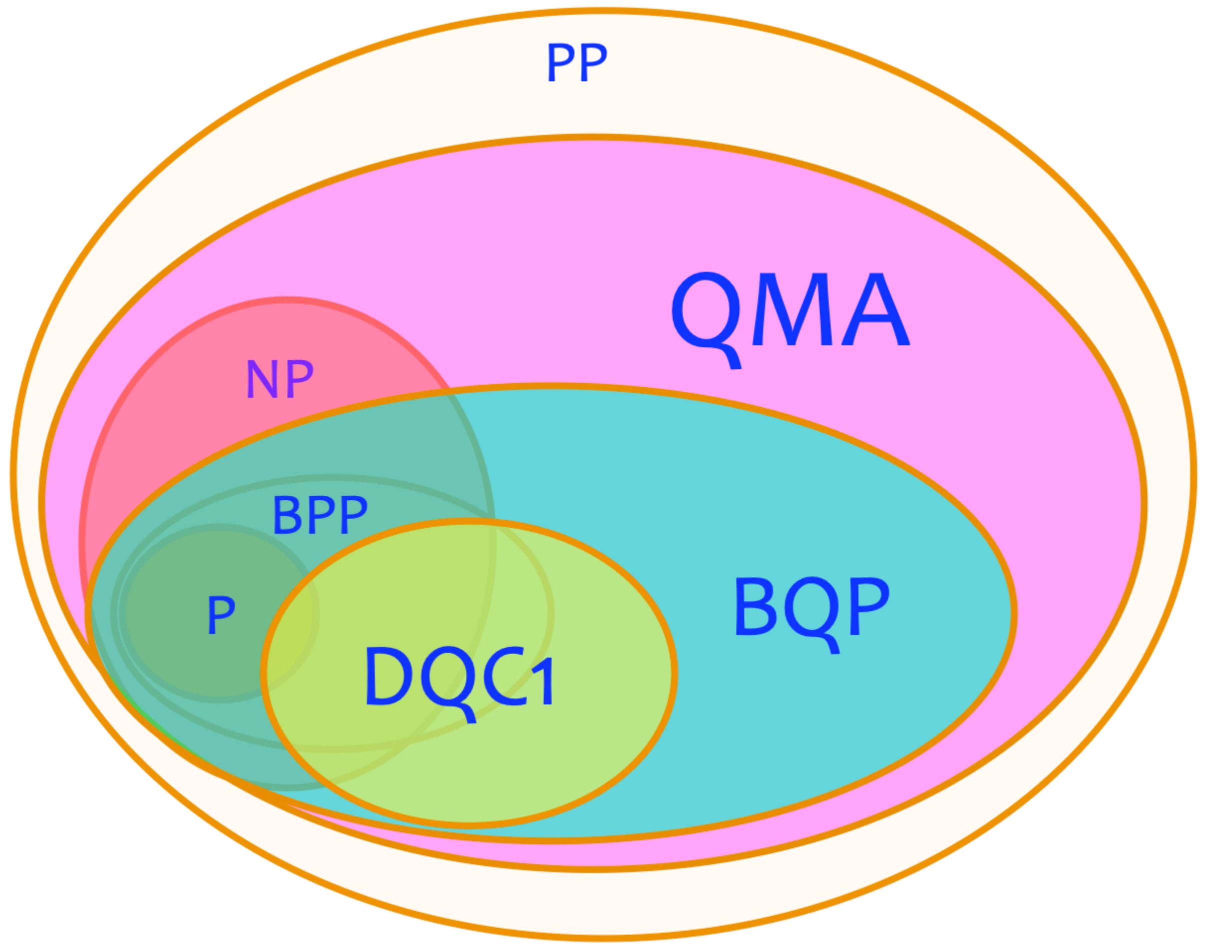

However, generalizations quite substantial. Also, not ***in BQP*** *for full range.*
Interestingly, QMA1 hardness of multiplicative error does not give DQC1 result.
Question unresolved. *Better classical algorithms emerging for* ***regions of interest.***

C. Gyurik, C. Cade, VD, Towards quantum advantage for topological data analysis (2020)
C. Cade, M. Crichigno,Complexity of Supersymmetric Systems (2021)

Happy? Unhappy?

There is a reason to be worried*

efficient (**additive** error) estimation of ***normalized*** *approximate Betti numbers in regimes* ***where clique sampling is efficient***

*(*but I think in some cases we will be fine*
*** arboricity properties will do, but let's not complicate)*

There is a reason to be worried*

efficient (**additive** error) estimation of ***normalized*** *approximate Betti numbers in regimes* ***where clique sampling is efficient***

normalized **=** *Betti / total-clique-number*

sampling is efficient **=** *dunno, have many cliques out of all possible, probably**?*

→*total-clique-number is* $\Omega(n^k/poly(n))$....

*(*but I think in some cases we will be fine*

*** arboricity properties will do, but let's not complicate)*

There is a reason to be worried*

Known results: Betti numbers tend to scale linearly with *n*
Normalized Betti … is zero (to inverse sub-exponential additive error).

*(*but I think in some cases we will be fine*
*** arboricity properties will do, but lets not complicate)*

Is there a *guaranteed* quantum advantage
***for what we do have*, and can it be relevant?**

- *hardness of TDA?*
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- ***…and new classical algorithms?***

Analyzing Prospects for Quantum Advantage in Topological Data Analysis

Dominic W. Berry,[1,*] Yuan Su,[2] Casper Gyurik,[3] Robbie King,[2,4] Joao Basso,[2] Alexander Del Toro Barba,[2] Abhishek Rajput,[5] Nathan Wiebe,[5,6] Vedran Dunjko,[3] and Ryan Babbush[2,†]

arXiv:2209.13581

Polynomial improvements in:
(i) Dicke state preparation, clique checking, (ii) filtering for Kernel projection,
(iii) better amplitude estimation for **optimal Toffoli gate count**

also moved to relative error scalings for the estimation of Betti numbers!

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	Amplitude estimation		Amplification		Preparing cliques
Cost for *relative error* r for β_{k-1}:	$2\frac{\ln(1/\delta)}{r}\sqrt{\frac{\#\text{k-cliques}}{\beta^G_{k-1}}}$	$\times$	$\frac{\pi}{4}\sqrt{\frac{\binom{n}{k}}{\#\text{k-cliques}}}$	$\times$	$(n\log^2 n + 6\lvert E\rvert)$

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Quantum dominating factor for β_{k-1}:

$$\sqrt{\frac{\binom{n}{k}}{\beta_{k-1}^{G}}}$$

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Classical vanilla cost: #k-cliques $\leq \binom{n}{k}$

Do graphs which are "**Betti dense**" (quantum easy)
but still **large in clique numbers** (classically hard) even exist?

(= can the **red** be superpolynomially larger than the **blue**?)

Do there exist graphs allowing a superpolynomial advantage?

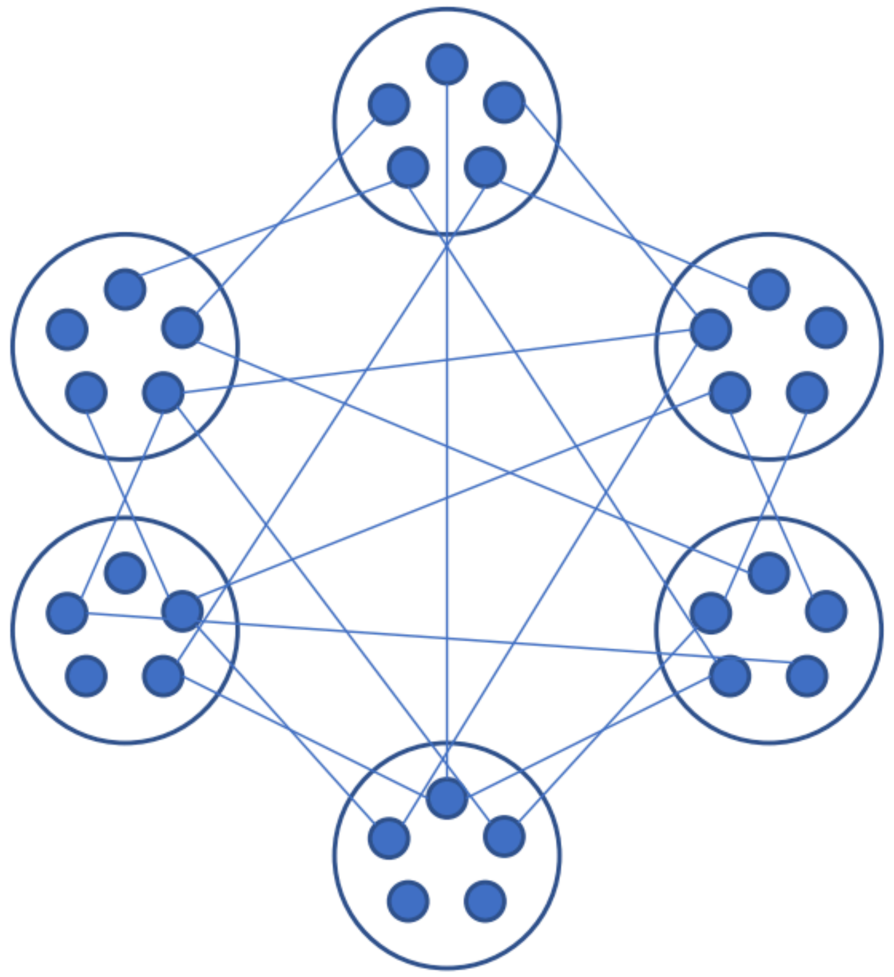
reminder k= dimension of holes we are counting (which Betti?)
n = # vertices

Regime $k \in O(1)$ - classical runtime is poly, no superpolynomial speed-up

Regime $k = cn,\ c \in [0,1)$ - classical runtime is exponential… ***but so is the quantum***

Do there exist graphs allowing a superpolynomial advantage?

Regime $k \in \Theta(polylog(n))$.



$K(n/k, k)$

Künneth formulas

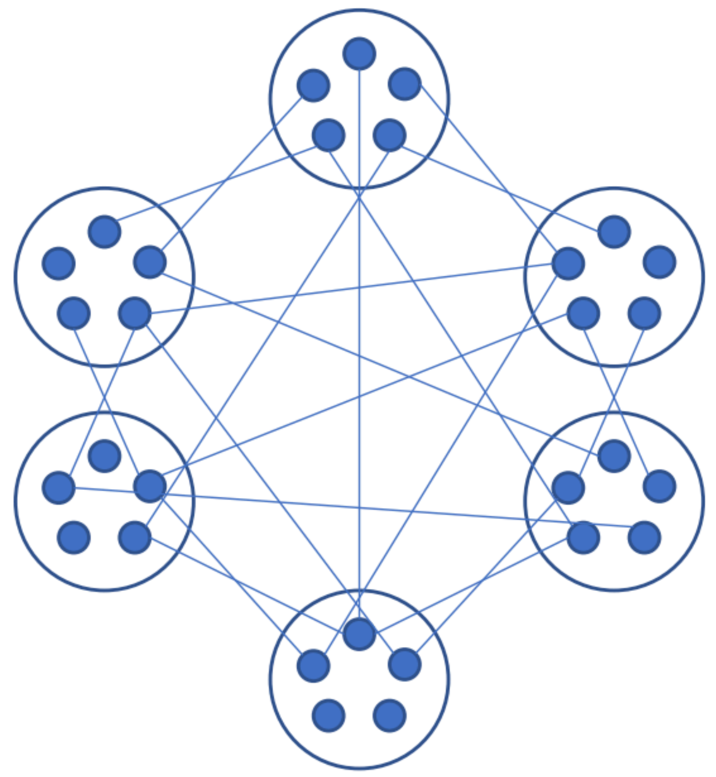
$$T_c \sim exp(k \times (1 + \ln\ n/k))$$

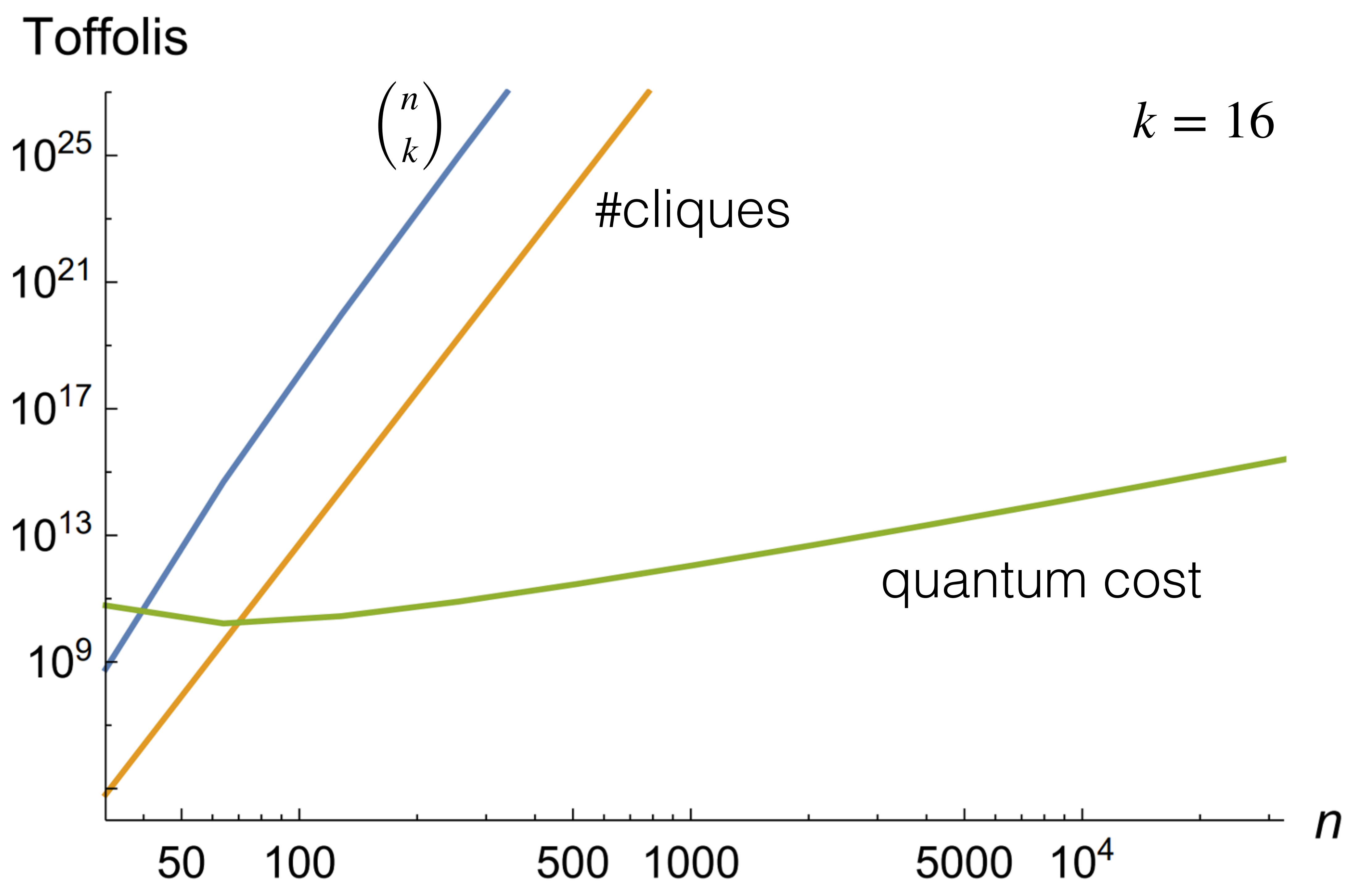
$$T_q \sim exp(k(1 + k/n)/2)$$

α s.t. $T_c = T_q^{\alpha}; \quad \alpha \in \Theta(\log(n))$

Superpolynomial speed-up

Computing actual numbers…





k=16, *n=256*

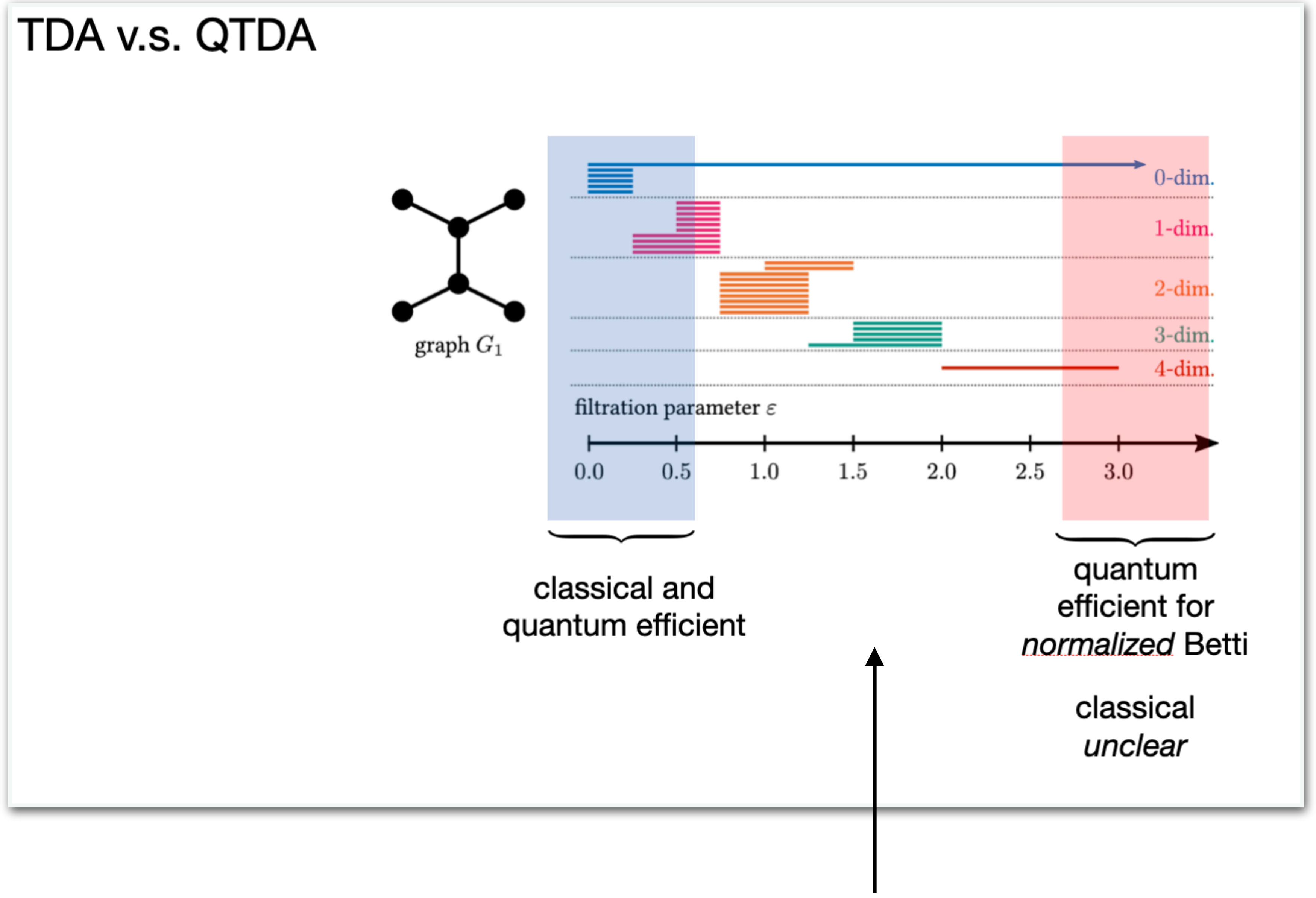
"speed-up"~ 7.5th power

Dimension $\binom{n}{k} \sim 10^{25}$

Classical: *#cliques* ~ 10^{19}

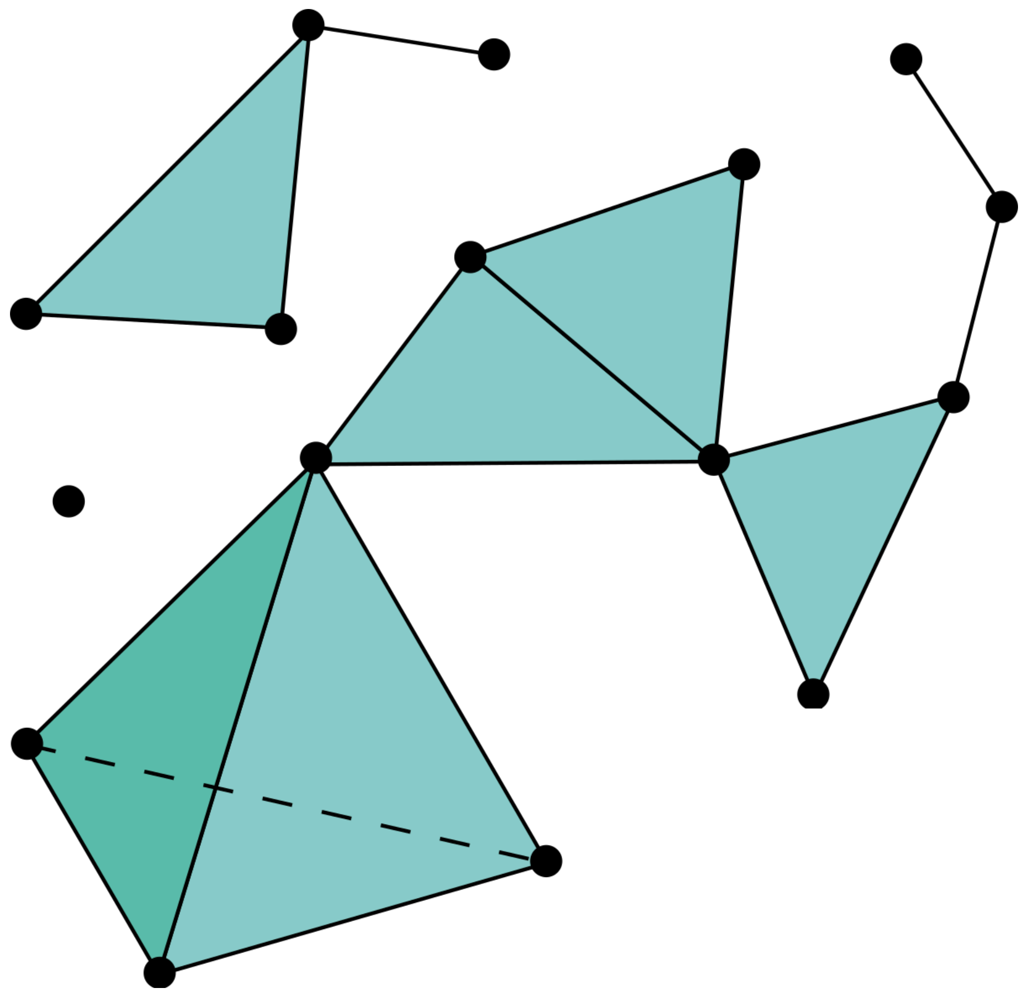
Quantum: 80 billion Toffolis

Clique sampling *circumvented*



Not accessible due to no efficient generic sampling method of cliques from graphs

"Maximal faces" representation:



QMA1 hardness persists.
Likely DQC1 hardness as well.

In the meantime…

A (simple) classical algorithm for estimating Betti numbers

Simon Apers[1], Sander Gribling[1], Sayantan Sen[2], and Dániel Szabó[1]

[1]Université Paris Cité, CNRS, IRIF, Paris, France
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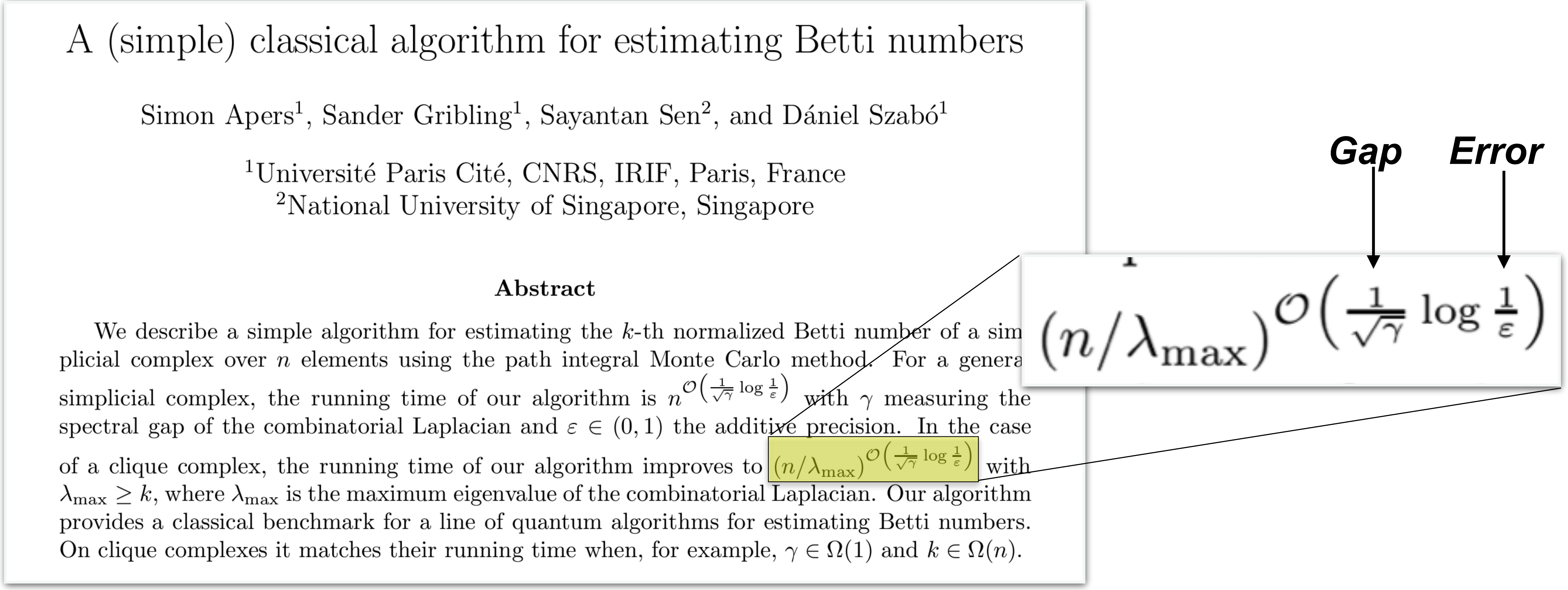
Abstract

We describe a simple algorithm for estimating the k-th normalized Betti number of a simplicial complex over n elements using the path integral Monte Carlo method. For a general simplicial complex, the running time of our algorithm is $n^{\mathcal{O}\left(\frac{1}{\sqrt{\gamma}}\log\frac{1}{\varepsilon}\right)}$ with γ measuring the spectral gap of the combinatorial Laplacian and $\varepsilon \in (0,1)$ the additive precision. In the case of a clique complex, the running time of our algorithm improves to $(n/\lambda_{\max})^{\mathcal{O}\left(\frac{1}{\sqrt{\gamma}}\log\frac{1}{\varepsilon}\right)}$ with $\lambda_{\max} \geq k$, where $\lambda_{\max}$ is the maximum eigenvalue of the combinatorial Laplacian. Our algorithm provides a classical benchmark for a line of quantum algorithms for estimating Betti numbers. On clique complexes it matches their running time when, for example, $\gamma \in \Omega(1)$ and $k \in \Omega(n)$.

In the meantime…



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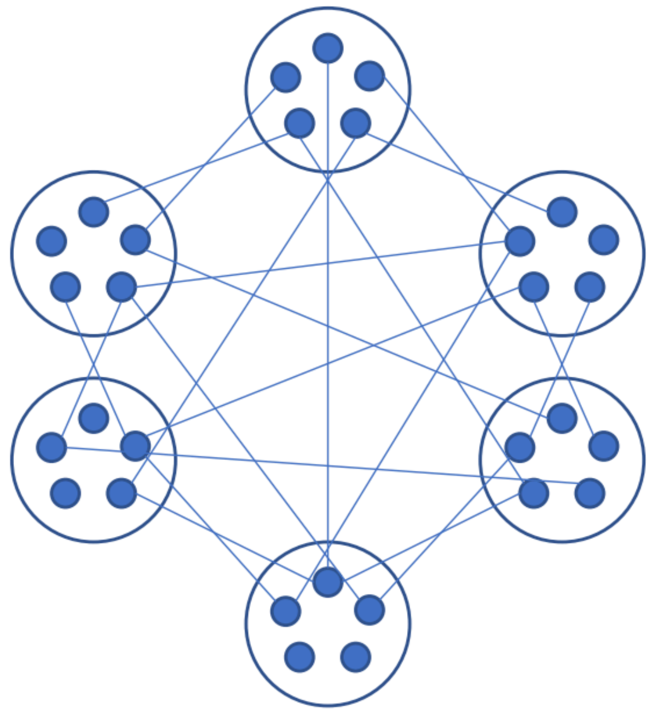
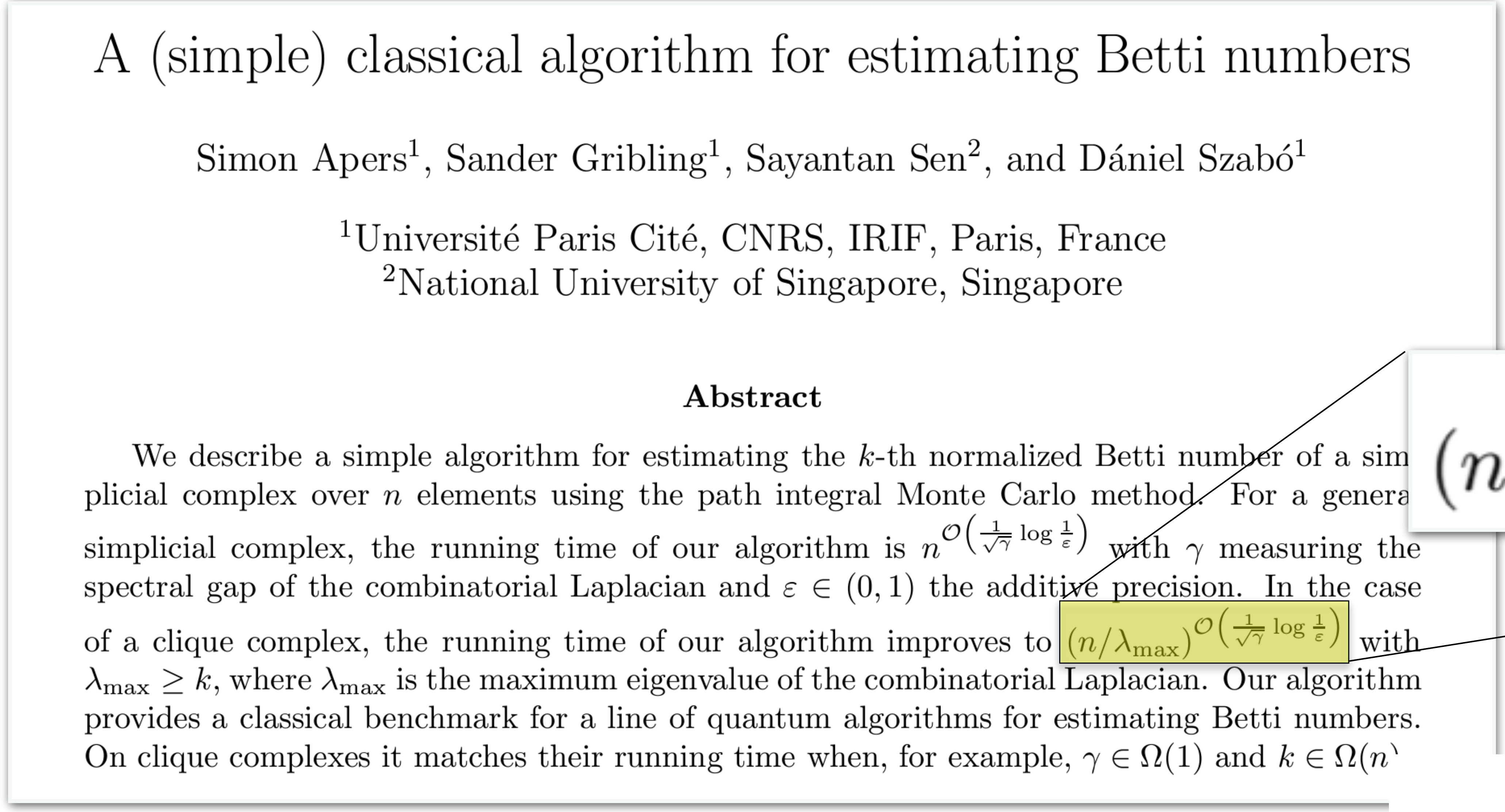


has constant gap $\gamma \in \Theta(1)$.

In the meantime…



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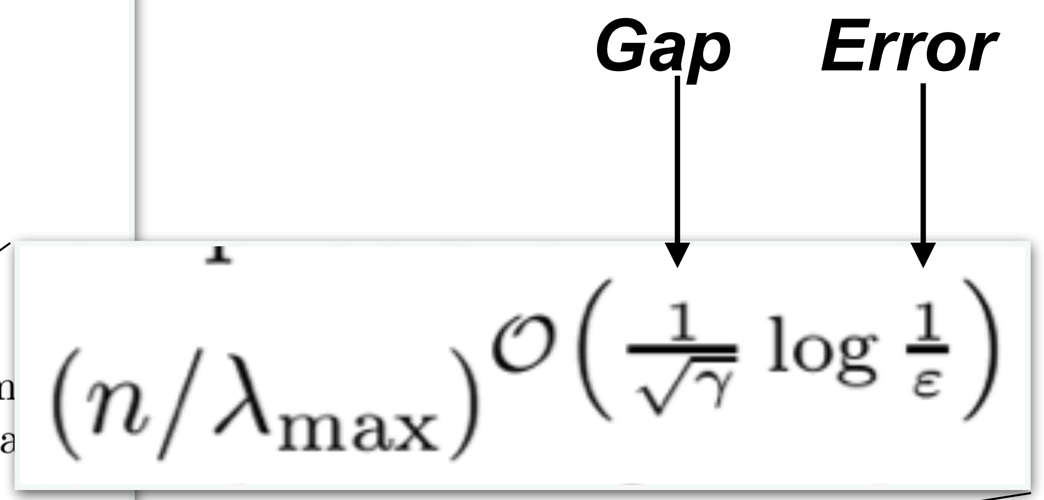

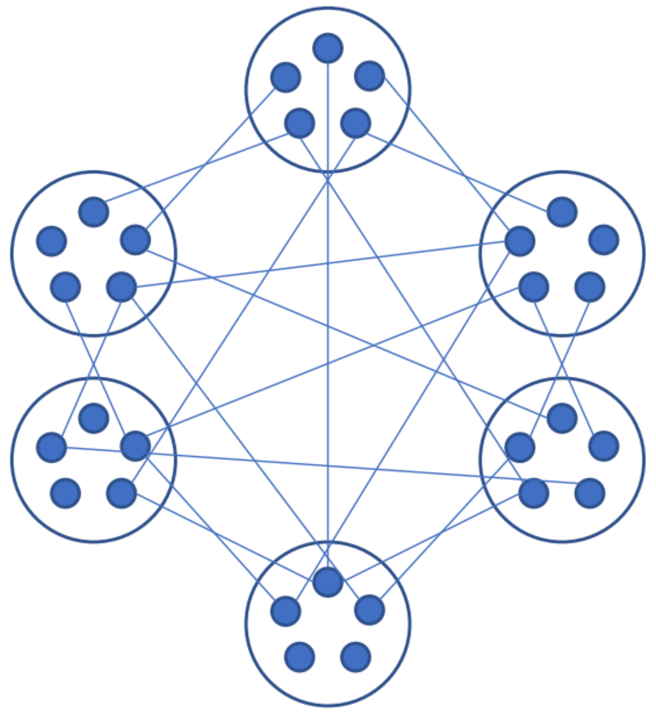

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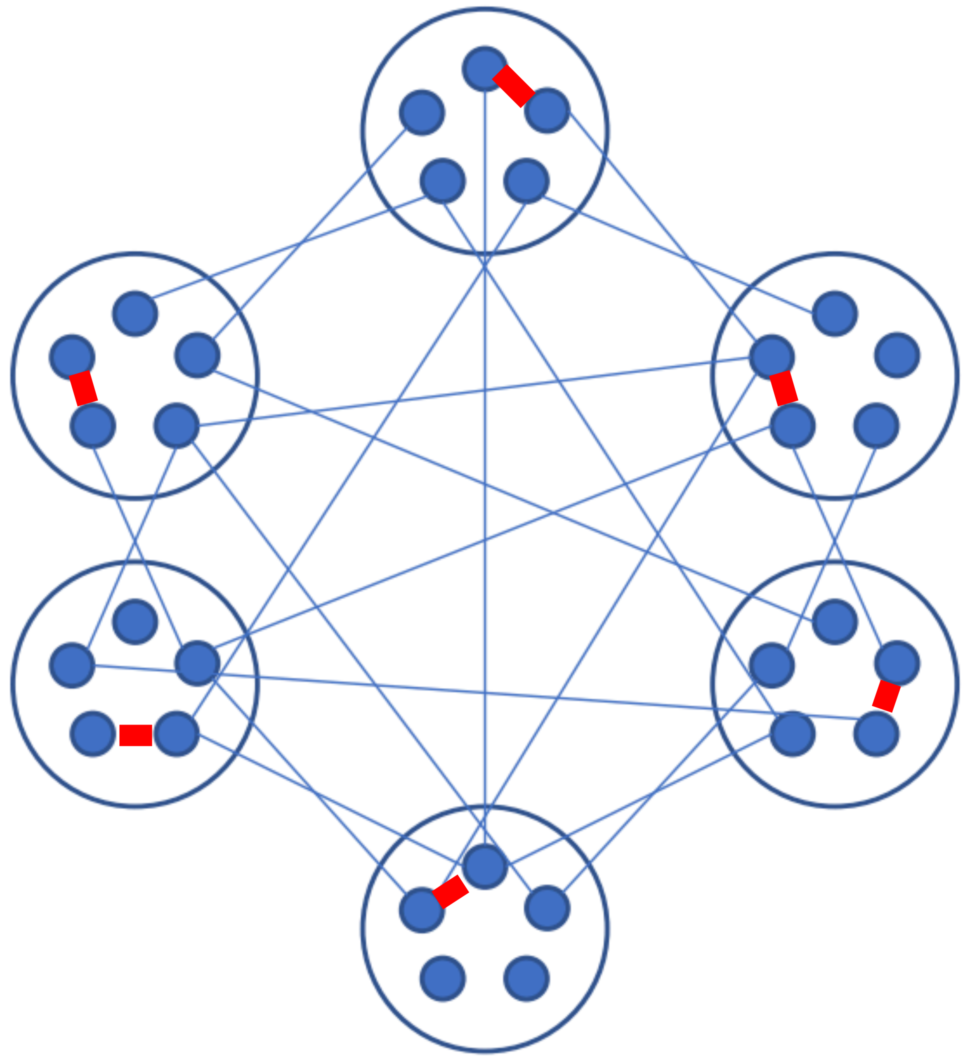

perturbation has $\gamma \leq \frac{1}{n}$

superpoly speed-up

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Gap
Error

$$(n/\lambda_{\max})^{\mathcal{O}\left(\frac{1}{\sqrt{\gamma}}\log\frac{1}{\varepsilon}\right)}$$

normalized ***approximate*** *Betti numbers*

approximate = counting *small* eigenvalues (not *silly* due to Chegeer-like arguments)

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Gap: $\frac{1}{\sqrt{\gamma}}$ — Error: $\log\frac{1}{\varepsilon}$

$$(n/\lambda_{\max})^{\mathcal{O}\left(\frac{1}{\sqrt{\gamma}}\log\frac{1}{\varepsilon}\right)}$$

normalized ***<u>approximate</u>*** *Betti numbers*

approximate = counting *small* eigenvalues (not *silly* due to Chegeer-like arguments)

small = small constant or $log^{-1}(n) \rightarrow$ no quantum advantage

small = $poly^{-1}(n) \rightarrow$ mebbe quantum advantage

Meta-problem

we prove separations… by *computing Betti number*

so there exists **an** efficient algorithm, *and we know it*!

Meta-problem

we prove separations… by *computing Betti number*

so there exists **an** efficient algorithm, *and we know it*!

but if we cannot compute it… we cannot claim quantum speed-up

Meta-problem

we prove separations… by *computing Betti number*

so there exists **an** efficient algorithm, *and we know it*!

but if we cannot compute it… we cannot claim quantum speed-up

Solution: prove certain perturbations sometimes perturb the Betti *measurably* by a small amount without *explicitly* computing it

(Likely) can be done, as alternatively adding edges could
always fully uncontrollably change Betti numbers, or do nothing…

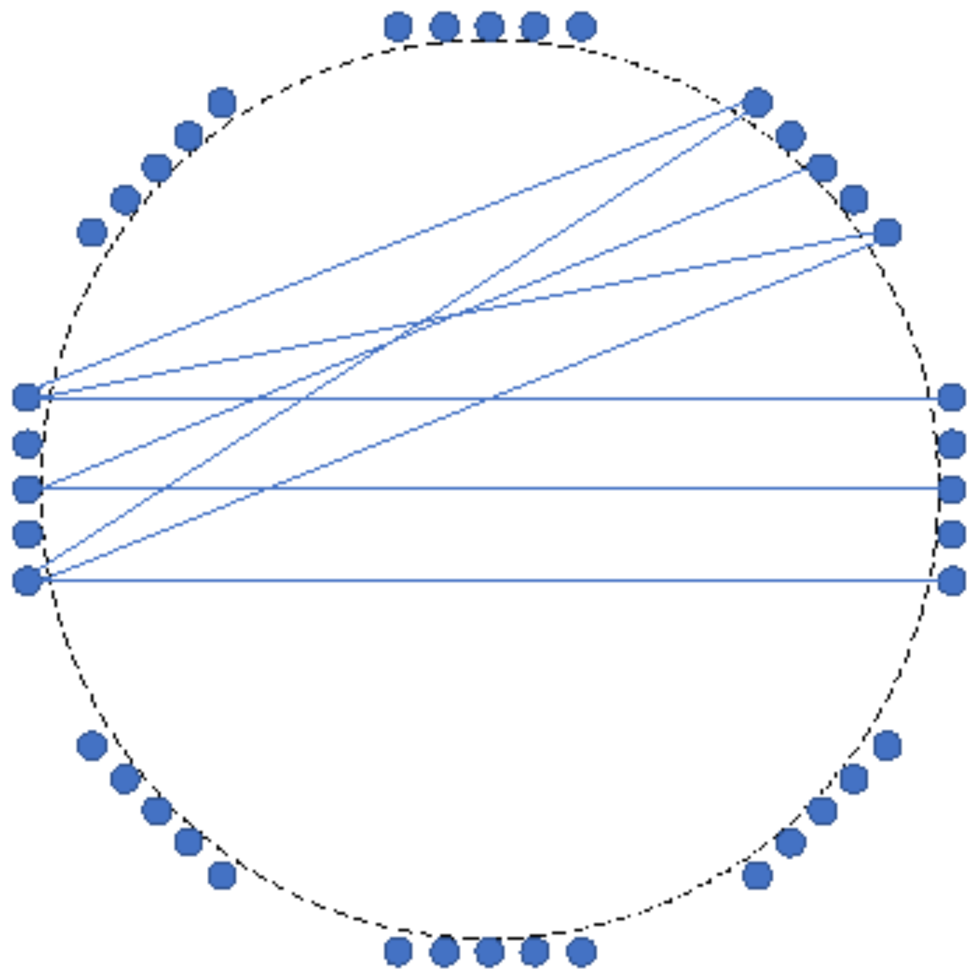
To be sure-sure: prove DQC1-hardness!

Are superpolynomial speedups for TDA likely generic?

Likely not. For *generic* graphs, we do not have the nice properties.…
For Erdős-Renyi graphs, there are regions with expected **quartic** speed-ups

For Vietoris-Rips complexes (i.e., the way I described the pipeline from point clouds) for data from *i.i.d.* distributions, ***avg***. Betti numbers do not scale fast enough.

But already in 2D (worst) case:



Large Betti and small gap….

Small summary

DQC1 hardness needed

high Betti problems needed

huge v.s. tiny Betti problems needed

non-iid point-clouds needed

graph density / max face input settings needed

good *approximate* Betti needed

but superpoly speed-ups still plausible
some high-poly speed-ups even *likely*

and there is much more to the overall topic

Beyond *obvious* TDA applications

Betti numbers
Persistence properties

Other complexes
Other homology problems

Supersymmetry ⟷ Homology

SUSY QM on lattices ⟷ Homology on (simplicial) complexes

$$H = \{\mathcal{Q}, \mathcal{Q}^\dagger\}$$

supercharge operators

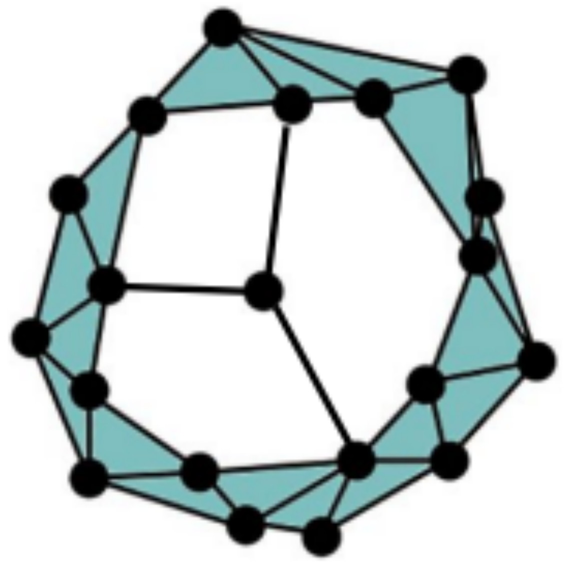


*Image credit: Scientific American, JSTOR

Witten '82
https://arxiv.org/pdf/2107.00011.pdf

$$H = \{\mathcal{Q}, \mathcal{Q}^\dagger\} \quad \mathcal{Q} = \textstyle\sum_i a_i^\dagger P_i$$

$$H = \sum_{(i,j)\in E} P_i a_i^\dagger a_j P_j + \sum_{i\in V} P_i \qquad P_i = \textstyle\prod_{j|(i,j)\in E}(1 - a_j^\dagger a_j)$$

#ℓ-dim holes = #ℓ-fermion ground states

*Image credit: Scientific American, JSTOR

fine print: the above is for the *independence complex*

Witten '89
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quantum computational physics ↔ computational physics ↔ theory ↔ quantum TDA methods

*Image credit: Scientific American, JSTOR

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Combinatorial Laplacians for complex networks

spectral entropy of higher-order representations of networks

Quantity: $$S_\alpha(\Delta_k^G) = \frac{1}{1-\alpha} \log \left(\sum_{j=0}^{d_k-1} p(\lambda_j)^\alpha \right)$$

Generalizations hard as Renyi entropies known to be DQC1-hard

Precision not as obviously problematic

Clique sampling still an issue, depending on the model

Maletić S., Rajković M., Eur. Phys. J. ST (1) , pp. 77-97(2012)
Boccaletti et al., Phys. Rep., vol. 1018, pp. 1-64 (2023)

Some topics in group + friends

Computational many-body methods & comp. topology

Heuristic and QML algorithms for comp. topology

Applications of (Q)TDA to HEP and other physics

Applications in biology (networks)

Other developments in complexity and new algos


Patrick


Alice

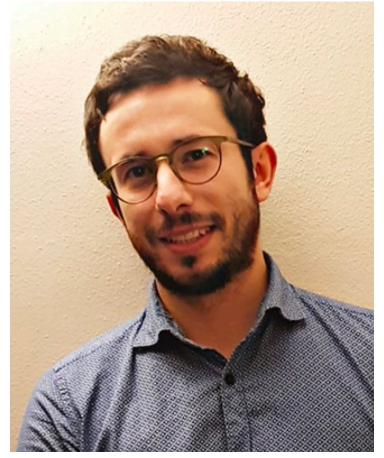

Casper


Mahtab

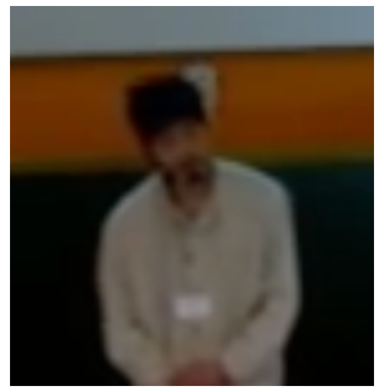

Vincent

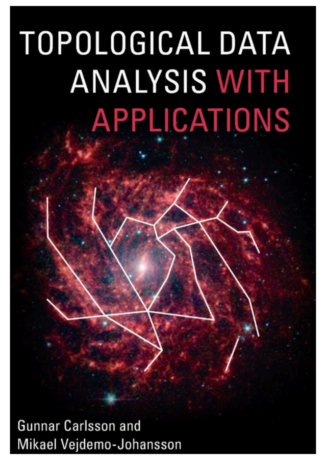

Waheeda


Sofia

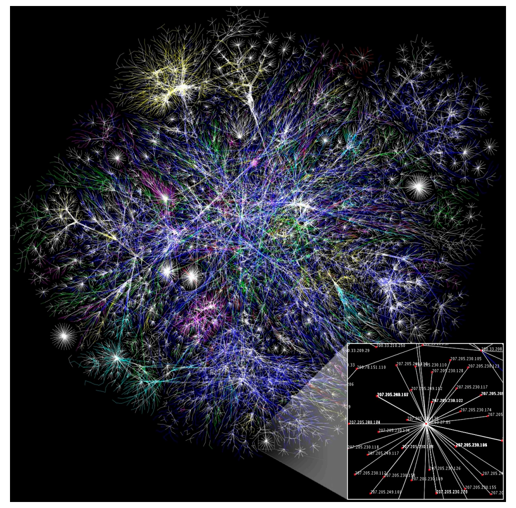

Michele

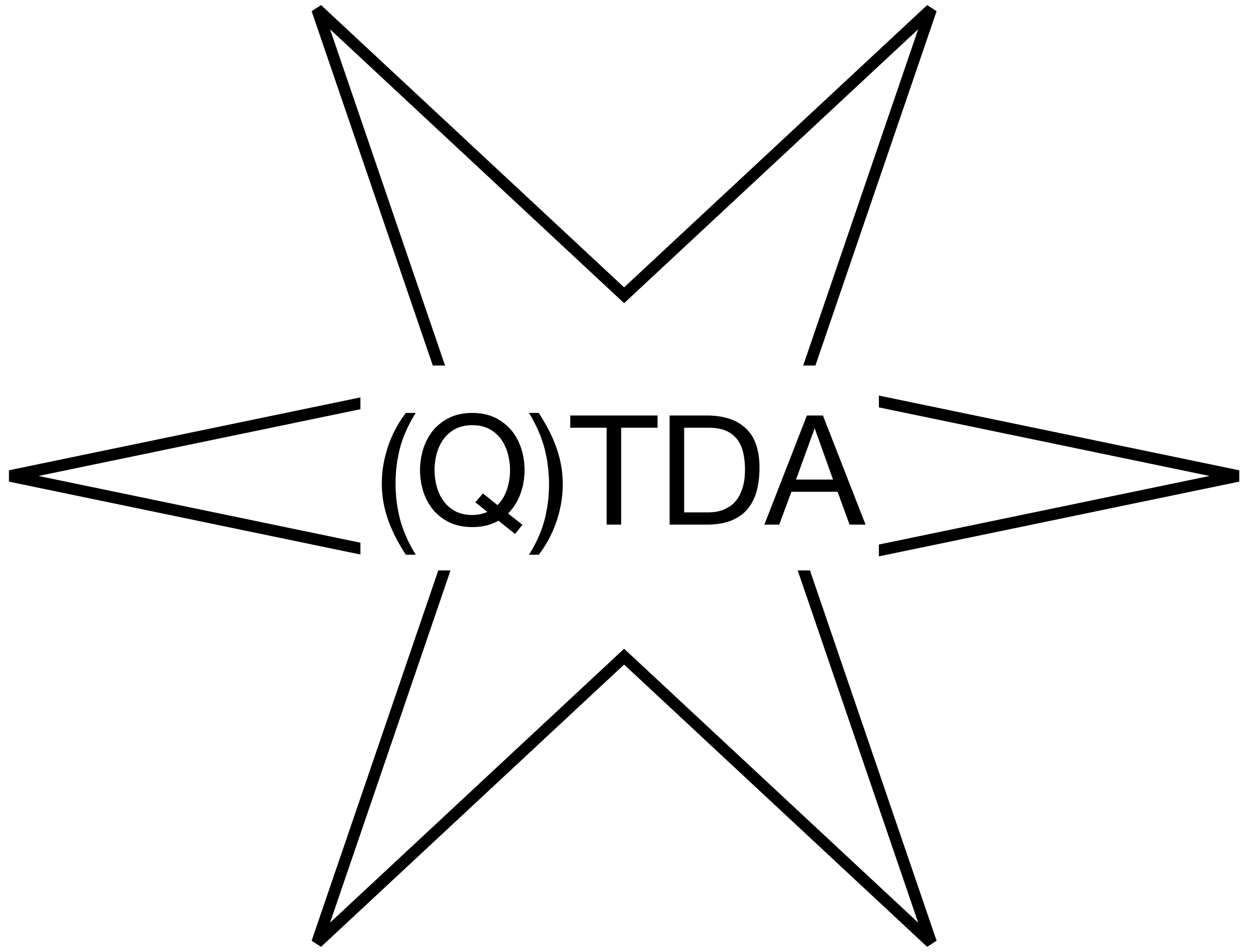

Robbie

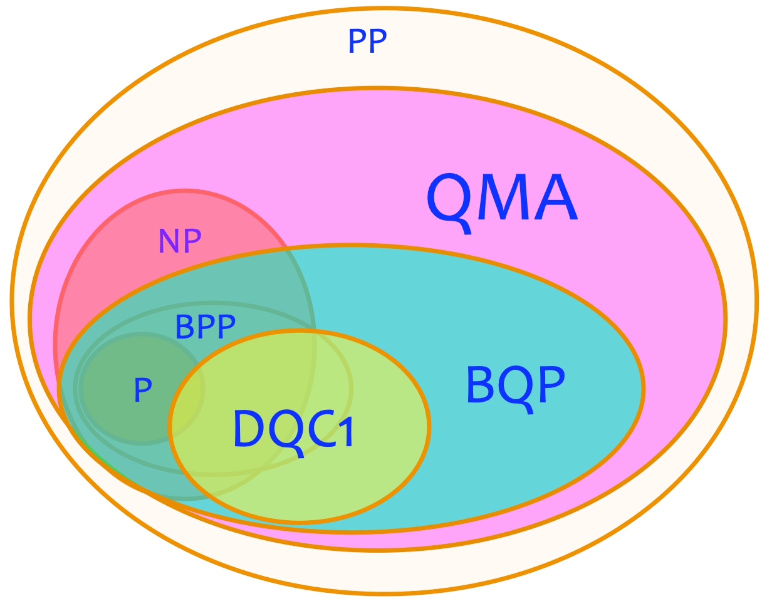

Hayakawa

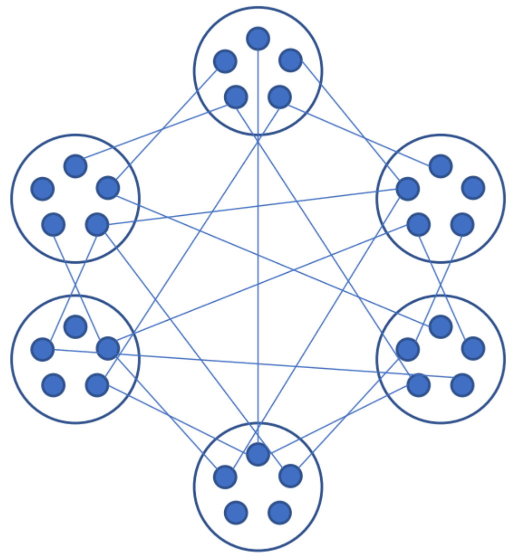


medicine, materials,
entanglement, time-series…









Quantum+Topology** is well-connected and *rich*

*Image credit: Scientific American, JSTOR

** Don't get me started on TQFT and knots!

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Quantum Delta the Netherlands

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