Efficient information recovery from Pauli noise via classical shadow

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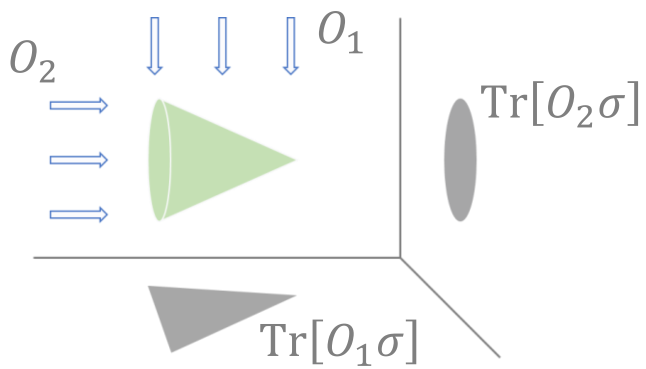
Shadow Information

- Results we usually want from quantum computing are still **classical information**
- **Shadow information** is the expectation value of some chosen observable: $\mathrm{Tr}[O\sigma]$



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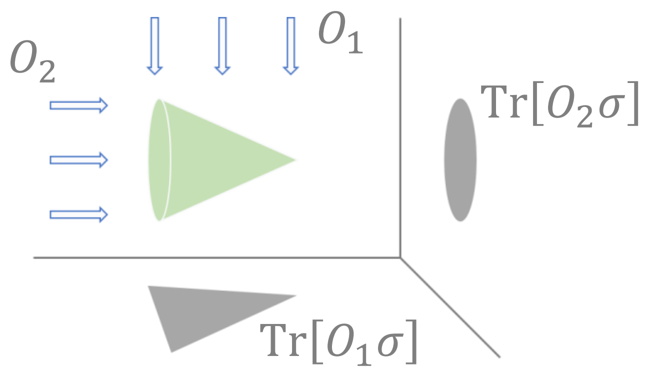

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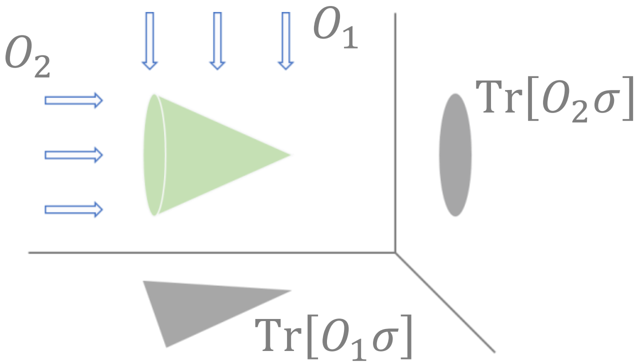
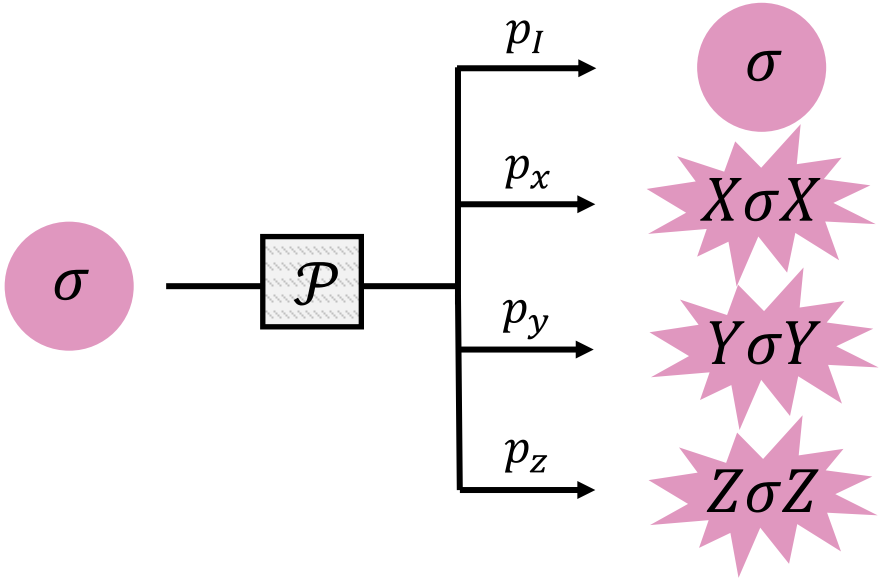


- Shadow information is the target of many quantum algorithms
- Estimation of the information $\mathrm{Tr}[O\sigma]$ is obtained via making **measurements** with the corresponding observable O and post-processing.

Pauli noise

- One of the most standard theoretical models for quantum noise in the study of quantum error correction and mitigation is Pauli noise.



Pauli noise Kraus representation:

$$\mathcal{P}(\sigma) = p_I\sigma + p_x X\sigma X + p_y Y\sigma Y + p_z Z\sigma Z,$$

where X, Y, Z are Pauli matrices and $p_I + p_x + p_y + p_z = 1$.

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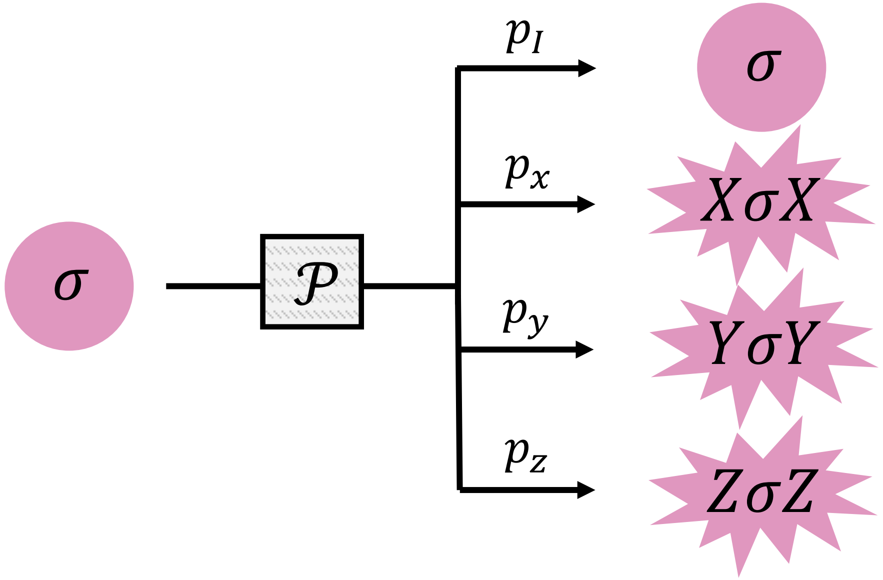

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- Pauli twirling can be applied to converts any noise channel to a Pauli channel

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- Given access to an unknown Pauli channel $\mathcal{P}$ and a noisy state $\mathcal{P}(\sigma)$.

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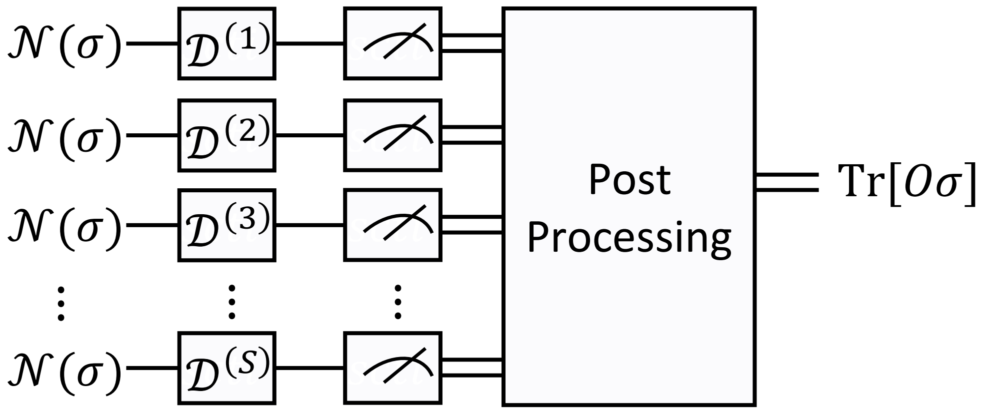
- Given access to an unknown Pauli channel $\mathcal{P}$ and a noisy state $\mathcal{P}(\sigma)$.
- For a known k-local observable O, $\|O\| = 1$, we want a function $f(\mathcal{P}(\sigma), O)$ that can predict the ideal value of $tr(O\sigma)$ up to accuracy ϵ, i.e.

$$|f(\mathcal{P}(\sigma), O) - tr(O\sigma)| \leq \epsilon$$

Known method

- Probabilistic Error Cancellation
  - Given description of the channel $\mathcal{P}$, find the map $\mathcal{D}$ such that $tr\big(O\ \mathcal{D} \circ \mathcal{P}(\sigma)\big) = tr(O\sigma)$.
  - $\mathcal{D}$ is not CPTP but can be written as a linear combination of CPTP maps.
  - Simulate action of $\mathcal{D}$ by probabilistic sampling.



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  - Tomography – computational inefficient and still an approximation
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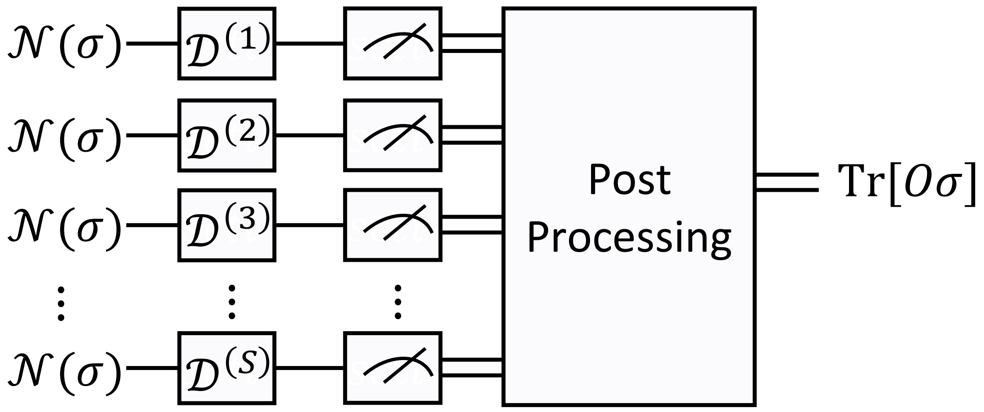

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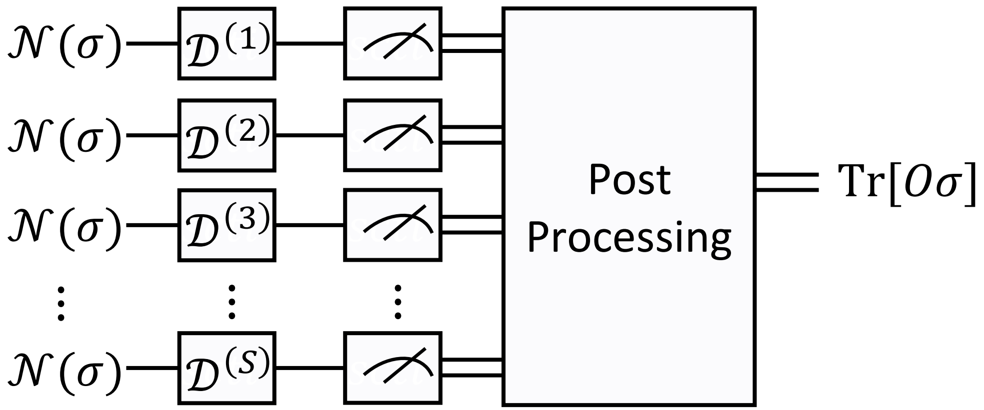
- Implementation of arbitrary CPTP map required



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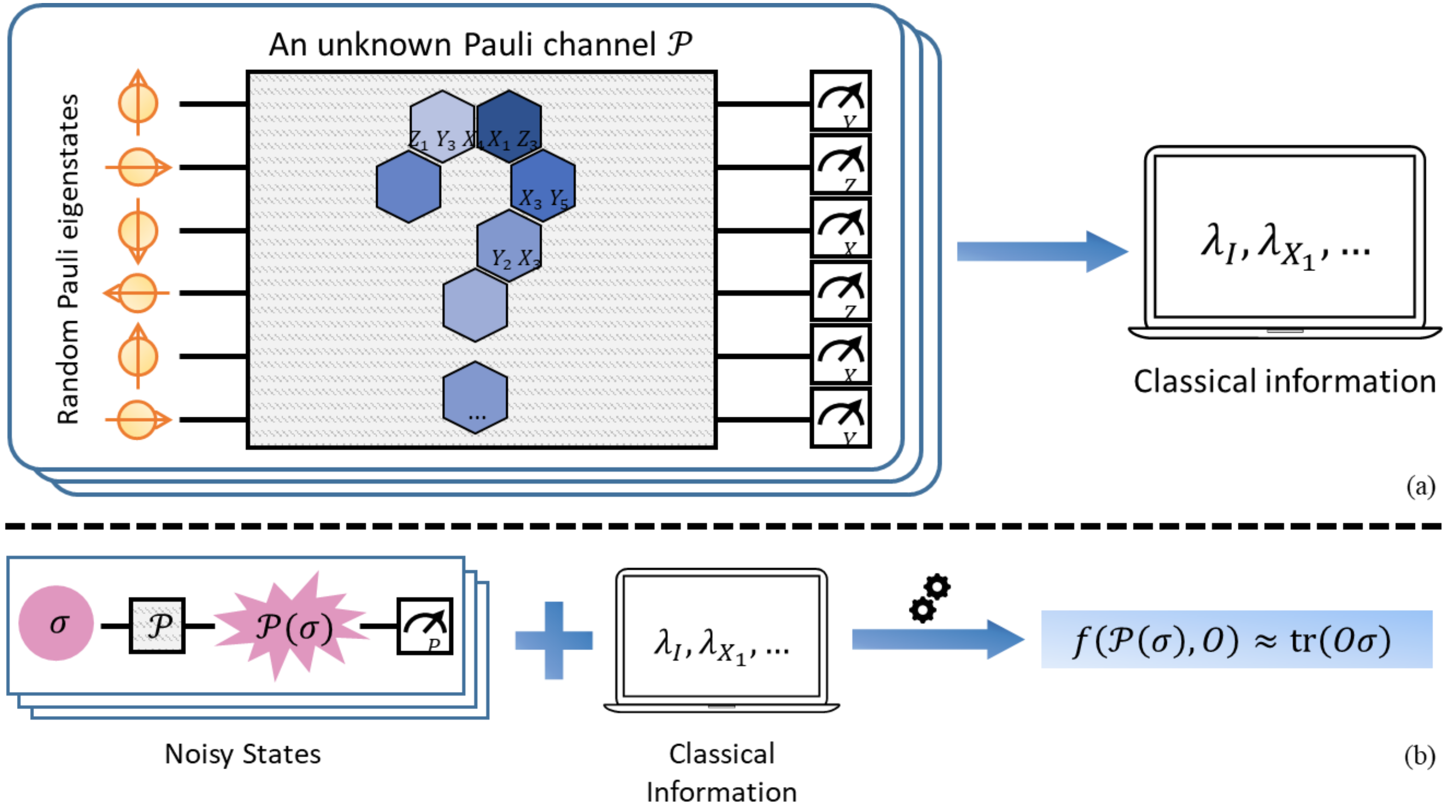
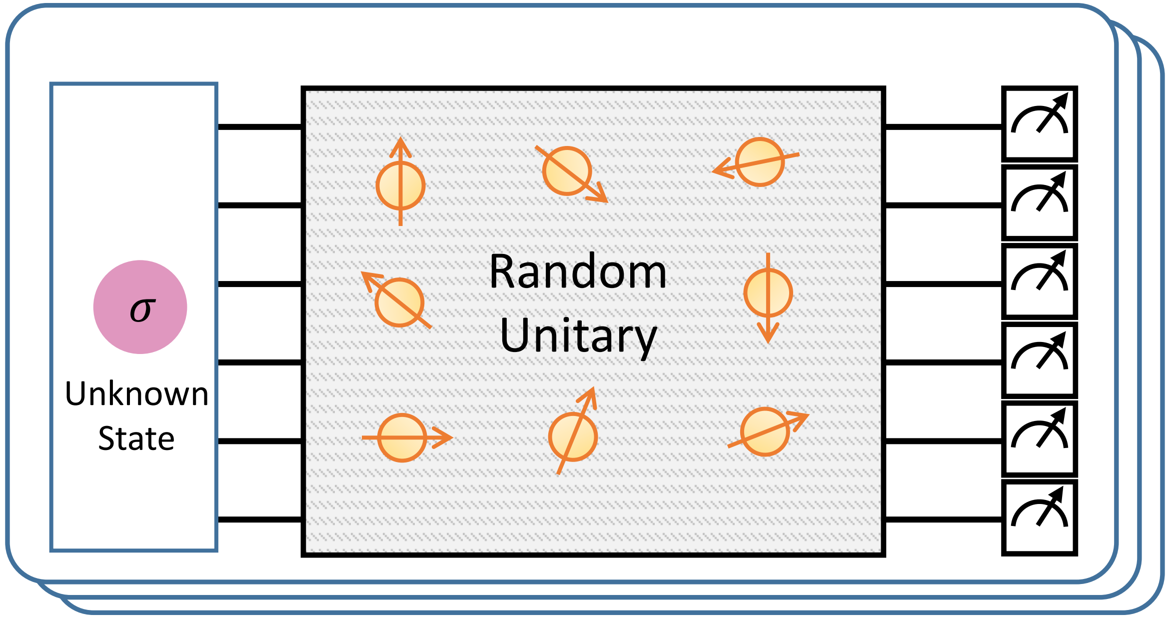


Learning classical information

Post-processing measurement of noisy states to recover true shadow information

Classical shadow

- Used to predict $tr(O\sigma)$ for a set of O **simultaneously**



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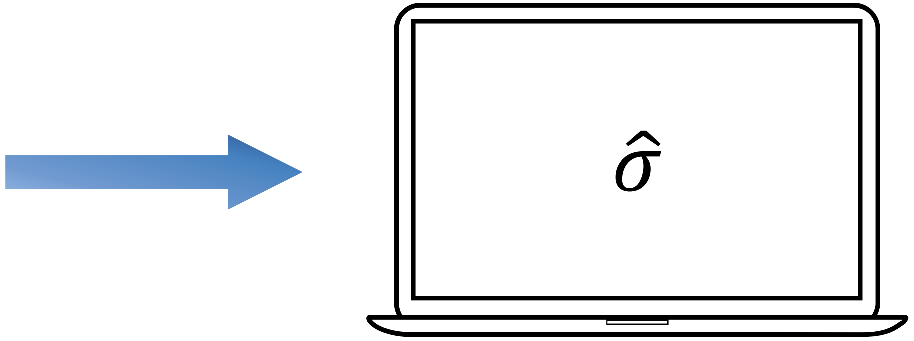

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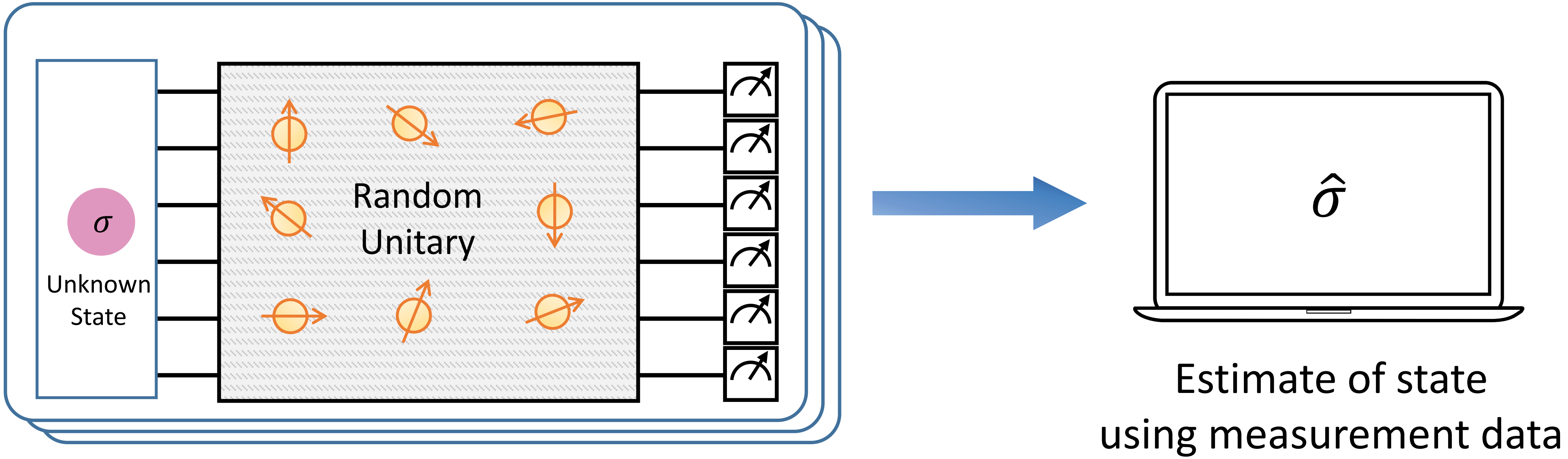

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$$\hat{\sigma} = \otimes_i \left(3U_i^{\dagger}|b_i\rangle\langle b_i|U_i - \mathbb{I}\right) = \otimes_i \left(3|t_i\rangle\langle t_i| - \mathbb{I}\right)$$

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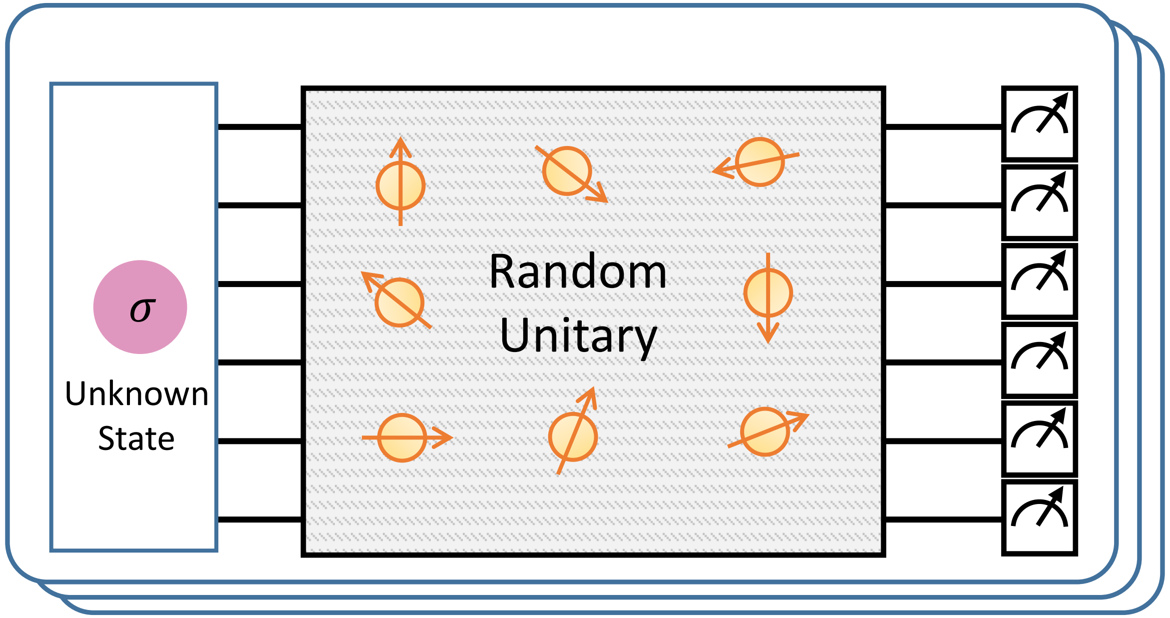

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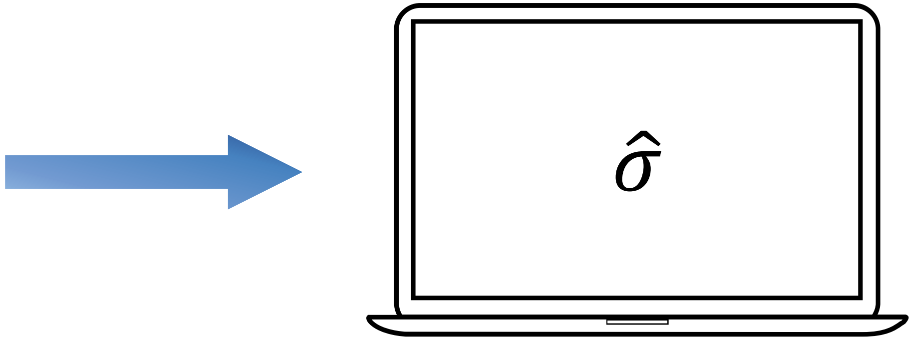

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Can estimate shadow information for M k-local observables to accuracy ϵ with $\boldsymbol{\mathcal{O}(\log(M)/\epsilon^2)}$

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Learning the Pauli channel

- Observation 1: $\mathrm{tr}\big(O\mathcal{P}(\sigma)\big) = tr\big(\mathcal{P}^\dagger(O)\sigma\big)$, where $\mathcal{P}^\dagger$ is the adjoint map of $\mathcal{P}$.
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$$\hat{\lambda}_Q = 3^{|Q|} \cdot \frac{1}{N}\sum_{i=1}^{N}\prod_{j=1}^{n} tr\left(Q_j(3|t_{ij}\rangle\langle t_{ij}| - \mathbb{I})\right) tr(Q_j|s_{ij}\rangle\langle s_{ij}|)$$

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$$\sum_{Q:|Q|\leq k} \frac{\alpha_Q}{\hat{\lambda}_Q} \widehat{tr}\left(Q\mathcal{P}(\sigma)\right)$$

Algorithm for information recovery

1. Prepare N random n-fold product Pauli eigenstates $\{\rho_i = \otimes_j |s_{ij}\rangle\langle s_{ij}|\}$.
2. Apply the unknown channel $\mathcal{P}$ on the N random states and perform random Pauli measurements on each qubit, obtaining data $\{\otimes_j |t_{ij}\rangle\langle t_{ij}|\}$.
3. For each n-qubit Pauli operator Q with $|Q| \leq k$, compute $\hat{\lambda}_Q$.

Learning

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4. For $O = \sum_{Q:|Q|\leq k} \alpha_Q Q$, let $\overleftarrow{\alpha}_Q = \alpha_Q / \hat{\lambda}_Q$ for $|Q| \leq k$.
5. Use classical shadows to estimate $tr\big(Q\mathcal{P}(\sigma)\big)$ for $|Q| \leq k$.
6. Construct estimator for $tr(O\sigma)$ as

$$f(\mathcal{P}(\sigma), O) = \sum_{Q:|Q|\leq k} \overleftarrow{\alpha}_Q \, \widehat{tr}\big(Q\mathcal{P}(\sigma)\big)$$

Post-processing

Complexity of the algorithm

- Three quantities to consider: number of channel used (N_1), number of noisy states used (N_2) and total computation time

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$$\mathcal{O}(\log(n)/\epsilon^2)$$

Computational complexity: $\mathcal{O}(n^k \log(n)/\epsilon^2)$

Numerical experiments

- Two-qubit product Pauli channel with p_I around 0.75
- Observable is a Heisenberg-typed Hamiltonian: $O = \sum_{j=1}^{n-1}\left(J_x\sigma_j^x\sigma_{j+1}^x + J_y\sigma_j^y\sigma_{j+1}^y + J_z\sigma_j^z\sigma_{j+1}^z + h\sigma_j^z\right)$
- To see how much our algorithm improves estimation, for 500 random states, $\{\sigma_1, \dots, \sigma_{500}\}$ subject them to noise. We compute

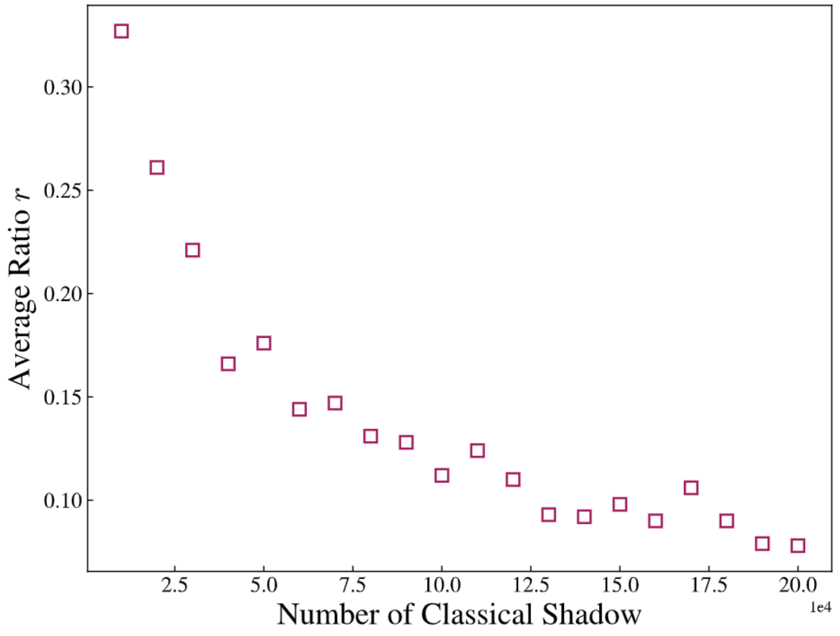
$$\text{MAE with no post-processing} = \frac{1}{500}\sum_{i=1}^{500}\left|tr\left(O\mathcal{P}(\sigma_i)\right) - tr(O\sigma_i)\right| \qquad \text{MAE with post-processing} = \frac{1}{500}\sum_{i=1}^{500}\left|f(\mathcal{P}(\sigma_i), O) - tr(O\sigma_i)\right|$$

- This allows us to compute the ratio between the two MAEs

$$r = \frac{\text{MAE with post–processing}}{\text{MAE with no post–processing}}$$

- We repeat this 10 times for N ranging from 10,000 to 200,000

As number of samples increases, more precise classical Information we are extracting, accuracy improves with samples.



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- Method:
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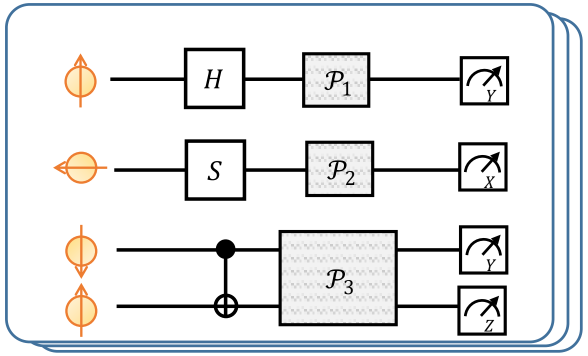

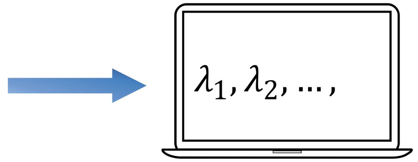

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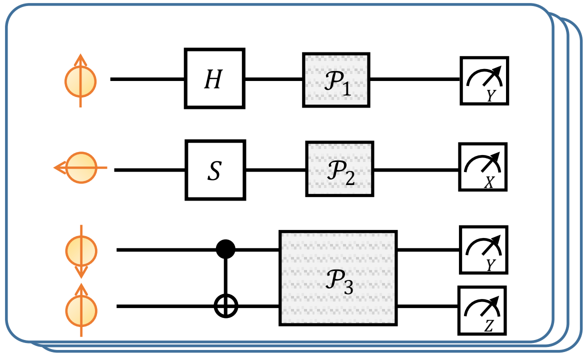

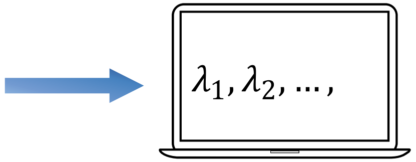

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  - Each α_Q is scaled by product of eigenvalues

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- Updated algorithm for weight contracting channel
- Channel access $\mathcal{O}(\boldsymbol{n^{2k}} \log(n)/\epsilon^2)$
- State copies $\mathcal{O}(\log(n)/\epsilon^2)$
- Computation time $\mathcal{O}\left(\boldsymbol{n^{4k}} \log(n)\ /\epsilon^2\right)$

Conclusion

- Main results:
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  - $\mathcal{O}(\log(n))$ sample complexity, $\mathcal{O}(n^k \log(n))$ time complexity
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