Quantum Algorithm for Robust Optimizaiton via Stochastic-Gradient Online Learning

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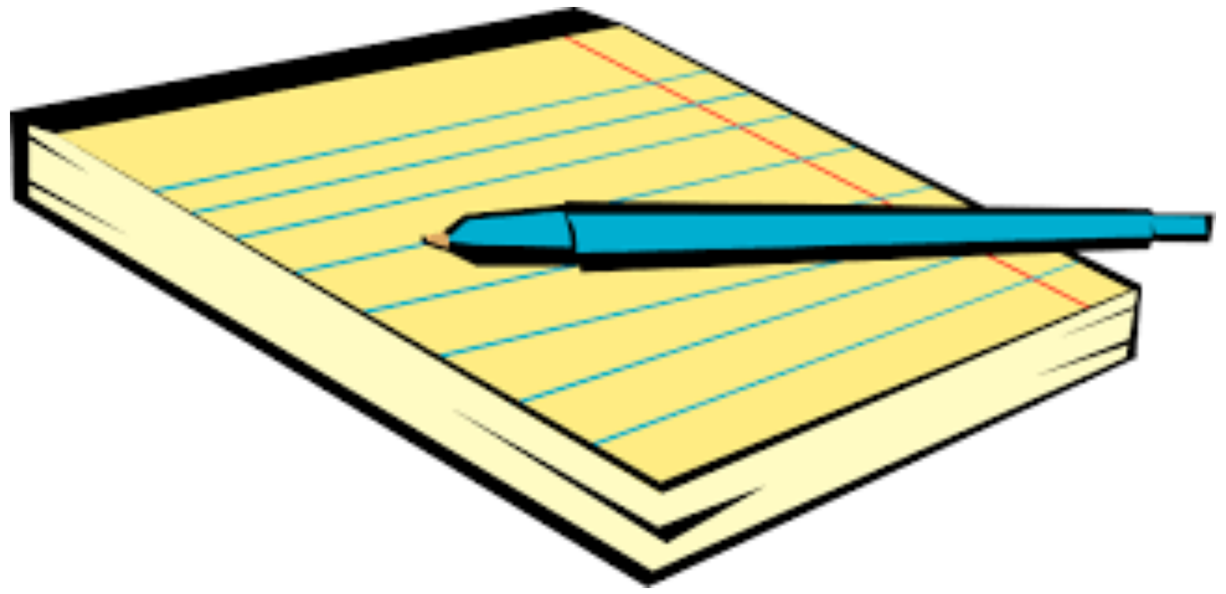
Outline



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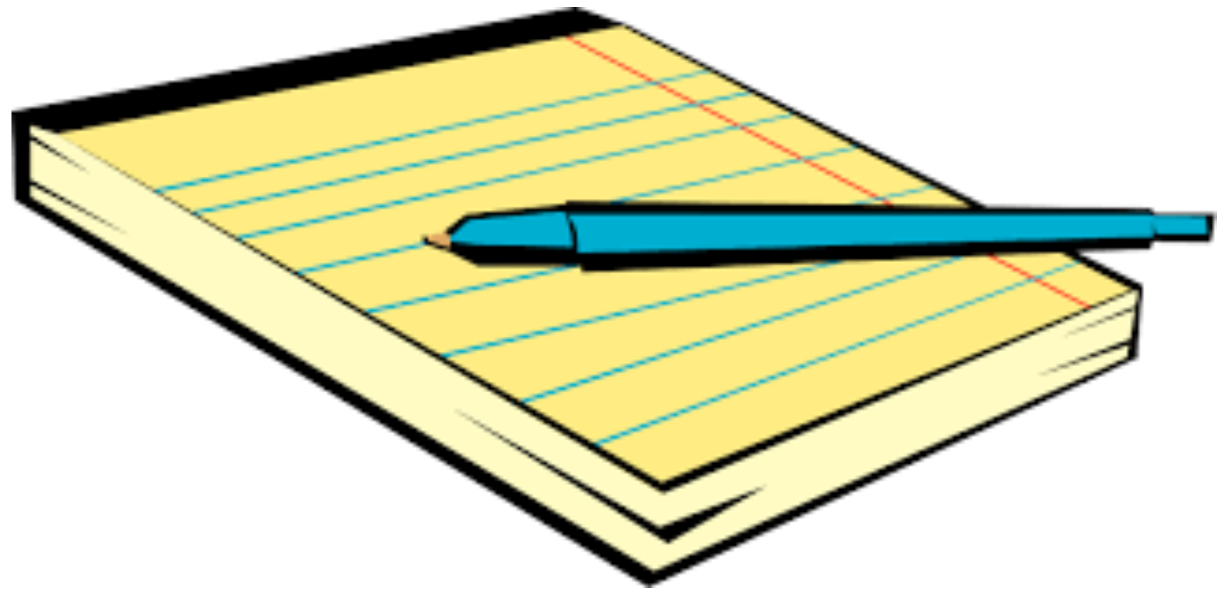
- Robust convex optimization



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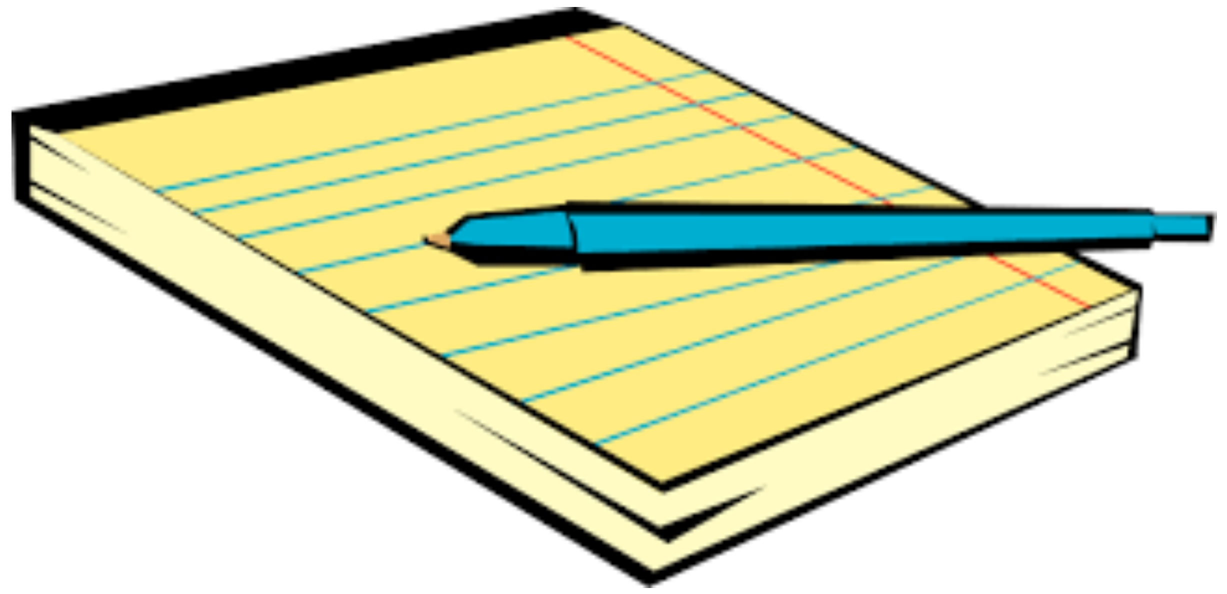

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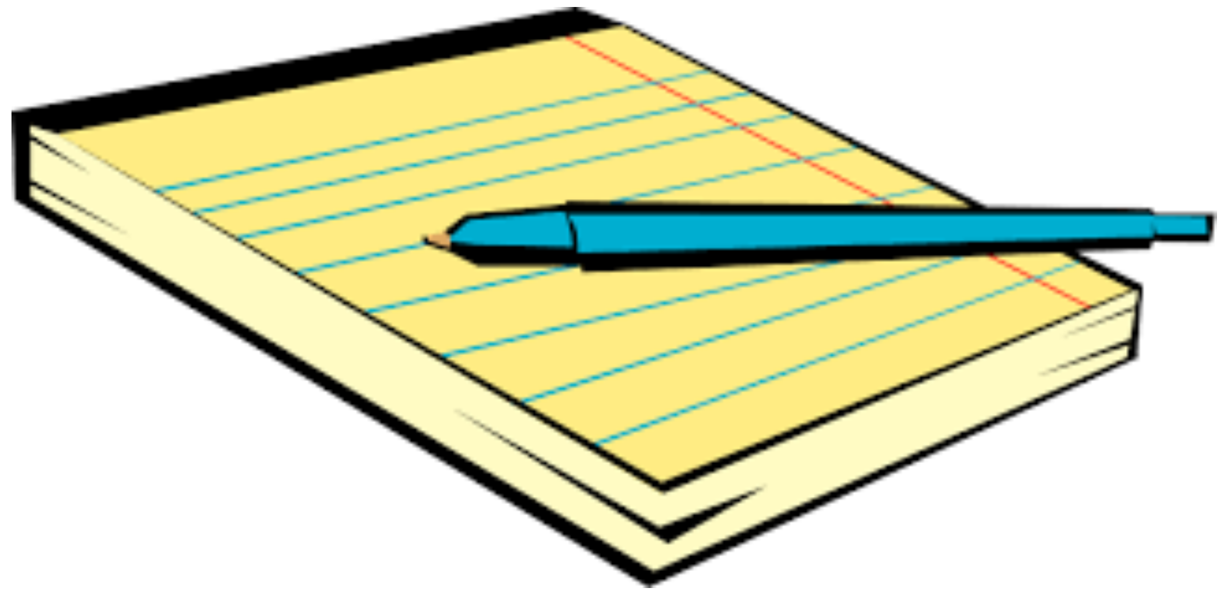

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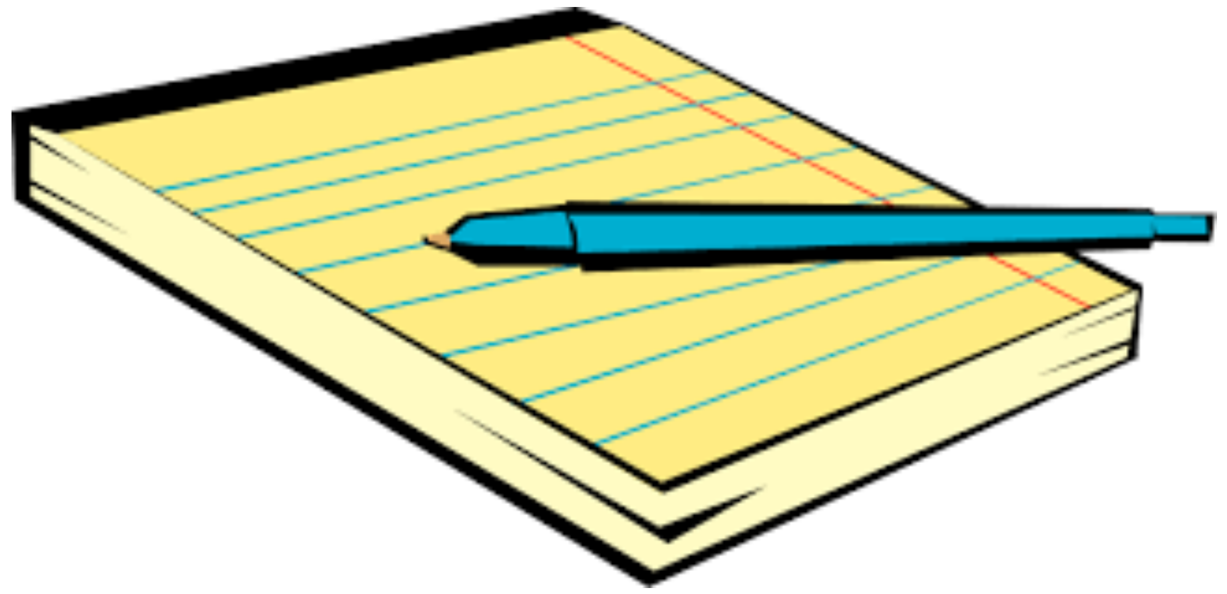

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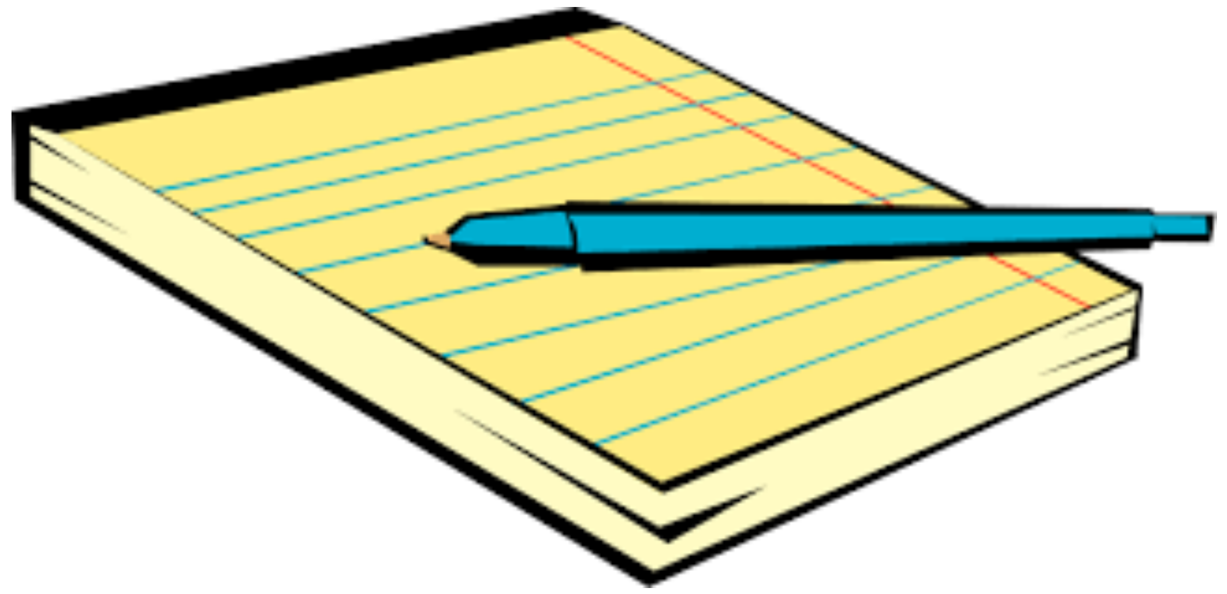

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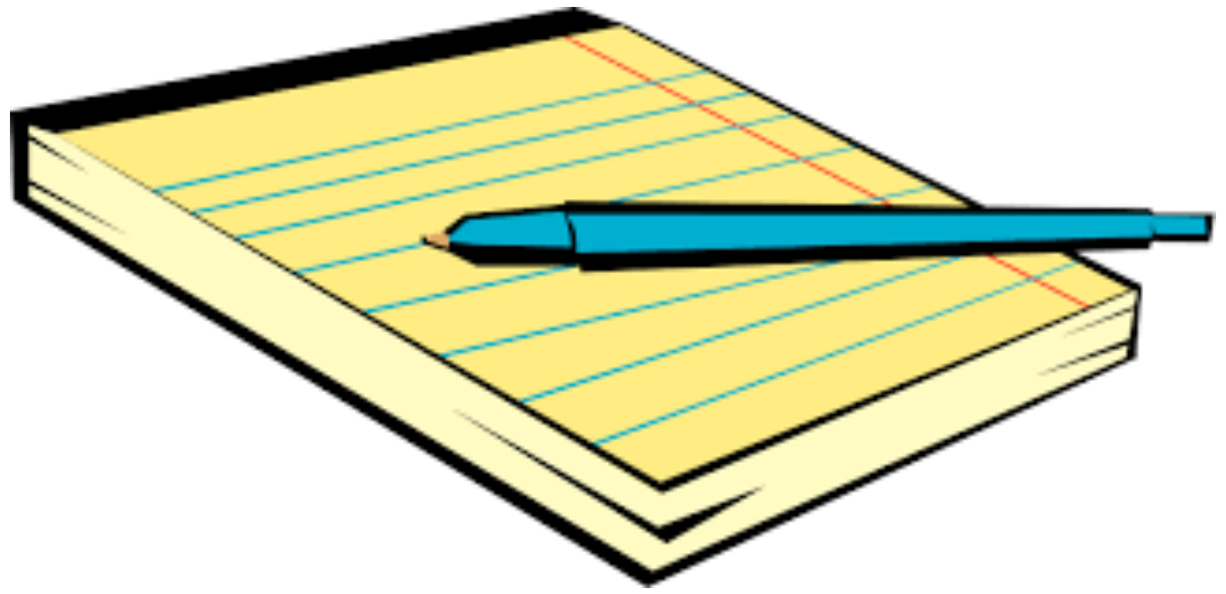

Robust convex optimization

Convex optimization

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\begin{aligned}
\textbf{minimize} \quad & f_0(x) & \\
\textbf{subject to} \quad & f_i(x, u_i) \le 0, & \forall i \in [m] \\
& x \in \mathcal{D} &
\end{aligned}
$$

- $u_1, \cdots, u_m \in \mathbb{R}^d$ are fixed parameters.
- $f_0, \cdots, f_m$ are convex in x.
- Domain $\mathcal{D} \subseteq \mathbb{R}^n$ is convex.

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- $f_0, \cdots, f_m$ are convex in x.
- $f_1, \cdots, f_m$ are concave in $u_1, \cdots, u_m$.
- Domain $\mathcal{D} \subseteq \mathbb{R}^n$ and uncertainty set $\mathcal{U} \subseteq \mathbb{R}^d$ are convex.

Optimization problem $\rightarrow$ feasibility problem

Optimization problem $\to$ feasibility problem

Use binary search over its optimal values

Optimization problem → feasibility problem

Use binary search over its optimal values

$$\exists? \qquad x \in \mathcal{D} : f_i(x, u_i) \leq 0 \qquad \forall u_i \in \mathcal{U}, \qquad \forall i \in [m]$$

Motivation

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- Computational cost for large scale problems can be highly prohibitive
- A meta-algorithm to approximately solve the robust counterpart of a convex optimisation problem, using only an algorithm for the original optimization formulation

Dual-subgradient robust optimization algorithm

Oracle-Based Robust Optimization via Online Learning

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- Initialize noise parameters $\left(u_1^{(0)}, \ldots, u_m^{(0)}\right) \in \mathscr{U}^m$ and $x^{(0)} \in \mathscr{D}$ arbitrarily;
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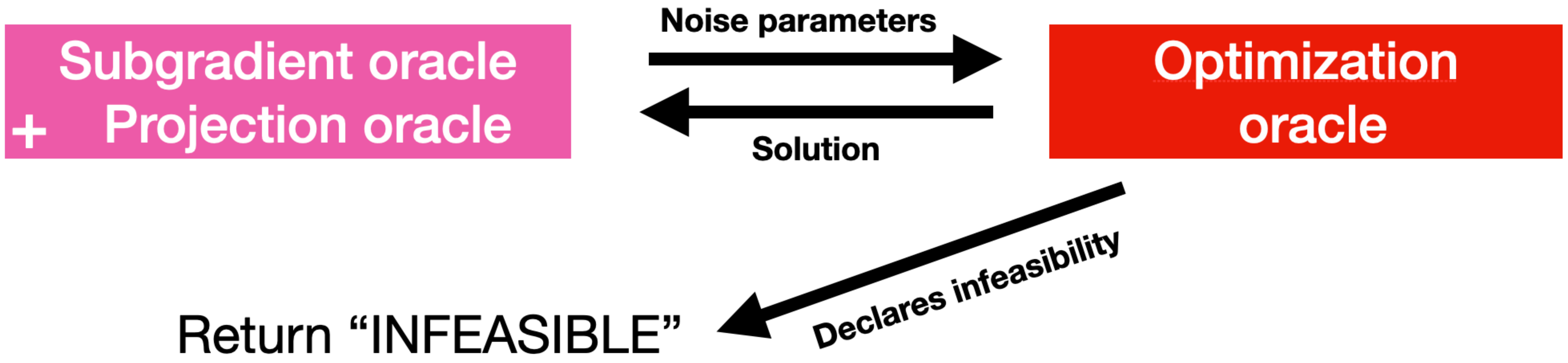

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Quantum Robust Optimization

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on a superposition of $|x_i\rangle$ in $O(1)$ time

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- Query the subgradient oracle on the **sampled** indices to create **stochastic** subgradients.

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Updating the noise parameters using subgradients	Update the noise parameters using **stochastic** subgradients.

Algorithm 3 Quantum online sampling-based dual subgradient robust optimization algorithm

Input: Target accuracy $\epsilon > 0$, failure probability $\delta \in (0,1)$, parameters D, G_1, G_2, G_∞, F;

Set $T = \left\lceil \frac{1}{\epsilon^2} \max\left\{4F\log(\frac{m}{\delta}), \frac{225D^2}{16}\left(G_2^2 + \frac{G_1G_\infty - G_2^2}{s}\right)\right\}\right\rceil$ and $\eta^{(t)} = \frac{4D}{3\sqrt{t+1}}\left(G_2^2 + \frac{G_1G_\infty - G_2^2}{s}\right)^{-1/2}$;

Initialize $(u_1^{(0)}, \ldots, u_m^{(0)}) \in \mathcal{U}^m$ and $x^{(0)} \in \mathcal{D}$ arbitrarily;

for $t = 0$ to $T-1$ **do**

  Sample s pairs $S^{(t)} = ((i_1, j_1), \ldots, (i_s, j_s)) \in ([m]\times[d])^s$ with probability at least $1-\delta/T$ by measuring s copies of the quantum state $\sum_{i=1}^m \sum_{j=1}^d \sqrt{p^{(t)}(i,j)}|i\rangle|j\rangle$ (Fact 1), where

$$p^{(t)}(i,j) = \frac{\left|\left(\nabla_u f_i(x^{(t)}, u_i^{(t)})\right)_j\right|}{\sum_{k=1}^m \left\|\nabla_u f_k(x^{(t)}, u_k^{(t)})\right\|_1};$$

  Compute an estimate $\Gamma^{(t)} \in \mathbb{R}$ of $\sum_{k=1}^m \left\|\nabla_u f_k(x^{(t)}, u_k^{(t)})\right\|_1$ with relative error $1/4$ (Fact 2);

  Query the oracle $\mathcal{O}_\nabla$ with inputs $(i,j) \in S^{(t)}$, $x^{(t)}$, and $u_i^{(t)}$, to prepare $g_i^{(t)} \in \mathbb{R}^d$ as

$$(g_i^{(t)})_j = \frac{|\{(k,\ell) \in S^{(t)} : k = i, \ell = j\}|}{s} \frac{\text{sign}\left[(\nabla_u f_i(x^{(t)}, u_i^{(t)}))_j\right]}{(\Gamma^{(t)})^{-1}};$$

  for i in $S^{(t)}$ **do**

    $u_i'^{(t+1)} \leftarrow u_i^{(t)} + \eta^{(t)} g_i^{(t)}$;

    $u_i^{(t+1)} \leftarrow \mathcal{P}_\mathcal{U}(u_i'^{(t+1)})$; ▷ Update noise memory

  end for

  $x^{(t+1)} \leftarrow \mathcal{O}_\epsilon(u_1^{(t+1)}, \ldots, u_m^{(t+1)})$;

  if oracle declares infeasibility **then return** `INFEASIBLE`;

  end if

end for

Output: $\bar{x} = \frac{1}{T}\sum_{t=1}^T x^{(t)}$;

Algorithm 3 Quantum online sampling-based dual subgradient robust optimization algorithm

Input: Target accuracy $\epsilon > 0$, failure probability $\delta \in (0,1)$, parameters D, G_1, G_2, G_∞, F;

Set $T = \left\lceil \frac{1}{\epsilon^2} \max\left\{4F \log(\frac{m}{\delta}), \frac{225D^2}{16}\left(G_2^2 + \frac{G_1 G_\infty - G_2^2}{s}\right)\right\}\right\rceil$ and $\eta^{(t)} = \frac{4D}{3\sqrt{t+1}}\left(G_2^2 + \frac{G_1 G_\infty - G_2^2}{s}\right)^{-1/2}$;

Initialize $(u_1^{(0)}, \ldots, u_m^{(0)}) \in \mathcal{U}^m$ and $x^{(0)} \in \mathcal{D}$ arbitrarily;

for $t = 0$ to $T-1$ **do**

Sample s pairs $S^{(t)} = ((i_1, j_1), \ldots, (i_s, j_s)) \in ([m] \times [d])^s$ with probability at least $1 - \delta/T$ by measuring s copies of the quantum state $\sum_{i=1}^m \sum_{j=1}^d \sqrt{p^{(t)}(i,j)}|i\rangle|j\rangle$ (Fact 1), where

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Compute an estimate $\Gamma^{(t)} \in \mathbb{R}$ of $\sum_{k=1}^m \left\|\nabla_u f_k(x^{(t)}, u_k^{(t)})\right\|_1$ with relative error $1/4$ (Fact 2);

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for i in $S^{(t)}$ **do**

$u_i'^{(t+1)} \leftarrow u_i^{(t)} + \eta^{(t)} g_i^{(t)}$;

$u_i^{(t+1)} \leftarrow \mathcal{P}_\mathcal{U}(u_i'^{(t+1)})$; ▷ Update noise memory

end for

$x^{(t+1)} \leftarrow \mathcal{O}_\epsilon(u_1^{(t+1)}, \ldots, u_m^{(t+1)})$;

if oracle declares infeasibility **then return** `INFEASIBLE`;

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Quantum multi-sampling: samples s numbers from $[n]$ in $O\left(\sqrt{sm}\log\left(\frac{1}{\delta}\right)\right)$ time.

  Compute an estimate $\Gamma^{(t)} \in \mathbb{R}$ of $\sum_{k=1}^m \left\|\nabla_u f_k(x^{(t)}, u_k^{(t)})\right\|_1$ with relative error $1/4$ (Fact 2);

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    $u_i'^{(t+1)} \leftarrow u_i^{(t)} + \eta^{(t)} g_i^{(t)}$;

    $u_i^{(t+1)} \leftarrow \mathcal{P}_{\mathcal{U}}(u_i'^{(t+1)})$; ▷ Update noise memory

  end for

  $x^{(t+1)} \leftarrow \mathcal{O}_\epsilon(u_1^{(t+1)}, \ldots, u_m^{(t+1)})$;

  if oracle declares infeasibility **then return** `INFEASIBLE`;

  end if

end for

Output: $\bar{x} = \frac{1}{T}\sum_{t=1}^T x^{(t)}$;

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    $u_i'^{(t+1)} \leftarrow u_i^{(t)} + \eta^{(t)} g_i^{(t)}$;

    $u_i^{(t+1)} \leftarrow \mathcal{P}_\mathcal{U}(u_i'^{(t+1)})$; ▻ Update noise memory

  end for

  $x^{(t+1)} \leftarrow \mathcal{O}_\epsilon(u_1^{(t+1)}, \dots, u_m^{(t+1)})$;

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  end if

end for

Output: $\bar{x} = \frac{1}{T}\sum_{t=1}^T x^{(t)}$;

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  Compute an estimate $\Gamma^{(t)} \in \mathbb{R}$ of $\sum_{k=1}^m \left\|\nabla_u f_k(x^{(t)}, u_k^{(t)})\right\|_1$ with relative error $1/4$ (Fact 2);

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    $u_i^{(t+1)} \leftarrow \mathcal{P}_\mathcal{U}(u_i'^{(t+1)})$; ▷ Update noise memory

  end for

  $x^{(t+1)} \leftarrow \mathcal{O}_\epsilon(u_1^{(t+1)}, \ldots, u_m^{(t+1)})$;

  if oracle declares infeasibility **then return** `INFEASIBLE`;

  end if

end for

Output: $\bar{x} = \frac{1}{T}\sum_{t=1}^T x^{(t)}$;

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Quantum norm estimation

Algorithm 3 Quantum online sampling-based dual subgradient robust optimization algorithm

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for i in $S^{(t)}$ **do**

$u_i'^{(t+1)} \leftarrow u_i^{(t)} + \eta^{(t)} g_i^{(t)}$;

$u_i^{(t+1)} \leftarrow \mathcal{P}_\mathcal{U}(u_i'^{(t+1)})$; ▻ Update noise memory

end for

$x^{(t+1)} \leftarrow \mathcal{O}_\epsilon(u_1^{(t+1)}, \ldots, u_m^{(t+1)})$;

if oracle declares infeasibility **then return** `INFEASIBLE`;

end if

end for

Output: $\bar{x} = \frac{1}{T}\sum_{t=1}^T x^{(t)}$;

Quantum multi-sampling: samples s numbers from $[n]$ in $O\left(\sqrt{sm}\log\left(\frac{1}{\delta}\right)\right)$ time.

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for i in $S^{(t)}$ **do**

$u_i^{\prime(t+1)} \leftarrow u_i^{(t)} + \eta^{(t)} g_i^{(t)}$;

$u_i^{(t+1)} \leftarrow \mathcal{P}_{\mathcal{U}}(u_i^{\prime(t+1)})$; ▷ Update noise memory

end for

$x^{(t+1)} \leftarrow \mathcal{O}_\epsilon(u_1^{(t+1)}, \ldots, u_m^{(t+1)})$;

if oracle declares infeasibility **then return** `INFEASIBLE`;

end if

end for

Output: $\bar{x} = \frac{1}{T}\sum_{t=1}^T x^{(t)}$;

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Quantum norm estimation

Stochastic gradient satisfies

$$\mathbb{E}\left[g_i^{(t)}\right] = \lambda \nabla_u f_i\left(x^{(t)}, u_i^{(t)}\right)$$

$$\mathbb{E}\left[\left\|g_i^{(t)}\right\|_2^2\right] \leq \tilde{G}_2^2$$

Algorithm 3 Quantum online sampling-based dual subgradient robust optimization algorithm

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Compute an estimate $\Gamma^{(t)} \in \mathbb{R}$ of $\sum_{k=1}^m \|\nabla_u f_k(x^{(t)}, u_k^{(t)})\|_1$ with relative error $1/4$ (Fact 2);

Query the oracle $\mathcal{O}_\nabla$ with inputs $(i,j) \in S^{(t)}$, $x^{(t)}$, and $u_i^{(t)}$, to prepare $g_i^{(t)} \in \mathbb{R}^d$ as

$$(g_i^{(t)})_j = \frac{|\{(k,\ell) \in S^{(t)} : k = i, \ell = j\}|}{s} \frac{\operatorname{sign}\left[(\nabla_u f_i(x^{(t)}, u_i^{(t)}))_j\right]}{(\Gamma^{(t)})^{-1}};$$

for i in $S^{(t)}$ **do**

$u_i'^{(t+1)} \leftarrow u_i^{(t)} + \eta^{(t)} g_i^{(t)}$;

$u_i^{(t+1)} \leftarrow \mathcal{P}_\mathcal{U}(u_i'^{(t+1)})$; ▷ Update noise memory

end for

$x^{(t+1)} \leftarrow \mathcal{O}_\epsilon(u_1^{(t+1)}, \dots, u_m^{(t+1)})$;

if oracle declares infeasibility **then return** `INFEASIBLE`;

end if

end for

Output: $\bar{x} = \frac{1}{T}\sum_{t=1}^T x^{(t)}$;

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Compute an estimate $\Gamma^{(t)} \in \mathbb{R}$ of $\sum_{k=1}^m \left\|\nabla_u f_k(x^{(t)}, u_k^{(t)})\right\|_1$ with relative error $1/4$ (Fact 2);

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$$(g_i^{(t)})_j = \frac{|\{(k,\ell) \in S^{(t)} : k = i, \ell = j\}|}{s} \frac{\text{sign}\left[(\nabla_u f_i(x^{(t)}, u_i^{(t)}))_j\right]}{(\Gamma^{(t)})^{-1}};$$

for i in $S^{(t)}$ **do**

$u_i'^{(t+1)} \leftarrow u_i^{(t)} + \eta^{(t)} g_i^{(t)}$;

$u_i^{(t+1)} \leftarrow \mathcal{P}_\mathcal{U}(u_i'^{(t+1)})$; **On sampled indices** ▷ Update noise memory

end for

$x^{(t+1)} \leftarrow \mathcal{O}_\epsilon(u_1^{(t+1)}, \ldots, u_m^{(t+1)})$;

if oracle declares infeasibility **then return** `INFEASIBLE`;

end if

end for

Output: $\bar{x} = \frac{1}{T}\sum_{t=1}^T x^{(t)}$;

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Summary of results

- When $T = \left\lceil \frac{1}{\epsilon^2} \max \left\{ 4F\left(\frac{m}{\delta}\right), \frac{9}{4}(1+4\nu)^2 D^2 \widetilde{G}_2^2 \right\} \right\rceil$ and $\eta^{(t)} = \frac{D}{(1-\nu)\widetilde{G}_2\sqrt{t+1}}$

Algorithm	Calls to $\mathcal{O}_\nabla$	Calls to $\mathcal{O}_\mathcal{P}$	Calls to $\mathcal{O}_\epsilon$
Ben-Tal *et. al*	$\frac{G_2^2 m d D^2}{\epsilon^2}$	$\frac{G_2^2 m D^2}{\epsilon^2}$	$\frac{G_2^2 D^2}{\epsilon^2}$

- $D \geq \max_{u,v \in \mathcal{U}} \|u - v\|_2$
- $F \geq \max_{x \in \mathcal{D}, u \in \mathcal{U}} |f_i(x,u)|$
- $G_2 \geq \max_{x \in \mathcal{D}, u \in \mathcal{U}} \|\nabla_u f_i(x,u)\|_2$
- $G_1 \geq \sum_{k=1}^{m} \|\nabla_u f_k(x, u_k)\|_1$
- $G_\infty \geq \max_{k \in [m]} \|\nabla_u f_k(x, u_k)\|_1$,

$\forall x \in \mathcal{D}, \forall u_1, \ldots, u_m \in \mathcal{U}$

- When $T = \left\lceil \frac{1}{\epsilon^2} \max \left\{ 4F \log\left(\frac{m}{\delta}\right), \frac{225 D^2}{16}\left(G_2^2 + \frac{G_1 G_\infty - G_2^2}{s}\right) \right\} \right\rceil$ and $\eta^{(t)} = \frac{4D}{3\sqrt{t+1}}\left(G_2^2 + \frac{G_1 G_\infty - G_2^2}{s}\right)^{-1/2}$

Algorithm	Calls to $\mathcal{O}_\nabla$	Calls to $\mathcal{O}_\mathcal{P}$	Calls to $\mathcal{O}_\epsilon$
Our work	$\frac{\sqrt{G_1 G_\infty} G_2 \sqrt{md} D^2}{\epsilon^2} \log\left(\frac{D G_2}{\epsilon\delta}\right)$	$\min\{G_1 G_\infty, G_2^2 m\}\frac{D^2}{\epsilon^2}$	$\frac{G_2^2 D^2}{\epsilon^2}$

Summary of results

- When $T = \left\lceil \frac{1}{\epsilon^2} \max \left\{ 4F \left(\frac{m}{\delta} \right), \frac{9}{4}(1 \ldots \right.\right.$

Algorithm	
Ben-Tal et …	

- When $T = \left\lceil \frac{1}{\epsilon^2} \max \ldots - G_2^2 \ldots s \right)^{-1/2}$

Algorithm	$\mathcal{O}_\nabla$		… calls to $\mathcal{O}_\epsilon$
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- $G_1 G_\infty = O\left(m G_2^2\right) \Rightarrow$ quadratic speedup in the dimension of the noise parameters d.
- $G_1 G_\infty$ is comparable to $G_2^2 \Rightarrow$ quadratic speedup in $m = \left| \mathcal{U} \right|, d$.

- $D \geq \max_{u,v \in \mathcal{U}} \|u - v\|_2$
- $F \geq \max_{x \in \mathcal{D}, u \in \mathcal{U}} |f_i(x, u)|$
- $G_2 \geq \max_{x \in \mathcal{D}, u \in \mathcal{U}} \|\nabla_u f_i(x, u)\|_2$
- $G_1 \geq \sum_{k=1}^{m} \|\nabla_u f_k(x, u_k)\|_1$
- $G_\infty \geq \max_{k \in [m]} \|\nabla_u f_k(x, u_k)\|_1,$

$\forall x \in \mathcal{D}, \forall u_1, \ldots, u_m \in \mathcal{U}$

Applications

Robust linear programs

$$\exists? \qquad x \in \mathscr{D}$$

$$\text{s.t.} \qquad \left(a_i + P_i u_i\right)^\top x - b_i \leq 0, \qquad \forall u_i \in \mathscr{U}, \quad i \in [m],$$

$$\mathscr{U} = \{u \in \mathbb{R}^d : \ \| u \|_2 \leq 1\}$$
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Find a portfolio that maximizes the return (without considering the variance)

Robust semidefinite programs

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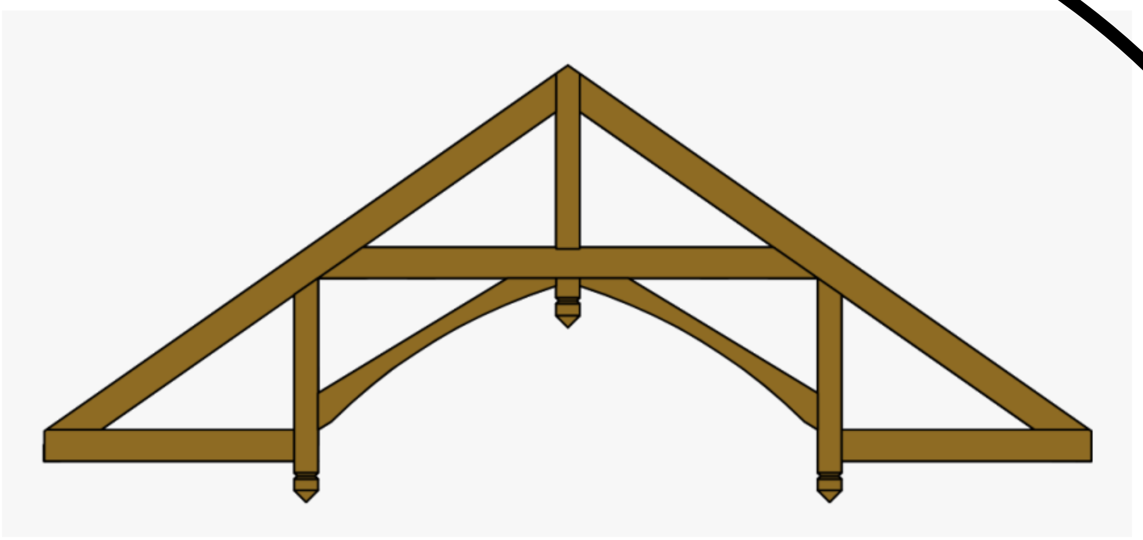
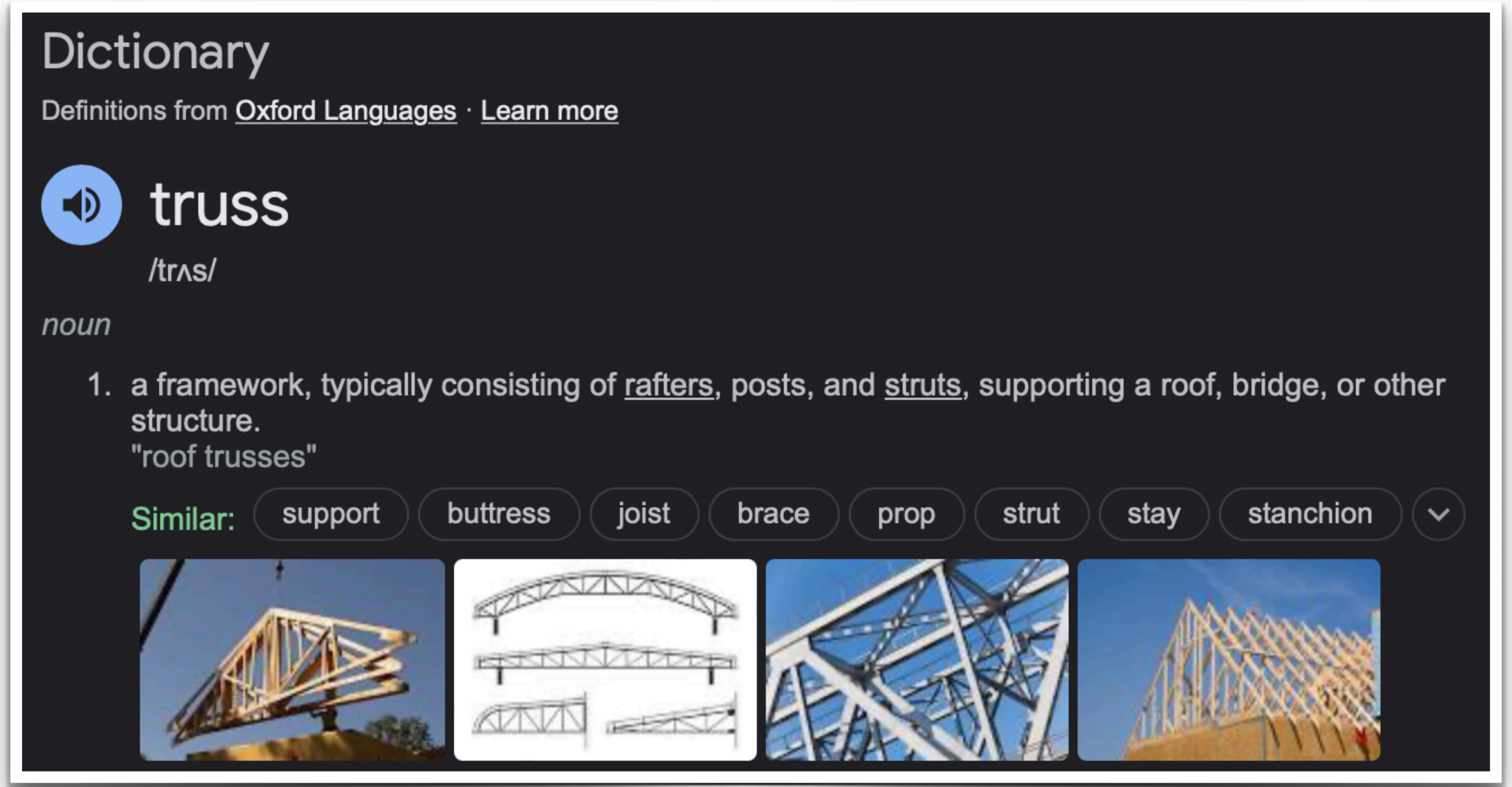


Truss Topological Design

Truss topological design (TTD)



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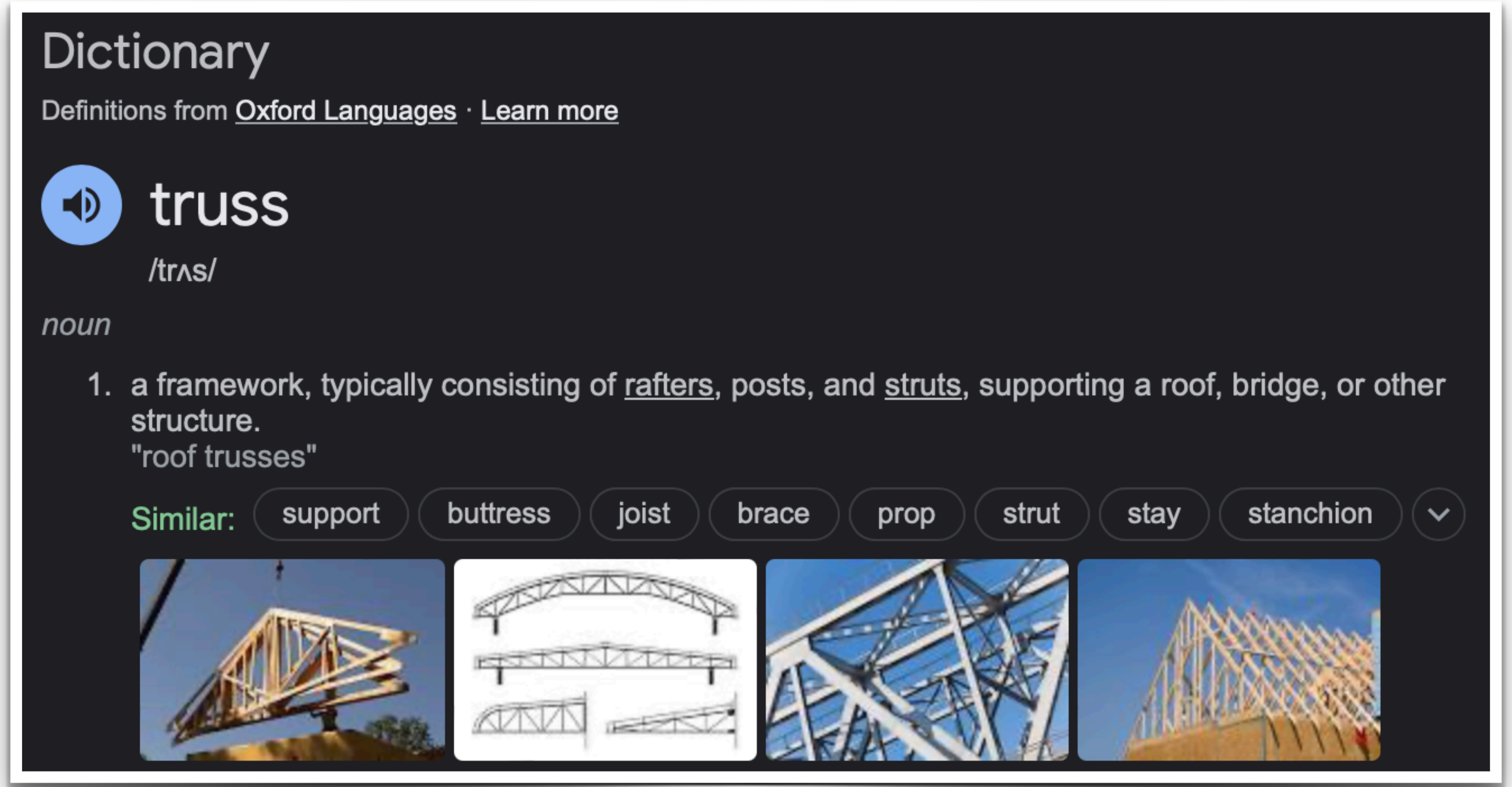

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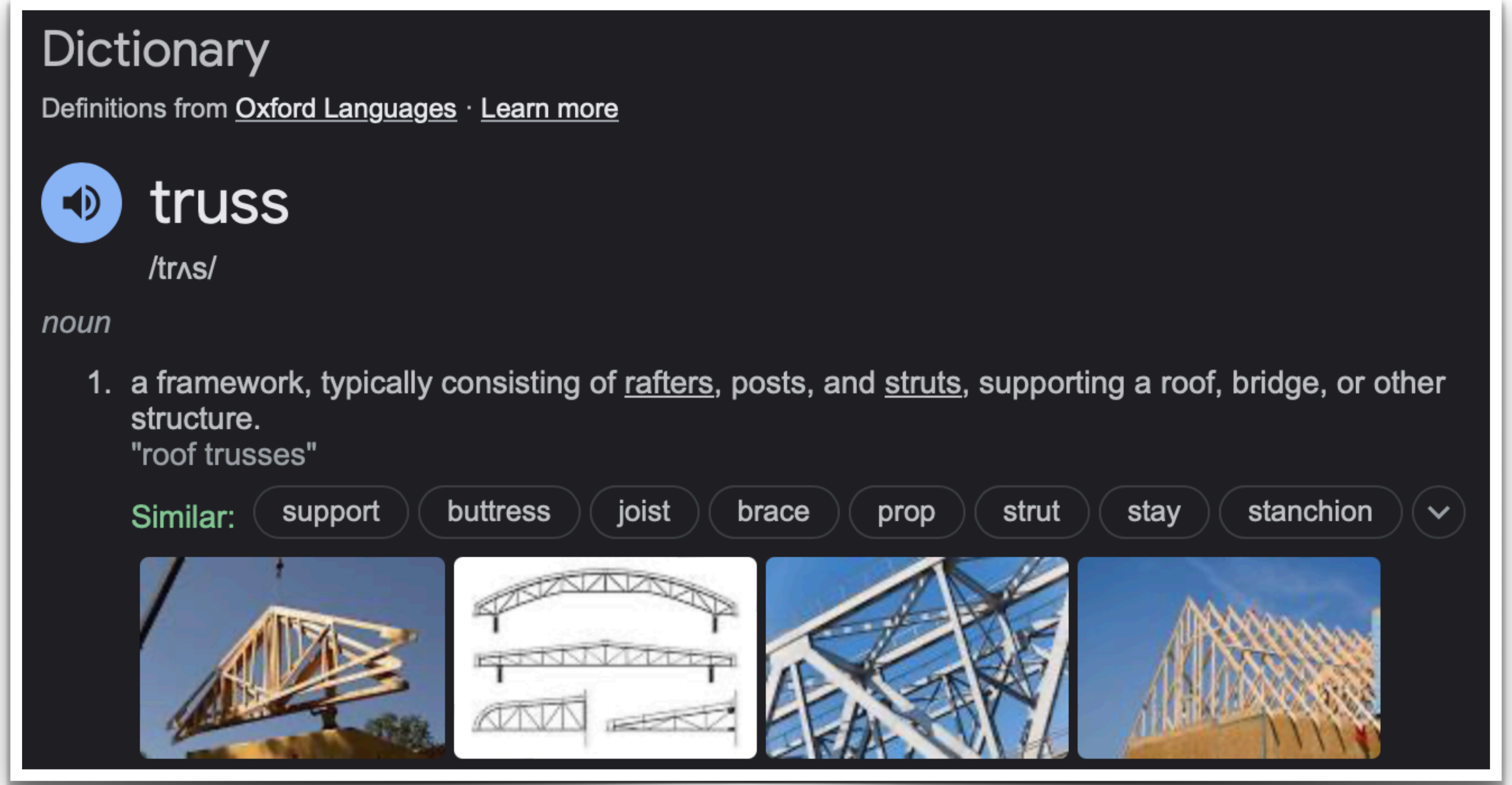

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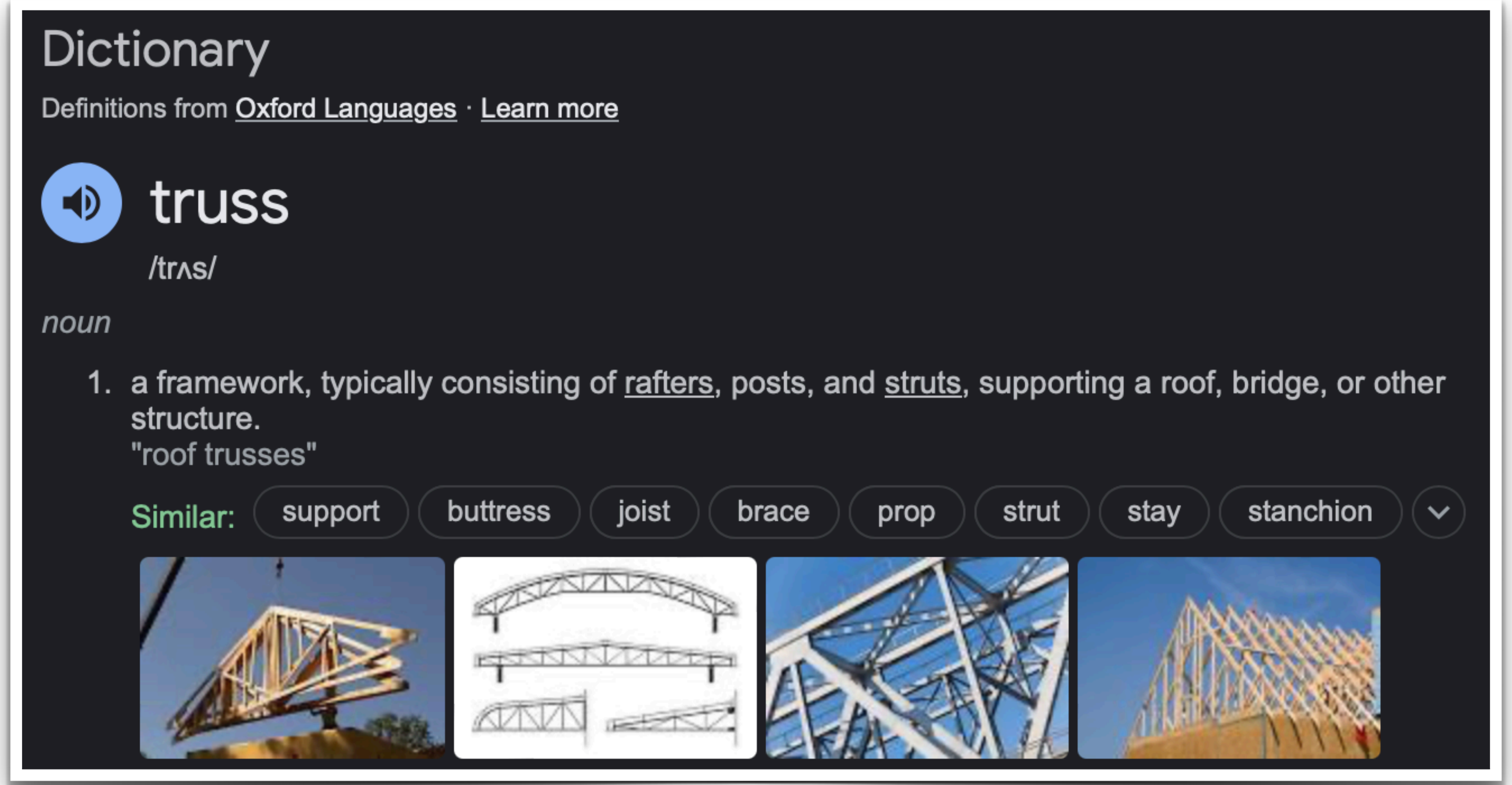

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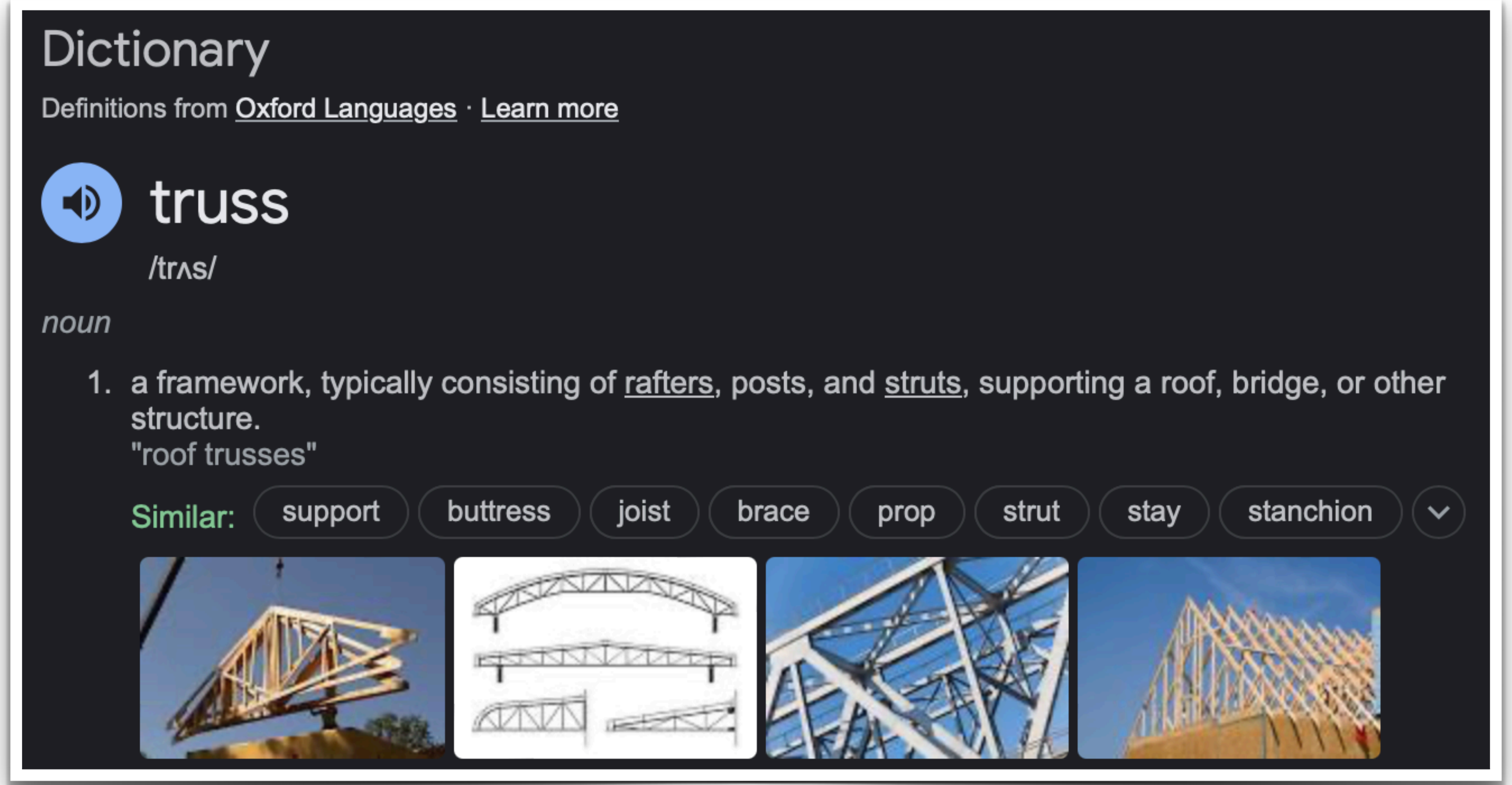

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Runtime dependent on the Frobenius, ℓ_∞-norm of the matrices that control the shape of the ellipsoid.

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Thank
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