Learning t doped stabilizer states

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Work based on on:

Learning t-doped stabilizer states ArXiv:2305.15398

Joint Work with

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Outline

- Introduction to the notion of learning.
- Clifford Group and Stabilizer states.
- *t*-doped stabilizer states.
- Learning *t*-doped stabilizer states.

Learning Theory

- Main idea: Learn information about an unknown system.
- General setup: Black box access to multiple copies of a state.
- Measure the state in a suitable way to learn information.
- Examples: Tomography, Shadow Tomography.
- Can the structure of a state help in its learning?

Learning Theory: Precision and sample complexity

- Learning tasks characterized by a trade-off between precision and sample complexity.
- Goal is to obtain an efficient learning with high precision.
- Tomography of a state involves $\mathcal{O}(\exp(n))$ samples.
- Two ways around:
  - Learn minimal amount of information for predictions: Classical shadows
  - Leverage on the structure of class of states.

Pauli group and stabilizer states

- The Pauli group $\mathcal{P}_n$ is defined as the group generated by $\langle \{\sigma_i^x, \sigma_i^z\}_{i=1}^n \rangle$ times a multiplicative factor of $\pm 1, \pm i$
- Focus on the quotient group $\mathbb{P}_n = \mathcal{P}_n / \{\pm \mathbb{1}_n, \pm i \mathbb{1}_n\}$
- Cardinality of the Pauli group $|\mathbb{P}_n| = 4^n$
- When given a state $|\psi\rangle$ it holds that $P|\psi\rangle = \pm|\psi\rangle$ with P a Pauli operator, stabilized by P.

$|\psi\rangle$

- A pure stabilizer state $|\sigma\rangle$ is defined as a state stabilized by 2^n commuting Pauli operators.

$$|\sigma\rangle\langle\sigma| = \frac{1}{2^n} \sum_P \sigma_P P \qquad \sigma_P = \pm 1$$

or

$$|\sigma\rangle\langle\sigma| = \frac{1}{2^n} \prod_i (\mathbb{I} + \sigma_i g_i) \qquad \sigma_i = \pm 1$$

Clifford Group and simulability

- The Clifford group $\mathcal{C}_n$ is the normalizer of $\mathcal{P}_n$

$$CPC^\dagger \in \mathcal{P}_n, \quad \forall C \in \mathcal{C}_n, \forall P \in \mathcal{P}_n$$

- $\mathcal{C}_n$ is generated by:

$$\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \; \mathrm{H} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \; \mathrm{S} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

- Efficient encoding of *n*-qubits Pauli strings on *2n*-bit-strings.

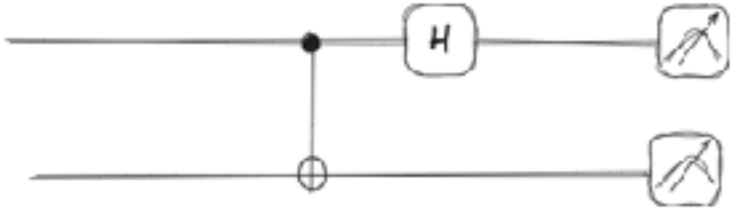
$$\mathbb{1} \to (00), \sigma^x \to (10), \sigma^z \to (01), \sigma^y \to (11)$$

- From classical encoding one can show classical simulability of stabilizer states, and Clifford group.

Learning of stabilizer states

- Two-qubit Bell basis corresponds to vectorization of single qubit Pauli operators. $|\mathbb{I}\rangle, \ |X\rangle, \ |Y\rangle, |Z\rangle$

- Measurement in the Bell basis realized by



- Given a stabilizer state $|\sigma\rangle$ and its conjugate $|\sigma^*\rangle$, measuring in the Bell basis returns outcome P with the following probability distribution

$$2^{-n}|\langle\sigma|P|\sigma\rangle|^2$$

- No access to $|\sigma^*\rangle$

Learning of stabilizer states

- Given two copies of $|\sigma\rangle$ we receive outcome P with probability

$$2^{-n}|\langle\sigma|P|\sigma^*\rangle|^2 = 2^{-n}|\langle\sigma|PP_Z|\sigma\rangle|^2$$

- Outcome P does not stabilize $|\sigma\rangle$ but given two outcomes P_1, P_2 their product P_1P_2 is a stabilizer.
- Sketch of the algorithm:
  - Estimate $2n+1$ Pauli operators P
  - Measure their phases.

- Failure probability 2^{-n}
- Number of required samples $5n+2$

Montanaro, Learning stabilizer states by Bell sampling, https://arxiv.org/abs/1707.04012

Non-Clifford Resources

- Addition of a non-Clifford gate to the set $\{\mathrm{CNOT}, \mathrm{H}, \mathrm{S}\}$ makes it *universal* for quantum computation.

- For example, the *T*-gate $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$

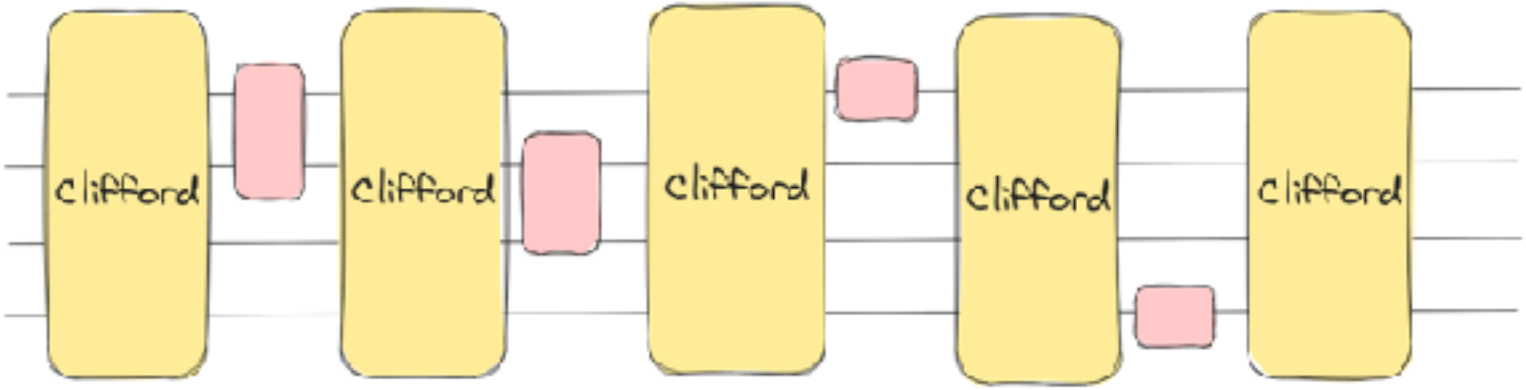
- Its action spreads Pauli operators $T\sigma_i^x T^\dagger = \frac{1}{\sqrt{2}}(\sigma_i^x + \sigma_i^y)$

- Classical algorithm to simulate action of Clifford circuit polluted with *T*-gate scales exponentially with the the number of *T*-gates*

*S. Bravyi and D. Gosset, PRL 116 (250501)

Doping of Clifford circuits.

- t-doped Clifford circuits

$C_t \equiv$ 

- A t-doped Clifford circuit possesses a preserved subsystem $G(C_t)$ of Pauli operators

$$G(C_t) := \{P \in \mathbb{P}(n) | C_t^\dagger P C_t \in \mathbb{P}(n)\}$$

- For $t = 0$ (Clifford circuit) $G(C_0 \in \mathcal{C}_n) \equiv \mathbb{P}(n)$

- Lower bound on cardinality $|G(C_t)| \geq 4^{n-t}$

Doping of Clifford circuits.

- There exists a Clifford operator $V \in \mathcal{C}_n$ such that

$$V^\dagger P V = C_t^\dagger P C_t \qquad \forall P \in G(C_t)$$

Definition: *(Diagonalizer)*
Give a subset of Pauli generators $\mathfrak{h} \equiv \{h_1, h_2, \ldots h_2m-1, h_{2m}\}$
The Diagonalizer is defined as $\mathcal{D}_{\mathfrak{h}} : \mathfrak{h} \mapsto \{\sigma_1^x, \sigma_1^z, \ldots, \sigma_m^x, \sigma_m^z\}$

- Given V and $\mathcal{D}_{g(C_t)}$, for $t < n$ the following decomposition holds

$$C_t \equiv \mathcal{D}_{g(C_t)}(\mathbb{I}^{\otimes n-t} \otimes c_t)V$$

- Sketch of the proof:

$$\mathcal{D}^\dagger_{g(U_t)} V U_t^\dagger \mathcal{D}_{g(U_t)} P_s \mathcal{D}^\dagger_{g(U_t)} U_t V^\dagger \mathcal{D}_{g(U_t)} = P_s \qquad s \in [1, \ldots, m]$$

Leone et al., Learning efficient decoders for quasi-chaotic quantum scramblers, https://arxiv.org/abs/2212.11338v3

t-doped stabilizer states

- Given a t-doped Clifford Circuit, a t-doped stabilizer state is defined as

$$|\psi_t\rangle \equiv C_t|0\rangle^{\otimes n}$$

- For any t holds the following decomposition

$$|\psi_t\rangle\langle\psi_t| = 2^{t-n}\underbrace{\left(\mathbb{I} + \sum_{i=1}^{k} \mathrm{tr}(h_i|\psi_t\rangle\langle\psi_t|)h_i\right)}_{H}\underbrace{\left(\prod_{j}^{|G_{\psi_t}|}\left(\frac{\mathbb{I}+\phi_j g_j}{2}\right)\right)}_{\Pi}$$

- Elements of H and Π commute.

- For $t < n$ it holds that: $|\psi_t\rangle \equiv \tilde{D}_{G(C_t)}|\phi_t\rangle \otimes |0\rangle^{n-t}$

Learning algorithm – main elements

- A t-doped stabilizer state $|\psi_t\rangle$ characterized by *m* generators g_j with *m* relative phases ϕ_j and *k* bad generators h_i with relative expectation values.
- Assumption: the ability to sample from the distribution

$$\Xi_t \equiv \{|\langle\psi_t|P|\psi_t\rangle|^2\}_{P\in\mathcal{P}(n)}$$

- Step 1: Learn g_js.
  - Probability of sample a $P \in G_{\psi_t}$ is

$$Pr(P \in G_{\psi_t}) = d^{-1}|G_{\psi_t}| \geq 2^{-t}$$

  - Check whether is a stabilizer or not. Measure M times P.
    Failure probability $\geq (1-2^{-t})(h_{max}/2+1/2)^M$
  - $n+m$ samples are required to get an exponential small probability of failure to build G_{ψ_t}
  - Total samples required to build G_{ψ_t}

$$\mathcal{O}(\left[2^t-1\right](n+m)(h_{\max}/2+1/2)^M+2^{-n})$$

Learning algorithm – main elements

- Step 2: Learn k h_is.
  - By Step 1, already found l bad generators.
  - Learn h_{l+1}s sampling from Ξ_t
  - Probability of sampling an element outside $G_{\psi_t} \cup h_1 G_{\psi_t} \cup \ldots \cup h_l G_{\psi_t}$

$$\pi_l = 1 - \frac{|G_{\psi_t}|}{d}\left(1 + \sum_{i=1}^{l} \mathrm{tr}^2(h_i \psi_t)\right) \geq 2^{-n}|G_{\psi_t}|(k-l)h_{\min}^2$$

  - Problem: is there an efficient way to check whether an element belong to $G_{\psi_t} \cup h_1 G_{\psi_t} \cup \ldots \cup h_l G_{\psi_t}$ or not?
    - Introduction of a diagonalizer operator $\mathcal{D} : G_{\psi_t} \to \tilde{G}_{\psi_t} \cap \mathbb{Z}$
    - Bad generators are now localized in l qubits.
    - Operation can now be achieved checking each qubit. Runtime: $\mathcal{O}(nl)$
  - Total number of samples $\mathcal{O}(2^t h_{min}^{-2}(\gamma + \log(k+1)))$
  - Computational steps $\mathcal{O}(n^2 k + nk)$

Finite Resolution – dependence on T gates doping

- Step 1 and 2 depend on h_{max} and h_{min}.
- *t*-doped stabilizer states are a discrete set $\Longrightarrow$ finite resolution δ_t.
- Resolution depends on the choice of the doping.
- In general $h_{min} \geq c2^{-bt}$ and $h_{max} \geq 1 - c2^{-bt}$
- For T gates $c = \sqrt{2}/6$ and $b \sim 2.27$
- Total number of samples $\mathcal{O}(2^{7t}t + 2^{4t}n(n+t))$
- Computational steps $\mathcal{O}(4^t n^2)$
- Failure probability $\mathcal{O}(n2^{-n})$

Sampling from probability distribution

- Sampling of Ξ_{ψ_t} requires to have access to both ψ_t and its conjugate ψ_t^* .
- What happens when we have only access to ψ_t ?
- The probability distribution given by Bell sampling is

$$\tilde{\Xi}_t \equiv 2^{-n}|\langle\psi_t|P|\psi_t^*\rangle|$$

- Differently from Montanaro there is no P_Z such that $P_Z|\psi_t\rangle = |\psi_t^*\rangle$
- However, statement holds for stabilizer part $P_Z \Pi P_Z = \Pi^*$
- Probability that the product of two samples P, P' is in G_{ψ_t}

$$Pr(PP' \in G_{\psi_t)}) \geq 2^{-4t}$$

- Total samples $\mathcal{O}(2^{4t}(n+m))$
- Learning of the state can be done with $\mathcal{O}(\exp(t)\,\mathrm{poly}(n))$

Conclusion and outlooks

- *t*-doped stabilizer states present a peculiar algebraic structure.
- Learning of a *t*-doped stabilizer state can be done for $poly(n)exp(t)$
- Learned generators and bad generators, can one synthesize a Clifford+T circuit to generate $|\psi_t\rangle$
- Can the algebraic structure be of help in estimating quantities?
- Can it be applied to *t-doped* fermionic and bosonic states?

Similar works:

Grewal et al., *Efficient Learning of Quantum States Prepared with Few Non-Clifford gates*, https://arxiv.org/pdf/2305.13409.pdf

Hangleiter et al., *Bell sampling from quantum circuits*, https://arxiv.org/abs/2306.00083