A Quadratic Speedup in Finding Nash Equilibria of Quantum Zero-Sum Games

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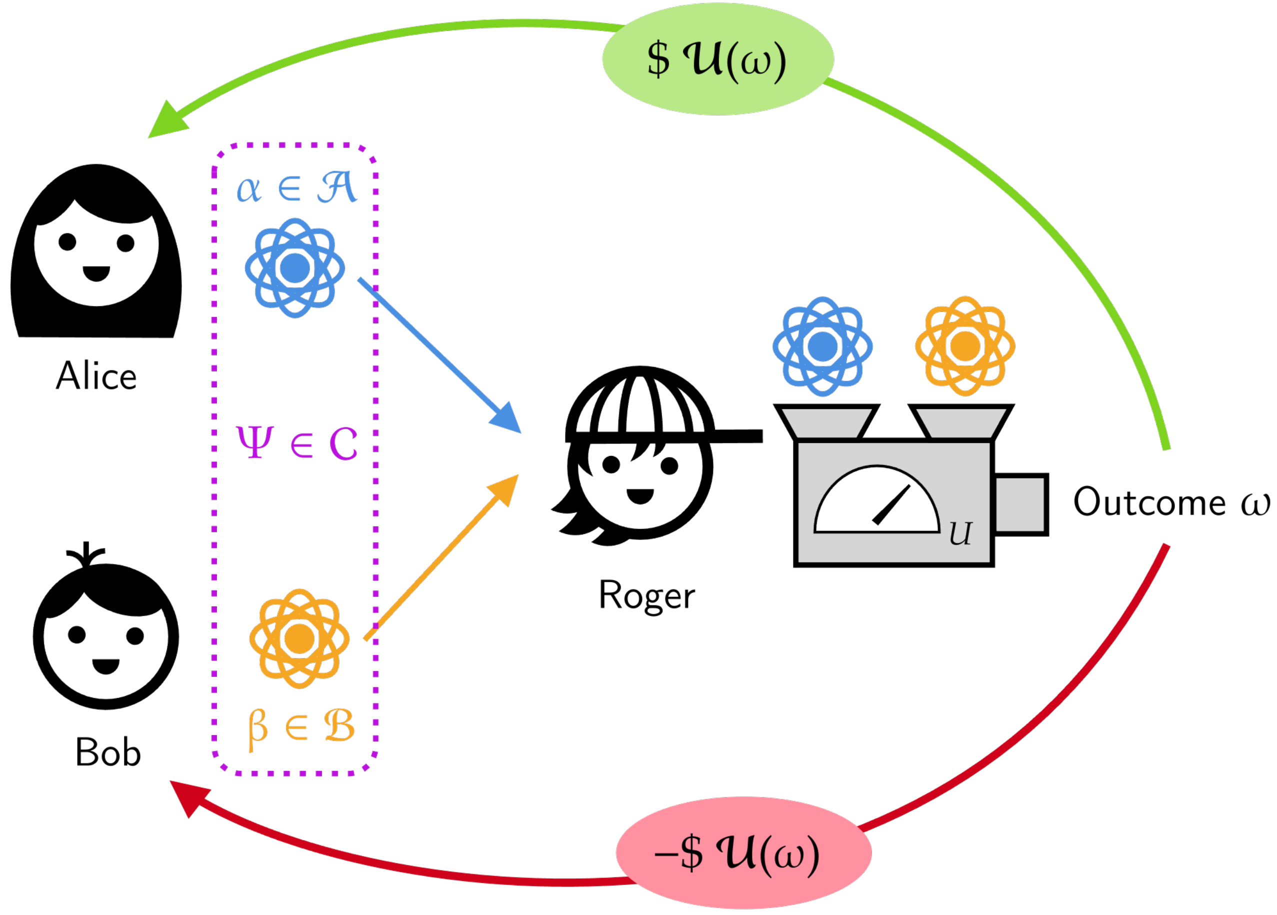
Overview of Results

- We propose a new algorithm (**Optimistic Matrix Multiplicative Weight Updates**) for finding approximate Nash equilibria of quantum zero-sum games.
  - To do so we leverage a **gradient-based** view of QZSG.
  - This allows us to easily leverage optimization techniques from the **classical** games literature.
- We prove that OMMWU achieves an $\mathcal{O}(1/\epsilon)$ iteration complexity.
  - This is a **quadratic speedup** relative to the best prior algorithm [JW09].
  - We leverage the proof technique of [EN20] for **monotone variational inequalities**.
- We further introduce a design framework for QZSG algorithms.
  - We use this to **unify** the QZSG algorithms landscape and **motivate** OMMWU.

A Quadratic Speedup in Finding Nash Equilibria of **Quantum Zero-Sum Games**

What is a Quantum Zero-Sum Game (QZSG)?



- A **two-player** game
- In each round, players play **unentangled** mixed states (**spectraplex**):

$$\alpha \in \mathcal{A} = \mathbb{C}^{2^n x 2^n}, \qquad \beta \in \mathcal{B} = \mathbb{C}^{2^m x 2^m}$$

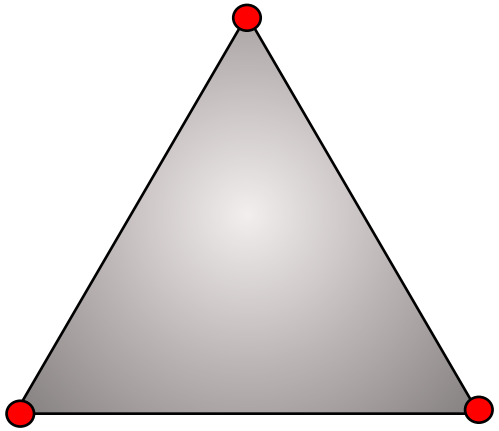
- They send **joint state** $\Psi = (\alpha, \beta)$ to a referee
- The referee makes a **joint measurement** w.r.t. payoff observable:

$$U = \sum_{\omega \in \Omega} \mathcal{U}(\omega)\, P_\omega$$

- Based on the measurement outcome, the referee **rewards** the players
  - **zero-sum** ⇒ one player's **win** is the other's **loss**

Game Strategies & Expected Payoff

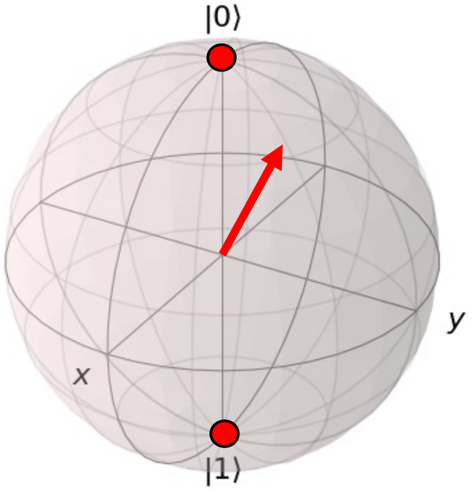
- CZSG are played in the **simplex**:



- Players play **indeterministic strategies**:
  - Probability vectors: $|\alpha\rangle, |\beta\rangle \in [0,1]^n$
- **Expected utility** for specific strategies:

$$u(\alpha, \beta) = \langle\beta|U|\alpha\rangle$$

- QZSG are played in the **spectraplex**:



- Players play **mixtures** of indeterministic strategies (**meta-strategies**):
  - Density matrices: $\alpha \in \mathbb{C}^{2^n x 2^n}$, $\beta \in \mathbb{C}^{2^m x 2^m}$
- **Expected utility** for specific strategies:

$$u(\alpha, \beta) = \mathrm{Tr}[U(\alpha \otimes \beta)]$$

The QZSG Objective

- If Alice's expected payoff is $u(\alpha, \beta)$,
  Bob's expected payoff is $-u(\alpha, \beta)$
- In playing the game, each player wants to **maximize** their expected payoff:
  - Alice wants: $\max_{\alpha} u(\alpha, \beta)$
  - Bob wants: $\max_{\beta} -u(\alpha, \beta) = \min_{\beta} u(\alpha, \beta)$
- These are **competing** interests, defining a **minimax** optimization problem:

$$\min_{\beta \in \mathcal{B}} \max_{\alpha \in \mathcal{A}} u(\alpha, \beta)$$

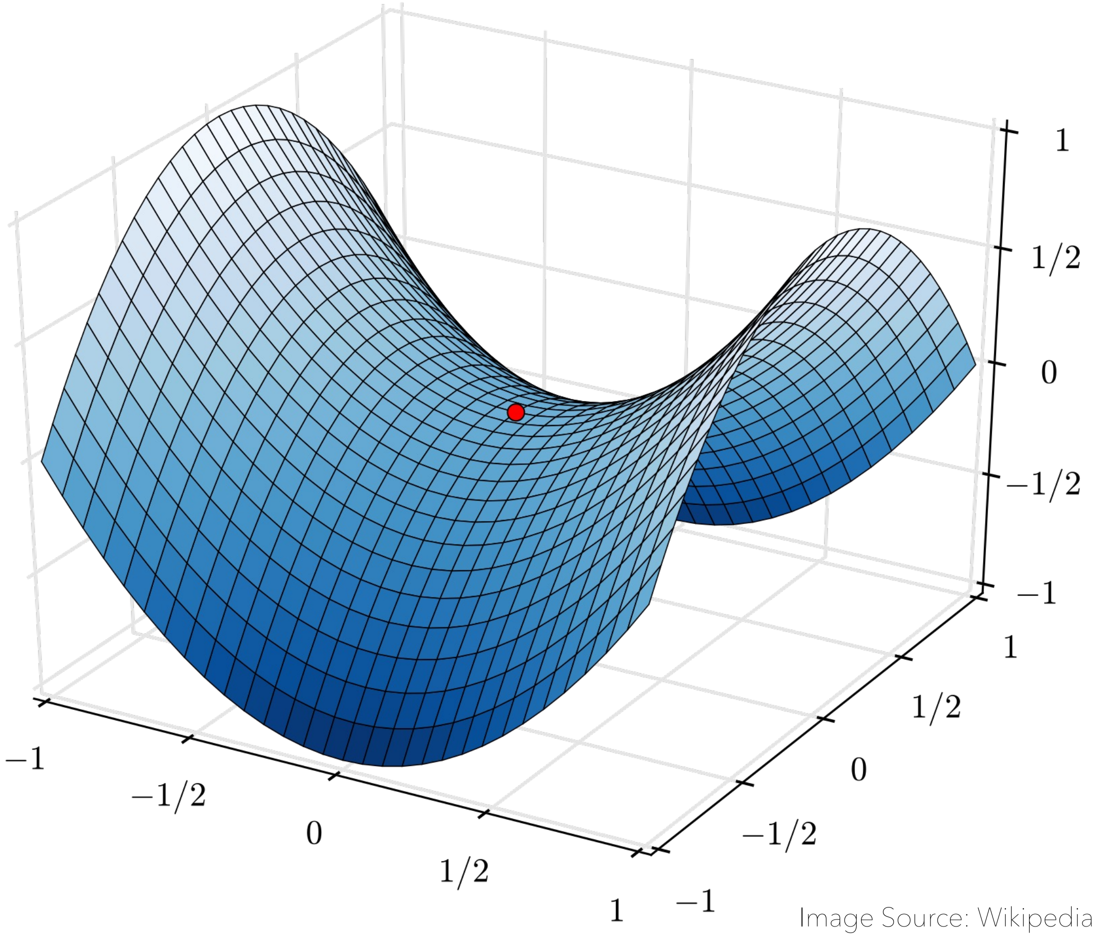


Image Source: Wikipedia

Why Study Quantum Games?

- General quantum games have emerged in many areas of quantum information:
  - Non-local games (Bell, CSHS, …, MIP*=RE)
  - Quantum interactive proofs (competitive refereed games)
  - Multi-prover quantum interactive proofs (cooperative games)
  - Quantum coin-flipping (two player game)

- However, optimization of general quantum games is **PPAD-complete** [BW22]

Why Study QZSG, Specifically?

- Meanwhile, as classically, QZSG optimization is **computationally tractable**
  - [JW09] proposed an explicit QZSG algo that converges to an ϵ-approx soln in $\mathcal{O}(1/\epsilon^2)$ iterations
- Uses of QZSG:
  - Game theory: proof that quantum strategies ≥ classical strategies [M99]
  - Complexity theory: proof that QRG(1) ⊆ PSPACE [JW09]
  - Machine learning: **Quantum Generative Adversarial Networks** [DK18]

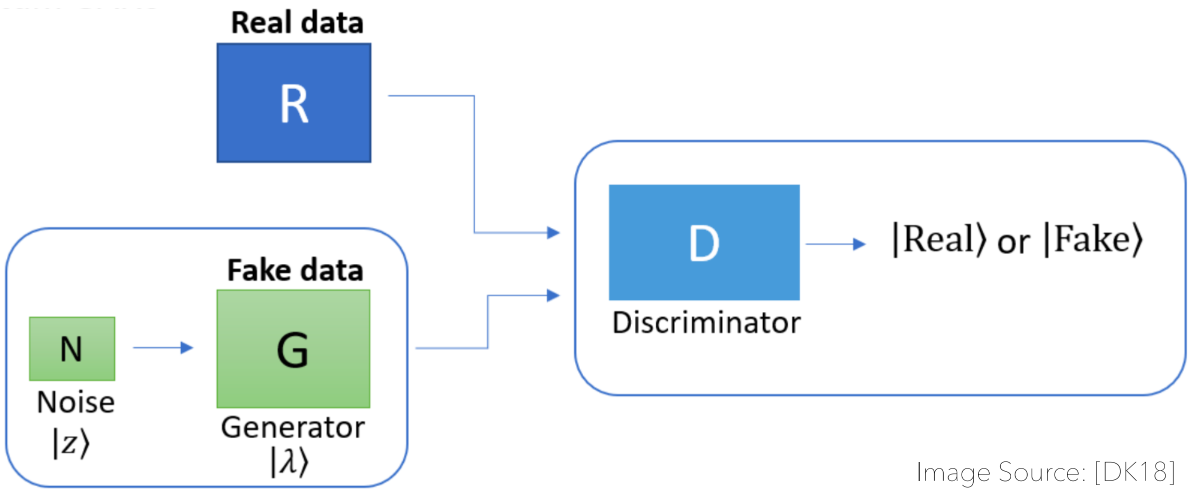


Image Source: [DK18]

A Quadratic Speedup in **Finding Nash Equilibria** of Quantum Zero-Sum Games

Nash Equilibria of QZSG

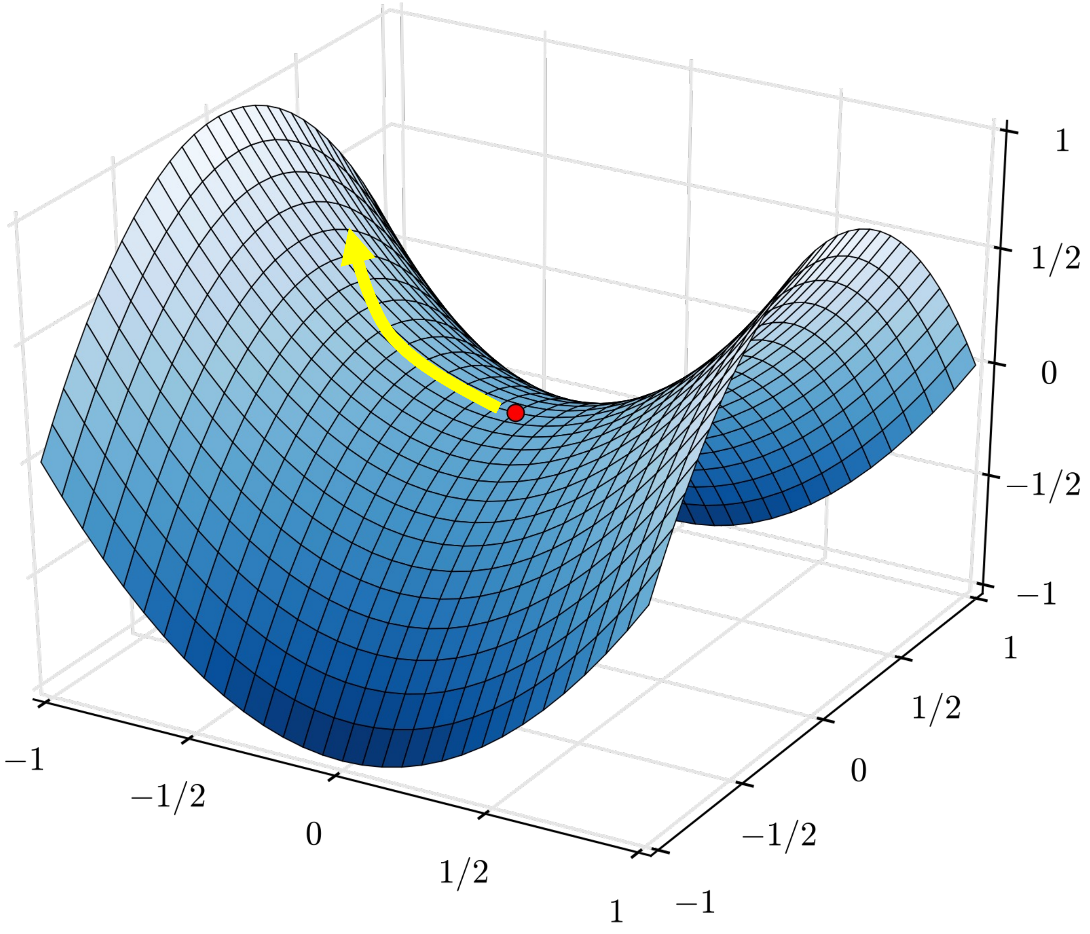
- The solutions (**fixed points**) of this minimax define the game's value:

$$u(\alpha^*, \beta^*) = \min_{\beta} \max_{\alpha} u(\alpha, \beta) = \max_{\alpha} \min_{\beta} u(\alpha, \beta)$$

von Neumann's Minimax Thm

- **Nash equilibria** are game states (α^*, β^*) such that neither player has an incentive to change to another state unilaterally:

$$u(\alpha^*, \beta^*) \geq u(\alpha, \beta^*), \qquad \forall \alpha \in \mathcal{A}$$
$$u(\alpha^*, \beta^*) \leq u(\alpha^*, \beta), \qquad \forall \beta \in \mathcal{B}$$



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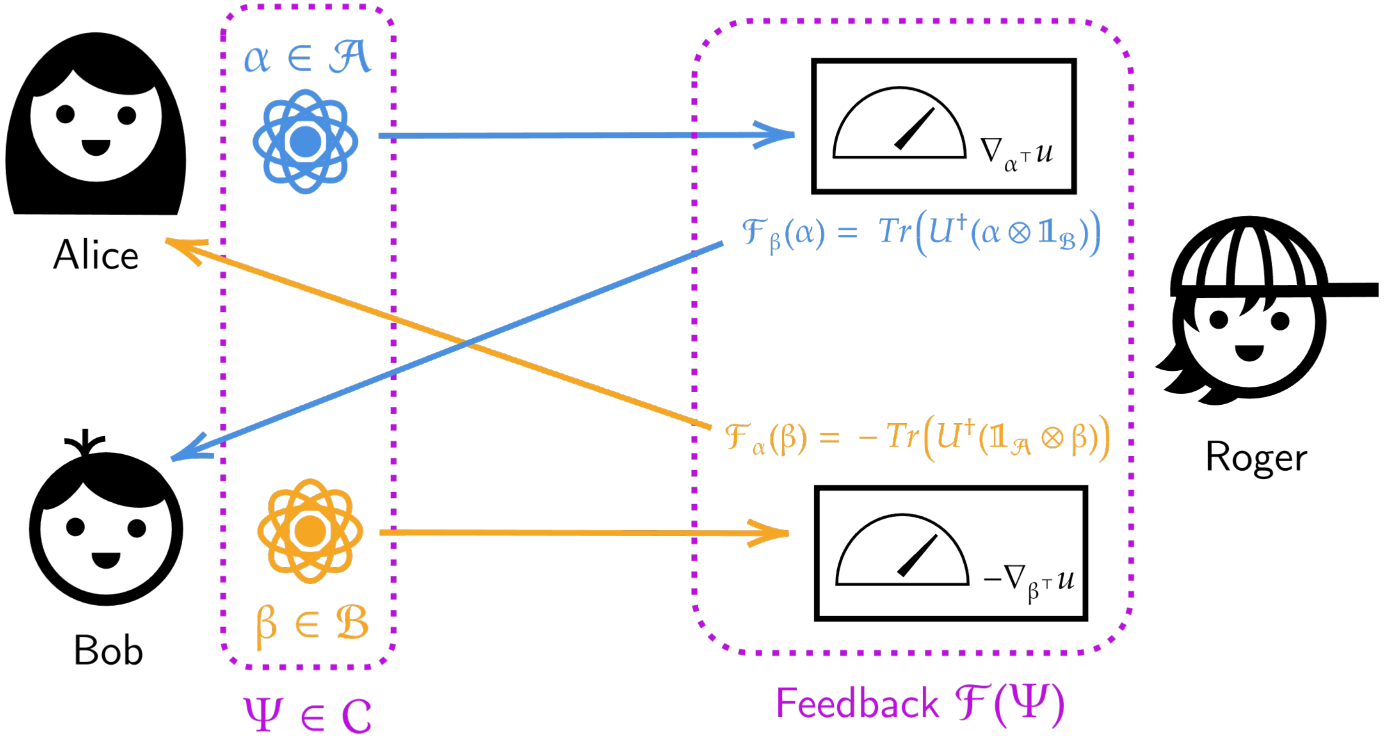
Algorithmic QZSG

- Algorithmically, we will view the game as an **online** learning problem
- In each round, each player **queries** the referee ("oracle") with a state
- The ref returns **feedback** to each player
- The players use this feedback to update and **improve** their states
- **Goal**: minimize the number of rounds until the players reach an ϵ-approx Nash equilibrium



Superoperator vs Gradient-Based Feedback

- Previous work on QZSG characterized the game's feedback via **superoperators**:
  - Alice's Feedback: $\Xi(\beta) = \mathrm{Tr}_{\mathcal{B}}[U(\mathbb{I}_{\mathcal{A}} \otimes \beta^{\mathrm{T}})]$
  - Bob's Feedback: $\Xi^*(\alpha) = \mathrm{Tr}_{\mathcal{A}}[U(\alpha^{\mathrm{T}} \otimes \mathbb{I}_{\mathcal{B}})]$
- We instead characterize the game's feedback via **gradient-based** operators:
  - Alice's Feedback: $\mathcal{F}_{\alpha}(\beta) = \nabla_{\alpha^{\mathrm{T}}} u(\alpha, \beta) = \mathrm{Tr}_{\mathcal{B}}[U(\mathbb{I}_{\mathcal{A}} \otimes \beta)]$
  - Bob's Feedback: $\mathcal{F}_{\beta}(\alpha) = -\nabla_{\beta^{\mathrm{T}}} u(\alpha, \beta) = -\mathrm{Tr}_{\mathcal{A}}[U(\mathbb{I}_{\mathcal{A}} \otimes \beta)]$
- The two characterizations are **equivalent**:
  - Feedback: $\mathcal{F}_{\alpha}(\beta) = \Xi(\beta^{\mathrm{T}})$, $\mathcal{F}_{\beta}(\alpha) = -\Xi^*(\alpha^{\mathrm{T}})$
  - Alice's expected payoff : $u(\alpha, \beta) = \mathrm{Tr}[\alpha\, \mathcal{F}_{\alpha}(\beta)] = \mathrm{Tr}[\alpha\, \Xi(\beta^{\mathrm{T}})]$
  - Bob's expected payoff: $-u(\alpha, \beta) = \mathrm{Tr}[\beta\, \mathcal{F}_{\beta}(\alpha)] = -\mathrm{Tr}[\beta\, \Xi^*(\alpha^{\mathrm{T}})]$

Formulating QZSG as a Variational Inequality

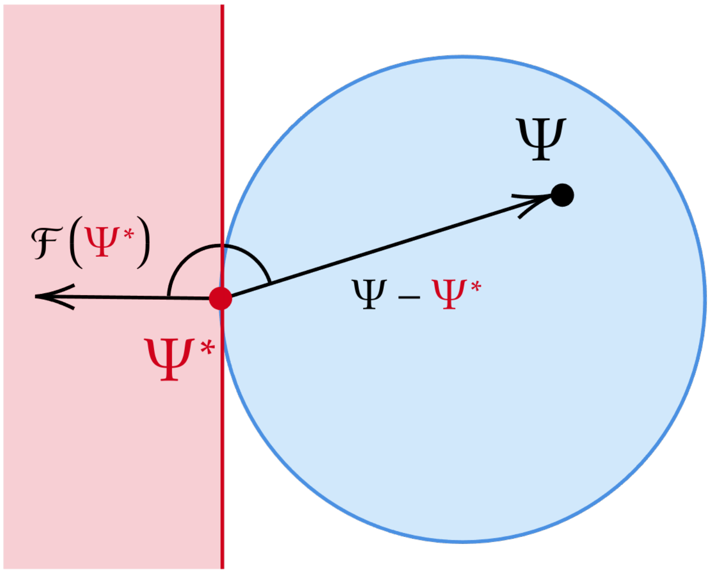
- With gradient-based feedback, the expected payoff is a **directional derivative**:
  - In Alice's direction: $u(\alpha,\beta) = \mathrm{Tr}[\alpha\,\mathcal{F}_\alpha(\beta)] = \mathrm{Tr}[\alpha\,\nabla_{\alpha^\mathsf{T}}\,u(\alpha,\beta)]$
  - In Bob's direction: $u(\alpha,\beta) = -\mathrm{Tr}[\beta\,\mathcal{F}_\beta(\alpha)] = \mathrm{Tr}[\beta\,\nabla_{\beta^\mathsf{T}}\,u(\alpha,\beta)]$

  $$u(\Psi) = \mathrm{Tr}[\Psi\,\mathcal{F}(\Psi)] = \mathrm{Tr}[\Psi\,\nabla_{\Psi^\mathsf{T}} u(\Psi)]$$

- With a directional derivative, we can characterize the game's equilibria as solutions of the **variational inequality (VI)**:

$$\mathrm{Tr}[(\Psi - \Psi^*)\,\mathcal{F}(\Psi^*)] \leq 0, \quad \forall\,\Psi \in \mathcal{A} \oplus \mathcal{B}$$

- We further prove that $\mathcal{F}(\Psi)$ is **monotone** and **Lipschitz**, which offers additional structure about the game that we can use to **leverage efficient classical algorithms** for solving such VIs.



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Gradient to Mirror Descent

- Classical **gradient descent (GD)**: $x_{t+1} = x_t - \eta \nabla F(x_t)$

- Equivalently, GD minimizes the 1st-order approx of F with **Euclidean regularizer** :

$$x_{t+1} = \underset{x}{\operatorname{argmin}} \left(F(x_t) + \nabla F(x_t)^T (x - x_t) + \frac{1}{2\eta} \|x - x_t\|^2 \right)$$

For $h(y) = \frac{1}{2}\|y\|^2$, $\nabla \mathrm{h}(\mathrm{y}) = \mathrm{y}$

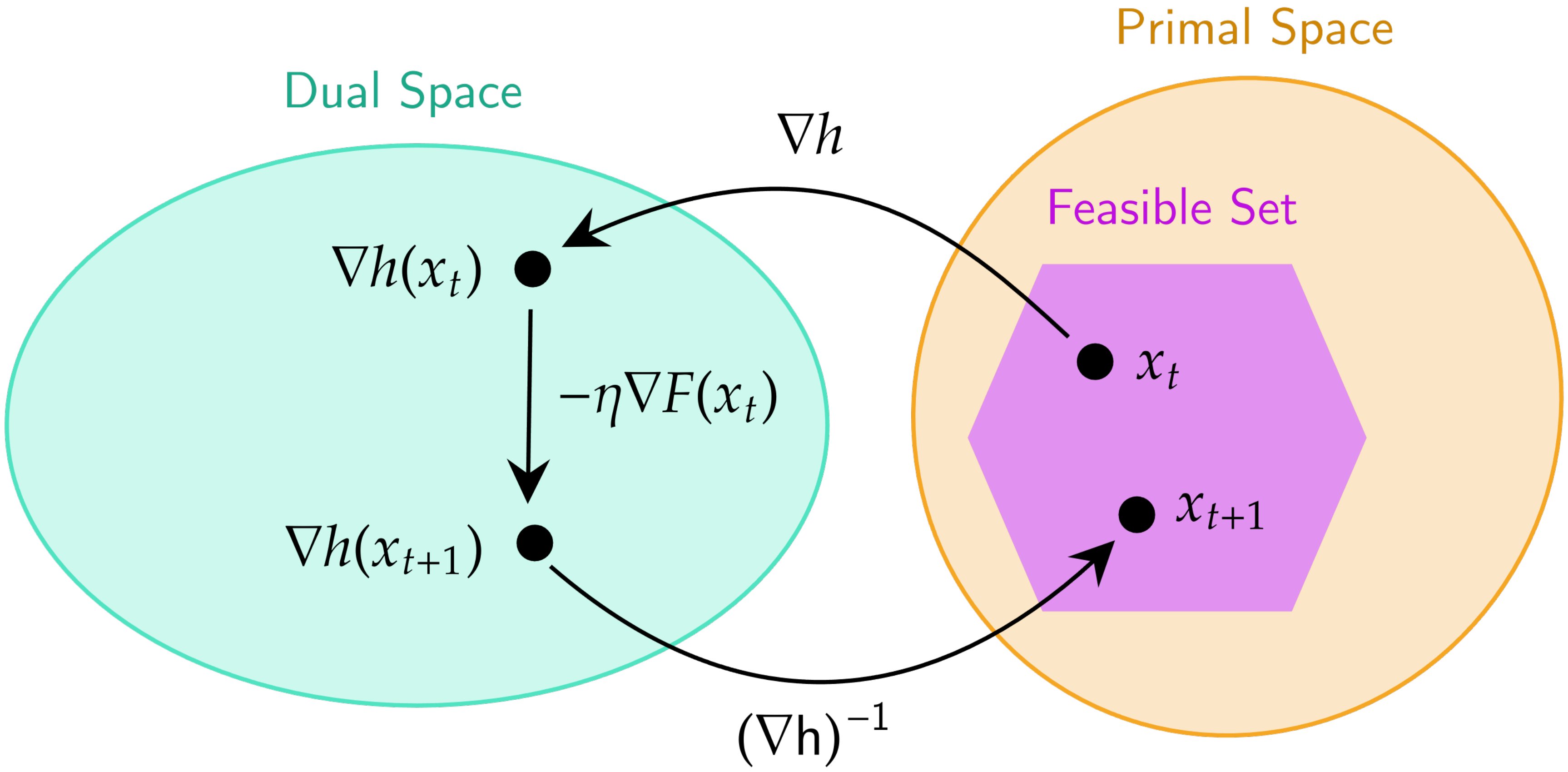
- To generalize GD to other regularizers h, perform **mirror descent (MD)**:

$$x_{t+1} = (\nabla h)^{-1}[\nabla h(x_t) - \eta \nabla F(x_t)]$$

Gradient to Mirror Descent

- To generalize GD to other regularizers h, perform **mirror descent (MD)**:

$$x_{t+1} = (\nabla h)^{-1}[\nabla h(x_t) - \eta \, \nabla F(x_t)\,]$$



Prior Work: The Jain-Watrous Algorithm [JW09]

- In 2009, Jain and Watrous proposed the **Matrix Multiplicative Weight Updates (MMWU)** algorithm, with the following update in each round t :

$$\alpha_t = \Lambda\left(\eta \sum_{i=0}^{t-1} \Xi(\beta_i^{\mathrm{T}})\right), \qquad \beta_t = \Lambda\left(-\eta \sum_{i=0}^{t-1} \Xi^*(\alpha_i^{\mathrm{T}})\right), \qquad \text{where } \Lambda(x) = \frac{\exp(x)}{\mathrm{Tr}(\exp(x))}$$

- We show MMWU is **"Lazy" Mirror Descent**, with a von Neumann **entropy** regularizer:

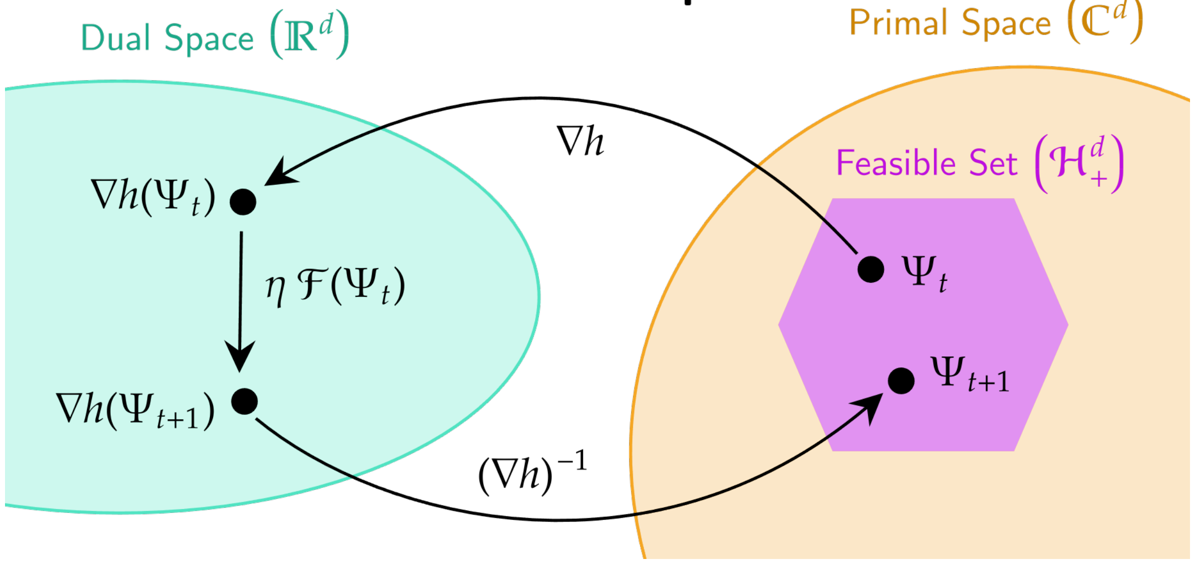
$$h(\psi) = Tr[\psi \log \psi]$$

- Like classical MWU, they prove an $\mathcal{O}(1/\epsilon^2)$ convergence.
  - However, in classical games, while this is optimal for classical **black-box** optimization, Nemirovski [N04] showed that $\mathcal{O}(1/\epsilon)$ can be achieved for monotone VIs.

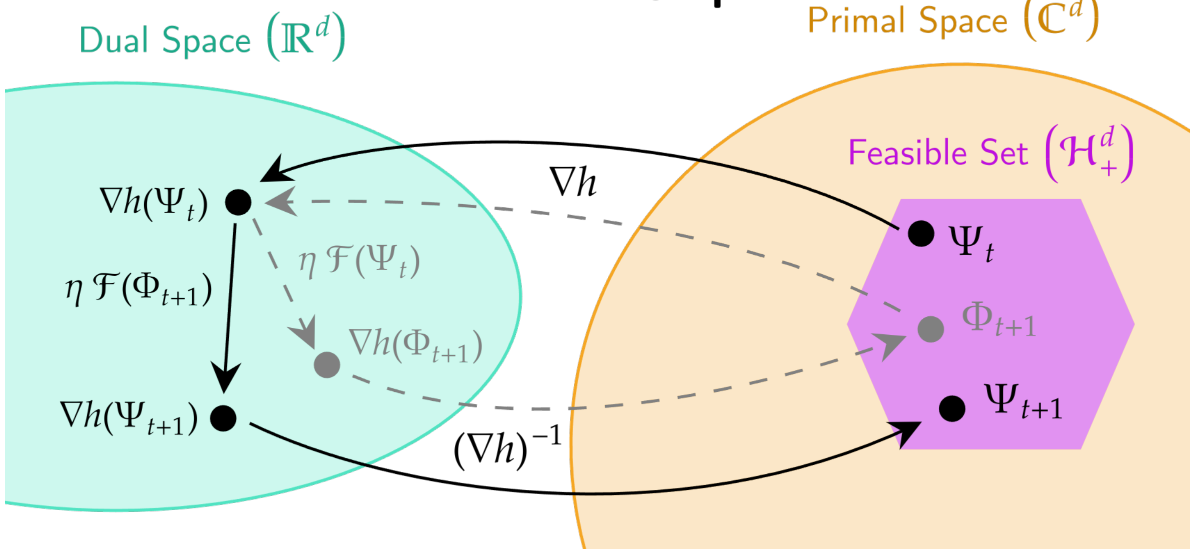
Achieving Speedup: Proximal Steps and Optimism

- To improve upon the MMWU, we leverage **proximal** instead of mirror steps
  - Proximal steps introduce "**momentum**"

Mirror Step



Proximal Step



- We further leverage **"optimism"** to reduce the total number of oracle calls from 2 to 1

Our Proposal: Optimistic Matrix Multiplicative Weight Updates

- We leverage proximal maps and optimism in our proposal of **Optimistic Matrix Multiplicative Weight Updates (OMMWU)** :

OMMWU

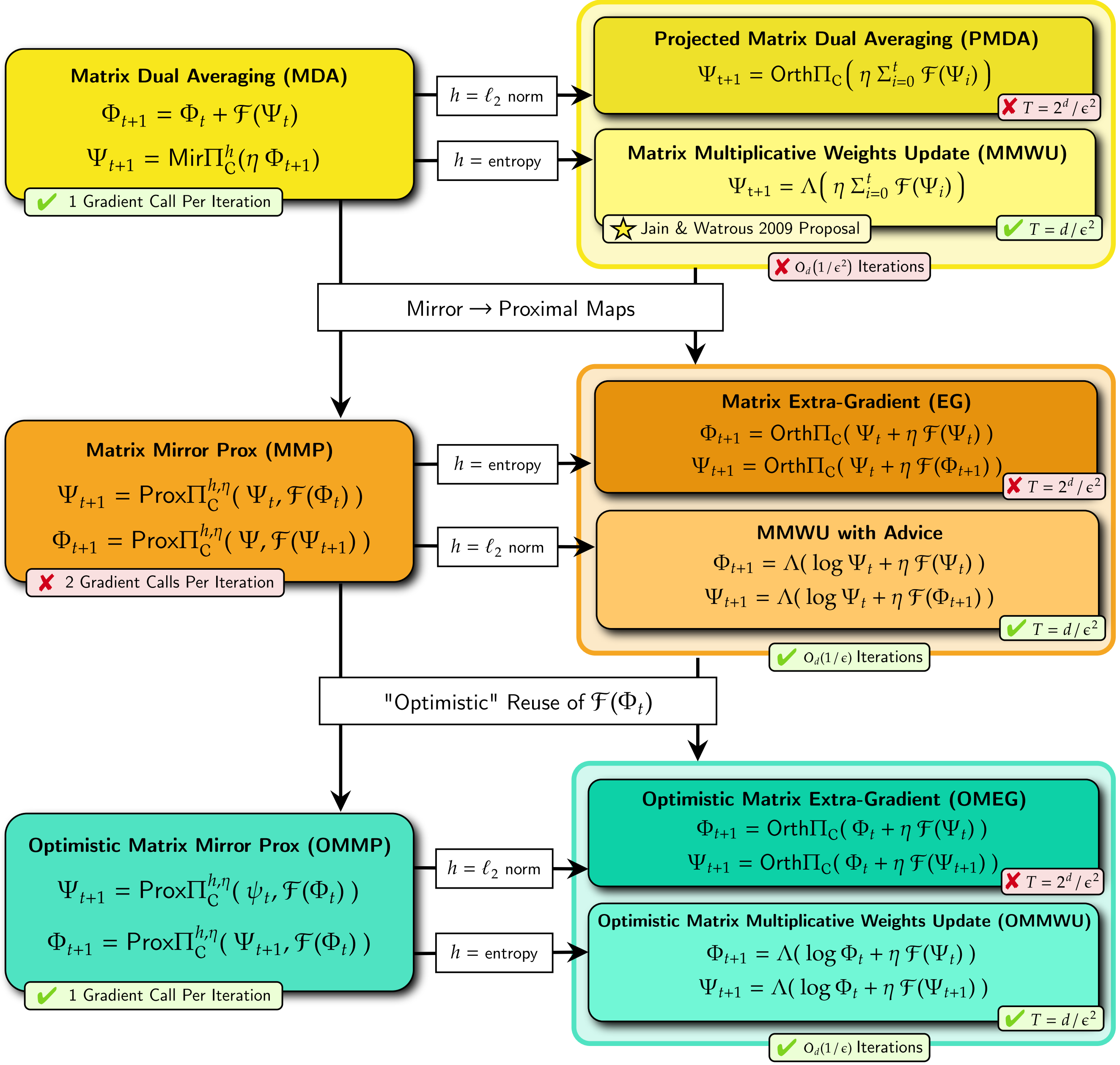
$$\Phi_{t+1} = \Lambda(\log \Phi_t + \eta\, \mathcal{F}(\Psi_t))$$
$$\Psi_{t+1} = \Lambda(\log \Phi_t + \eta\, \mathcal{F}(\Psi_{t+1}))$$

MMWU

$$\Psi_{t+1} = \Lambda\Big(\eta\, \Sigma_{i=0}^{t}\, \mathcal{F}(\Psi_i)\Big)$$

- Like classical OMWU, we prove an $\mathcal{O}(1/\epsilon)$ convergence.
  - The proof follows the proof structure of [EN20] for monotone VIs.
  - We leverage notions of strong convexity, smoothness, and Fenchel conjugacy.

QZSG Algorithm Design



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Thank you! Questions?

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