Quantum algorithms: What's quantum complexity theory got to do with it?

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Mindset

Quantum computers

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Quantum computers

vs.

Classical computers

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   - Relatively early in hardware development

With an eye on...

Computational Complexity Theory

- What resources (e.g. time, space) required to solve a computational problem?
- Complexity classes such as P, NP, BQP, QMA

Outline

Origins

Wiesner's quantum money (late 1970's): "Unforgeable" notion of money

- Security: Proven via semidefinite programming [MVW12]
- Adaptive attack: Scheme insecure if bank *returns* banknote after checking it! [BNSU14]

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Encapsulates many traits of quantum computing:

- Possible to do things which are impossible classically (due, e.g., to no-cloning theorem)
- Such feats do *not* require a universal quantum computer
- Beware the details hiding in quantum claims

The 1990's: The first "gamechangers"

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Lloyd's Hamiltonian simulation algorithm (1996)

- Efficiently simulates quantum systems governed by local Hamiltonians $H = \sum_i H_i \in \mathcal{L}(\mathbb{C}^2)^{\otimes n}$
- Introduced use of Trotterization/Lie Product Formula:

$$e^{H_1+H_2} = \lim_{n\to\infty} \left(e^{\frac{H_1}{n}} e^{\frac{H_2}{n}} \right)^n .$$

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  1. Use Hamiltonian simulation to simulate unitary $U = e^{iA}$
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- BQP-complete: Matrix inversion precisely captures the power of efficient quantum computation

The 2010's: A general quantum algorithms framework

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- Simulate Hamiltonian H for time t and error ϵ, i.e. unitary $U = e^{iHt}$ in time $O(t + \log(1/\epsilon))$

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  - Embed H in top-left block of some unitary U
  - Use "qubitization" to map $H \mapsto p(H)$ for some appropriate polynomial p such that $p(H) \approx e^{iHt}$
  - Use "post-selection" to probabilistically map

$$U|\psi\rangle \mapsto p(H)|\psi\rangle \approx e^{iHt}|\psi\rangle.$$

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Quantum Singular Value Transformation (Gilyén, Su, Low, and Wiebe 2019)

- Generalizes Low and Chuang's qubitization approach to *non-square* matrices A
- Given non-square A (embedded as block of unitary U), polynomial p, simulates mapping

$$|\psi\rangle \mapsto p\left(\sqrt{A^\dagger A}\right)|\psi\rangle.$$

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  - [Martyn, Rossi, Tan and Chuang 2021] *iteratively* apply QSVT to simulate Fourier Transform

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Which computing model?

Universal models:

- Quantum Turing Machines
- Quantum circuits
- Quantum adiabatic computing
- One-way measurement based computing
- Quantum walks
- Quantum Approximate Optimization Algorithm (QAOA)

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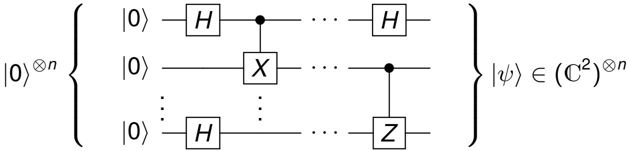
Here: Work with $\mathrm{poly}(n)$-size quantum circuit implementing n-qubit unitaries U, e.g.



What counts as an efficient quantum algorithm?

Bounded-error quantum polynomial-time (BQP)

Promise problem $\mathbb{A} = (A_{\mathrm{yes}}, A_{\mathrm{no}}) \in \mathrm{BQP}$ if $\exists$ P-uniform quantum circuit family $\{Q_n\}$ and polynomial q as below. The first output qubit of Q_n is measured in the standard basis and returned. For any input $x \in \{0,1\}^*$:

- (YES case) If $x \in A_{\mathrm{yes}}$, then Q_n outputs 1 with probability at least $2/3$.
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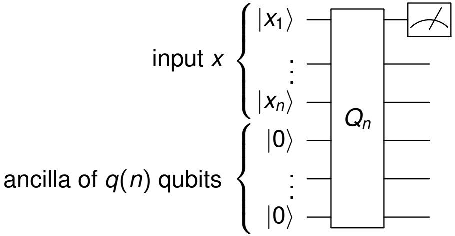

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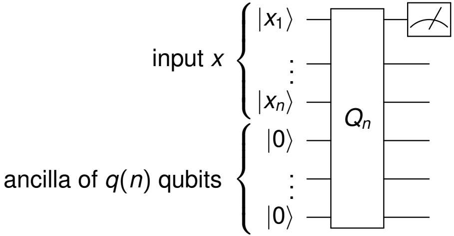

Exercise: Why do we require a P-uniform quantum circuit family?

Assumption

All quantum operations are noise-free, i.e. perfect

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The Pikachu of BQP

Linear system solving:

- Input: Invertible $A \in \mathbb{C}^{N\times N}$ and target vector $\mathbf{b} \in \mathbb{C}^N$
- Output: $\mathbf{x} \in \mathbb{C}^N$ such that $A\mathbf{x} = \mathbf{b}$.

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- If A and $\mathbf{x}$ given explicitly in matrix form $\Rightarrow \mathbf{x} = A^{-1}\mathbf{b}$ classically in time $\mathrm{poly}(N)$
- If A represented "succinctly" via query-access and $\mathbf{b}$ given via quantum circuit?

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Input: $O(1)$-sparse row-computable invertible Hermitian matrix $A \in \mathbb{C}^{N\times N}$.

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- MI is BQP-hard, i.e. BQP computation can be reduced to an instance of MI.

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Hamiltonian simulation [Low, Chuang 2017]

Given d-sparse H, simulation time $t \geq 0$, and $\epsilon > 0$, can simulate e^{iHt} up to error ϵ and success probability at least $1 - 2\epsilon$ in time[a]

$$O\left(td\,\|H\|_{\max} + \frac{\log(1/\epsilon)}{\log\log(1/\epsilon)}\right).$$

[a]Query complexity. Gate complexity has $O(n)$ overhead.

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Exercise: Given n-qubit unitary U, can we efficiently compute U^{2^n} in general?

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$$|b\rangle = \sum_{j=1}^{N} \alpha_j |\psi_j\rangle \in \mathbb{C}^N,$$

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- Apply QPE to unitary e^{iA} with an n-qubit ancilla:
$$\sum_{j=1}^{N} \alpha_j |0^n\rangle |\psi_j\rangle \quad \mapsto \quad \sum_{j=1}^{N} \alpha_j |\lambda_j\rangle |\psi_j\rangle \in (\mathbb{C}^2)^{\otimes n} \otimes \mathbb{C}^N.$$

Step 2: Eigenvalue processing

- Conditioned on the first register, rotate a new single-qubit ancilla as follows:

$$\sum_{j=1}^{N}\alpha_j|\lambda_j\rangle|\psi_j\rangle|0\rangle \quad\mapsto\quad \sum_{j=1}^{N}\alpha_j|\lambda_j\rangle|\psi_j\rangle\left(\sqrt{1-\frac{1}{\lambda_j^2\kappa^2(A)}}|0\rangle+\left(\frac{1}{\lambda_j\kappa(A)}\right)|1\rangle\right)\in(\mathbb{C}^2)^{\otimes n}\otimes\mathbb{C}^N\otimes\mathbb{C}^2.$$

Key parameter: Condition number $\kappa(A) := \|A^-1\|_\infty\|A\|_\infty$.

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Key parameter: Condition number $\kappa(A) := \|A^-1\|_\infty \|A\|_\infty$.

Exercise:
Assume $\|A\|_\infty = 1$. Show

$$\frac{1}{\kappa(A)} \leq \frac{1}{\lambda_j \kappa(A)} \leq 1.$$

Thus, amplitudes above well-defined.

Step 3: Eigenvalue reinsertion

- Uncompute eigenvalues via inverse QPE:

$$\begin{aligned} &\sum_{j=1}^{N}\alpha_j{\color{red}|\lambda_j\rangle}|\psi_j\rangle\left(\sqrt{1-\frac{1}{\lambda_j^2\kappa^2(A)}}|0\rangle+\left(\frac{1}{\lambda_j\kappa(A)}\right)|1\rangle\right)\\ \mapsto\quad &\sum_{j=1}^{N}\alpha_j{\color{red}|0\rangle}|\psi_j\rangle\left(\sqrt{1-\frac{1}{\lambda_j^2\kappa^2(A)}}|0\rangle+\left(\frac{1}{\lambda_j\kappa(A)}\right)|1\rangle\right). \end{aligned}$$

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- Measure third register in standard basis, postselect on outcome 1, discard third register:

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Exercise. Prove that probability of obtaining outcome 1 is at least $1/\kappa^2(A)$.

Runtime

To compute unit vector proportional to $A^{-1}|b\rangle$ within error ϵ:

$$\widetilde{O}(\log(N)s^2\kappa^2(A)/\epsilon) \quad \text{where}$$

- N the dimension of A,
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Implication:

- When $\kappa(A), s \in \text{polylog}(N)$, exponentially faster than classically solving $N \times N$ system.
- But this solves a different problem than classical linear systems solvers!

Outline

Matrix inversion problem (MI)

Input: $O(1)$-sparse row-computable invertible Hermitian matrix $A \in \mathbb{C}^{N\times N}$

- $O(1)$-sparse: At most $O(1)$ non-zero entries per row.
- Row-computable: $\exists$ polylog(N)-time classical algorithm which, given $r \in [N]$, outputs entries of row r.

Output: Let $|x\rangle \propto A^{-1}|0^N\rangle$ be a unit vector, and $\Pi = |1\rangle\langle 1|$ a projector onto the first qubit of $|x\rangle$. Then:

- If $\langle x|\Pi|x\rangle \geq 2/3$, output YES.
- If $\langle x|\Pi|x\rangle \leq 1/3$, output NO.

Theorem [Harrow, Hassidim, Lloyd, 2008]

MI is BQP-complete under poly-time many-one reduction, i.e.:

- MI is in BQP, i.e. can be efficiently solved in polylog(N) time on a quantum computer,
- MI is BQP-hard, i.e. BQP computation can be reduced to an instance of MI.

MI is BQP-hard

Goal: Show that any BQP computation V poly-time reducible to an instance A of MI.

Arbitrary BQP circuit

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Moral: If you can solve MI, you can simulate any BQP circuit

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Starting point: Let $V = V_m \cdots V_1$ be a BQP circuit on n qubits, $N = 2^n$. Assume WLOG m is power of 2.

Problem: Need to tie matrix inverse with action of V.

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Exercise: Check that U is unitary.

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Exercise: Check that $U^m|0^{\log m}\rangle|0^n\rangle = |m\rangle V|0^n\rangle$.

Implication: Measuring first qubit of second register of $U^m|0^{\log m}\rangle|0^n\rangle$ simulates measuring output qubit of V!

- We could apply this to any normal matrix U with $\|U\|_\infty < 1$ to get $(I - U)^{-1} = \sum_{l=0} U^l$.
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Exercise: I cheated less slightly somewhere else on this slide. Where did I make a bigger boo boo?

Final exercises for MI

Construction *almost* works, but for 3 issues to check:

1. A must be $O(1)$-sparse (by def of MI).

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4. I cheated again. There is a 4th issue — A must be Hermitian. But I will spare you these details.

Outline

Open season

Observation: For linear systems, given as input Hermitian A, we:

- Showed how to simulate A^{-1} by "manually" inverting eigenvalues, i.e. $\mathbb{A}^{-1} = \sum_i \frac{1}{\lambda_i} |\psi_i\rangle\langle\psi_i|$.
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- A not unitary $\rightarrow$ post-selection still needed.

Remember this?

Quantum Singular Value Transformation (Gilyén, Su, Low, and Wiebe 2019)

- Generalizes Low and Chuang's qubitization approach to *non-square* matrices A
- Given non-square A (embedded as block of unitary U), polynomial p, simulates mapping

$$|\psi\rangle \mapsto p\left(\sqrt{A^\dagger A}\right)|\psi\rangle.$$

- Unified framework for a host of quantum algorithms:
  - Hamiltonian simulation, linear systems, amplitude amplification, quantum machine learning algorithms, and essentially all "quantum matrix linear algebra"
  - [Martyn, Rossi, Tan and Chuang 2021] *iteratively* apply QSVT to simulate Fourier Transform

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Solutions:

1. Use block encodings of A (more generally, projected unitary encodings).
2. Use Quantum Signal Processing (QSP), i.e. qubitization.

Step 1: Block encodings

Recall key step of HHL algorithm ($A = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$:

$$\sum_{j=1}^{N} \alpha_j |0\rangle_R |\lambda_j\rangle|\psi_j\rangle \quad \mapsto \quad \sum_{j=1}^{N} \alpha_j \left(\sqrt{1 - \frac{1}{\lambda_j^2 \kappa^2(A)}} |0\rangle + \left(\frac{1}{\lambda_j \kappa(A)} \right) |1\rangle \right)_R |\lambda_j\rangle|\psi_j\rangle.$$

Postselecting on $|1\rangle$ in register R simulated application of A^{-1}, i.e. eigenvector $|\psi_j\rangle$ hit with coefficient λ_j^{-1}.

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Postselecting on $|1\rangle$ in register R simulated application of A^{-1}, i.e. eigenvector $|\psi_j\rangle$ hit with coefficient λ_j^{-1}.

Effective unitary HHL implements (before measuring R):

$$U = \begin{pmatrix} A^{-1} & ? \\ ? & ? \end{pmatrix} = |0\rangle\langle 0|_R \otimes A^{-1} + |0\rangle\langle 1|_R \otimes ? + |1\rangle\langle 0|_R \otimes ? + |1\rangle\langle 1|_R \otimes ?$$

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1. Prepare initial state: $|0\rangle_R |b\rangle$
2. Use QPE to simulate $U|0\rangle_R|b\rangle$.
3. Measure R and postselect on outcome 0:

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Block encoding

A *block encoding* of matrix A on n qubits is any unitary U s.t.

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More generally:

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A *projected unitary encoding* of matrix A on n qubits is (Π_L, U, Π_R) s.t.

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Exercise. What are Π_L and Π_R in case of block encoding?

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Given polynomial p, how to obtain projected unitary encoding of $p(A)$?

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Tool: Quantum Signal Processing (QSP), in two steps:

1. QSP on single qubit system
2. Embed into QSP on larger systems

QSP on single qubit system

- Consider following block encoding of 1×1 Hermitian matrix $[x]$ with $x \in [-1, 1]$:

$$R(x) := \begin{pmatrix} x & \sqrt{1-x^2} \\ \sqrt{1-x^2} & -x \end{pmatrix} \in \mathcal{L}(\mathbb{C}^2).$$

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- Question:
  - Suppose we can apply R, $R^\dagger$, and $e^{i\theta Z}$ for Pauli Z and any $\theta \in [0, 2\pi]$.
  - For which polynomials p can the mapping above be done, and what is the cost?

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Alternatively, for all $x \in \mathbb{R}$, $p(-x) = -p(x)$.

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Exercise: Why do we prefer *low*-degree polynomials?

On to the general case: A acting on n qubits

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Question: Thus far, mapped real $x \in [-1, 1]$ to $p(x)$. What is high-dimensional analogue of this?

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Any matrix $A \in \mathcal{L}(\mathbb{C}^d)$ has singular value decomposition

$$A = \sum_{i=1}^{d} s_i |l_i\rangle\langle r_i|, \text{ for}$$

- $s_i \geq 0$ are *singular values*,
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Exercise. Why does this generalize our single-qubit setup? (i.e. previously we had $x \in [-1, 1]$)

QSVT by alternating phase modulation (Gilyén, Su, Low, and Wiebe 2018)

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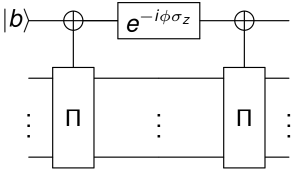
- $O(m)$ uses of following circuit which implements map $|b\rangle\langle b| \otimes e^{(-1)^b i\phi(2\Pi - I)}$:



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Exercise: Aren't you forgetting to ask a very important question? (Hint: The sequence $(\phi_1, \ldots, \phi_d)$.)

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Key insight: Can decompose all operators into direct sum of 1- and 2-dimensional subspaces.

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Recall: Given $A = \Pi_L U \Pi_R = \sum_{i=1}^{d} s_i |l_i\rangle\langle r_i|$. Order singular values:

$$\underbrace{\underbrace{s_1 \geq s_2 \geq \cdots \geq s_k}_{=1} \geq \underbrace{s_{k+1} \geq \cdots \geq s_r}_{0<\cdot<1}}_{r=\mathrm{rank}(A)} \geq \underbrace{s_{r+1} \geq \cdots \geq s_d}_{=0}$$

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Theorem (Invariant subspaces)

$$U = \bigoplus_{i\in[k]} [s_i]_{\tilde{\mathcal{H}}_i}^{\mathcal{H}_i} \oplus \bigoplus_{i\in[r]\setminus[k]} \begin{bmatrix} s_i & \sqrt{1-s_i^2} \\ \sqrt{1-s_i^2} & -s_i \end{bmatrix}_{\tilde{\mathcal{H}}_i}^{\mathcal{H}_i} \oplus \bigoplus_{i\in[d]\setminus[r]} [1]_{\tilde{\mathcal{H}}_i^R}^{\mathcal{H}_i^R} \oplus \bigoplus_{i\in[d]\setminus[r]} [1]_{\tilde{\mathcal{H}}_i^L}^{\mathcal{H}_i^L} \oplus [\cdot]_{\tilde{\mathcal{H}}_\perp}^{\mathcal{H}_\perp}$$

$$e^{i\phi(2\Pi - I)} = \bigoplus_{i\in[k]} \left[e^{i\phi}\right]_{\mathcal{H}_i}^{\mathcal{H}_i} \oplus \bigoplus_{i\in[r]\setminus[k]} \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}_{\mathcal{H}_i}^{\mathcal{H}_i} \oplus \bigoplus_{i\in[d]\setminus[r]} \left[e^{i\phi}\right]_{\mathcal{H}_i^R}^{\mathcal{H}_i^R} \oplus \bigoplus_{i\in[d]\setminus[r]} \left[e^{-i\phi}\right]_{\mathcal{H}_i^L}^{\mathcal{H}_i^L} \oplus [\cdot]_{\mathcal{H}_\perp}^{\mathcal{H}_\perp},$$

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Recall: k largest index with $s_k = 1$, r largest index with $s_r > 0$ (i.e. $r = \mathrm{rank}(A)$ for $A = \Pi_L U \Pi_R$).

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$$1 \leq i \leq k \qquad \mathcal{H}_i := \operatorname{Span}(|r_i\rangle) \qquad \tilde{\mathcal{H}}_i := \operatorname{Span}(|l_i\rangle),$$

$$k < i \leq r \qquad \mathcal{H}_i := \operatorname{Span}\left(|r_i\rangle, |r_i^\perp\rangle\right) \qquad |r_i^\perp\rangle := \frac{(I-\Pi)U^\dagger|l_i\rangle}{\|(I-\Pi)U^\dagger|l_i\rangle\|} = \frac{(I-\Pi)U^\dagger|l_i\rangle}{\sqrt{1-s_i^2}},$$

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Theorem (Invariant subspaces)

$$U = \bigoplus_{i\in[k]} [s_i]^{\mathcal{H}_i}_{\tilde{\mathcal{H}}_i} \oplus \bigoplus_{i\in[r]\setminus[k]} \begin{bmatrix} s_i & \sqrt{1-s_i^2} \\ \sqrt{1-s_i^2} & -s_i \end{bmatrix}^{\mathcal{H}_i}_{\tilde{\mathcal{H}}_i} \oplus \bigoplus_{i\in[d]\setminus[r]} [1]^{\mathcal{H}_i^R}_{\tilde{\mathcal{H}}_i^R} \oplus \bigoplus_{i\in[d]\setminus[r]} [1]^{\mathcal{H}_i^L}_{\tilde{\mathcal{H}}_i^L} \oplus [\cdot]^{\mathcal{H}_\perp}_{\tilde{\mathcal{H}}_\perp}$$

$$e^{i\phi(2\Pi - I)} = \bigoplus_{i\in[k]} \left[e^{i\phi}\right]^{\mathcal{H}_i}_{\mathcal{H}_i} \oplus \bigoplus_{i\in[r]\setminus[k]} \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}^{\mathcal{H}_i}_{\mathcal{H}_i} \oplus \bigoplus_{i\in[d]\setminus[r]} \left[e^{i\phi}\right]^{\mathcal{H}_i^R}_{\mathcal{H}_i^R} \oplus \bigoplus_{i\in[d]\setminus[r]} \left[e^{-i\phi}\right]^{\mathcal{H}_i^L}_{\mathcal{H}_i^L} \oplus [\cdot]^{\mathcal{H}_\perp}_{\mathcal{H}_\perp},$$

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Alternating these two yields:

$$U_\Phi = \bigoplus_{i\in[k]} [P(s_i)]_{\tilde{\mathcal{H}}_i}^{\mathcal{H}_i} \oplus \bigoplus_{i\in[r]\setminus[k]} \begin{bmatrix} P(s_i) & \cdot \\ \cdot & \cdot \end{bmatrix}_{\tilde{\mathcal{H}}_i}^{\mathcal{H}_i} \oplus \bigoplus_{i\in[d]\setminus[r]} \left[e^{i\phi_0}\right]_{\tilde{\mathcal{H}}_i^R}^{\mathcal{H}_i^R} \oplus \bigoplus_{i\in[d]\setminus[r]} \left[e^{-i\phi_0}\right]_{\tilde{\mathcal{H}}_i^L}^{\mathcal{H}_i^L} \oplus [\cdot]_{\tilde{\mathcal{H}}_\perp}^{\mathcal{H}_\perp}.$$

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QSVT by alternating phase modulation

Consider projected unitary encoding $A = \Pi_L U \Pi_R$, and let $p \in \mathbb{C}[x]$ be odd polynomial of degree d, s.t.

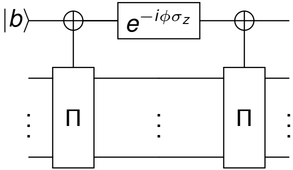
1. for all $x \in [-1, 1]$, $|p(x)| \leq 1$, and
2. for all $x \in [-\infty, -1] \cup [1, \infty]$, $|p(x)| \geq 1$.

There exists sequence of d angles $\Phi := (\phi_1, \ldots, \phi_d) \in \mathbb{R}^d$ s.t.

$$p(A) = \Pi_L U_\Phi \Pi_R = \Pi_L \left(e^{i\phi_1(2\Pi_L - I)} U e^{i\phi_2(2\Pi_R - I)} U^\dagger \cdots e^{i\phi_d(2\Pi_L - I)} U \right) \Pi_R.$$

What is the cost of implementing U_Φ?

- $O(m)$ uses of U and $U^\dagger$
- $O(m)$ uses of following circuit which implements map $|b\rangle\langle b| \otimes e^{(-1)^b i\phi(2\Pi - I)}$:



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$$A^+ := \sum_{i=1}^{d} \frac{1}{s_i} |r_i\rangle\langle l_i| \quad \text{(for clarity, we invert only } s_i > 0\text{).}$$

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   - More generally, cost is $O(\|A\|_{\mathrm{F}} / \delta)$ (i.e. if we don't assume $\|A\| \leq 1$).

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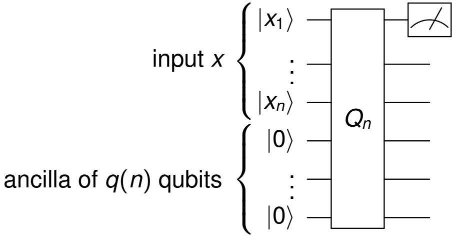
Goal:

- Given BQP circuit U which outputs 1 with probability $\geq p$.
- Compile new circuit U' which outputs 1 with probability ≈ 1.



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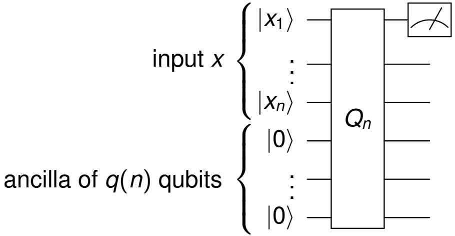

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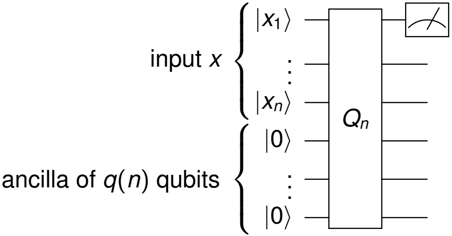

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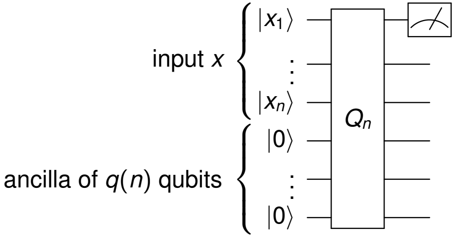

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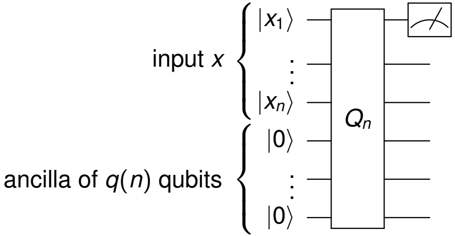

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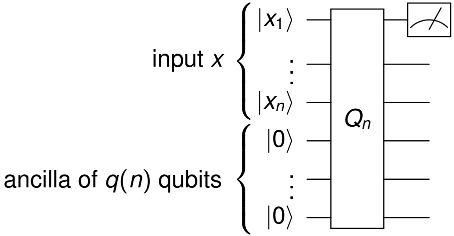

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- Suffices to choose p of degree $O\left(\frac{1}{\sqrt{p}} \log(\frac{1}{\epsilon})\right)$.

Outline

The story of dequantization

Quantum recommendation systems (Kerenidis, Prakash, 2016)

- Recommendation system (used by, e.g., Netflix):
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A quantum-inspired classical algorithm for recommendation systems (Tang, STOC 2019):

We give a classical analogue to Kerenidis and Prakash's quantum recommendation system, previously believed to be one of the strongest candidates for provably exponential speedups in quantum machine learning. Our main result is an algorithm that, given an $m \times n$ matrix in a data structure supporting certain ℓ^2-norm sampling operations, outputs an ℓ^2-norm sample from a rank-k approximation of that matrix in time $O(\mathrm{poly}(k)\log(mn))$, only polynomially slower than the quantum algorithm. As a consequence, Kerenidis and Prakash's algorithm does not in fact give an exponential speedup over classical algorithms. Further, under strong input assumptions, the classical recommendation system resulting from our algorithm produces recommendations exponentially faster than previous classical systems, which run in time linear in m and n.

The main insight of this work is the use of simple routines to manipulate ℓ^2-norm sampling distributions, which play the role of quantum superpositions in the classical setting. This correspondence indicates a potentially fruitful framework for formally comparing quantum machine learning algorithms to classical machine learning algorithms.

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Guided local Hamiltonian problem (GLH) [G, Le Gall 2022]

- Input: sparse Hamiltonian H on n qubits, $\alpha < \beta$, samplable $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$
- Promise: $\lambda_{\min}(H) \leq \alpha$ or $\lambda_{\min}(H) \geq \beta$, $\| \Pi_H |\psi\rangle \|_2 \geq \delta$
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ζ-samplable state for $\zeta \in [0, 1)$

Have ζ-sampling-access to $|\psi\rangle \in \mathbb{C}^{2^n}$ if all three hold:

- (query access) For any $i \in [2^n]$, can compute $\psi_i \in \mathbb{C}$ in $\mathrm{poly}(n)$ classical time
- (sampling access) Can sample in $\mathrm{poly}(n)$ classical time from distribution $p\colon [2^n] \to [0, 1]$ such that

$$\forall j \in [2^n] \qquad p(j) \in \left[(1-\zeta)\frac{|\psi_j|^2}{\| |\psi\rangle \|^2}, (1+\zeta)\frac{|\psi_j|^2}{\| |\psi\rangle \|^2} \right]$$

- (norm approximation) Have m s.t. $|m - \| |\psi\rangle \| \,| \leq \zeta \, \| |\psi\rangle \|$.

Note: When $\zeta = 0$, recover [Tang 2019]'s definition from dequantization of recommender systems

$n = \#$ of qubits

Theorem: GLH "tractable" in $O(1)$-precision setting

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$$\Uparrow$$

Theorem (informal)

The sparse "Guided Singular Value Estimation" problem is efficiently solvable to $O(1)$ precision.

$$\Uparrow$$

choose constant-degree polynomial P in QSVT to "process" singular values

$\rightarrow$ possible in $O(1)$-precision setting

Theorem (informal)

The sparse Quantum Singular Value Transform (QSVT) can be "dequantized" for $O(1)$ precision.

Dequantizing the QSVT in the sparse setting

Singular Value Transform (SVT)

Input: (1) query-access to s-sparse matrix $A \in \mathbb{C}^{M \times N}$ with $\|A\| \leq 1$
(2) query-access to $u \in \mathbb{C}^N$ s.t. $\|u\| \leq 1$
(3) ζ-samplable $v \in \mathbb{C}^N$ s.t. $\|v\| \leq 1$
(4) even polynomial $P \in \mathbb{R}[x]$ of degree d (even $\implies$ for all $x \in \mathbb{R}$, $P(x) = P(-x)$)

Output: estimate $\hat{z} \in \mathbb{C}$ s.t. $|\hat{z} - v^\dagger P(\sqrt{A^\dagger A})u| \leq \epsilon$

Lemma: Dequantizing SVT

$\forall \epsilon \in (0, 1]$ and $\zeta \leq \epsilon/8$, SVT solvable classically with probability $1 - 1/\operatorname{poly}(N)$ in $O^*((s^{2d+1})/\epsilon^2)$ time.

Proof sketch for dequantizing SVT

SVT(s, ϵ, ζ) (singular value transform)

Input: (1) query-access to s-sparse matrix $A \in \mathbb{C}^{M\times N}$ with $\|A\| \leq 1$
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Output: estimate $\hat{z} \in \mathbb{C}$ s.t. $|\hat{z} - v^\dagger P(\sqrt{A^\dagger A})u| \leq \epsilon$

Proof sketch.

Idea (à la [Tang 2019]): Compute r random entries of $\langle v, P(\sqrt{A^\dagger A})u\rangle$, take arithmetic mean:

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1. Set avg $= 0$
2. Repeat $r \in \Theta(1/\epsilon^2)$ times:

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   - Via query access, compute entry v_j

Proof sketch for dequantizing SVT

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   - Via s-sparsity of A, compute entry j of $w := P(\sqrt{A^\dagger A})u$ (do this recursively)

Proof sketch for dequantizing SVT

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   - Via s-sparsity of A, compute entry j of $w := P(\sqrt{A^\dagger A})u$ (do this recursively)
   - Update avg $=$ avg $+ (w_j m^2)/(v_j r)$

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Correctness: High probability bound obtained via Chebyshev's inequality

Summary

- Overview of quantum algorithms over the decades
- Quantum algorithm for solving linear systems
- Quantum Singular Value Transform
- Dequantization - beware the power of state preparation

References

- Harrow, Hassadim, Lloyd. Quantum algorithm for solving linear systems of equations, 2009.
- Gilyén, Su, Low, Wiebe. Quantum singular value transformation and beyond: exponential improvements for quantum matrix arithmetics, 2019.
- Tang. A quantum-inspired classical algorithm for recommendation systems, 2019.
- Gharibian, Le Gall. Dequantizing the Quantum Singular Value Transformation: Hardness and Applications to Quantum Chemistry and the Quantum PCP Conjecture, 2022.

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- See `https://groups.uni-paderborn.de/fg-qi/teaching.html`.

References

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