“Lattice Boltzmann Method approach for the simulation of fluid flows on a spherical surface”

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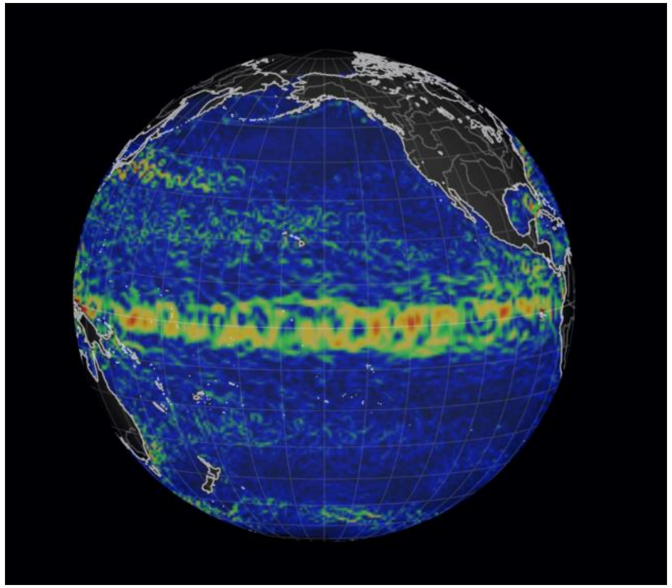
Motivation



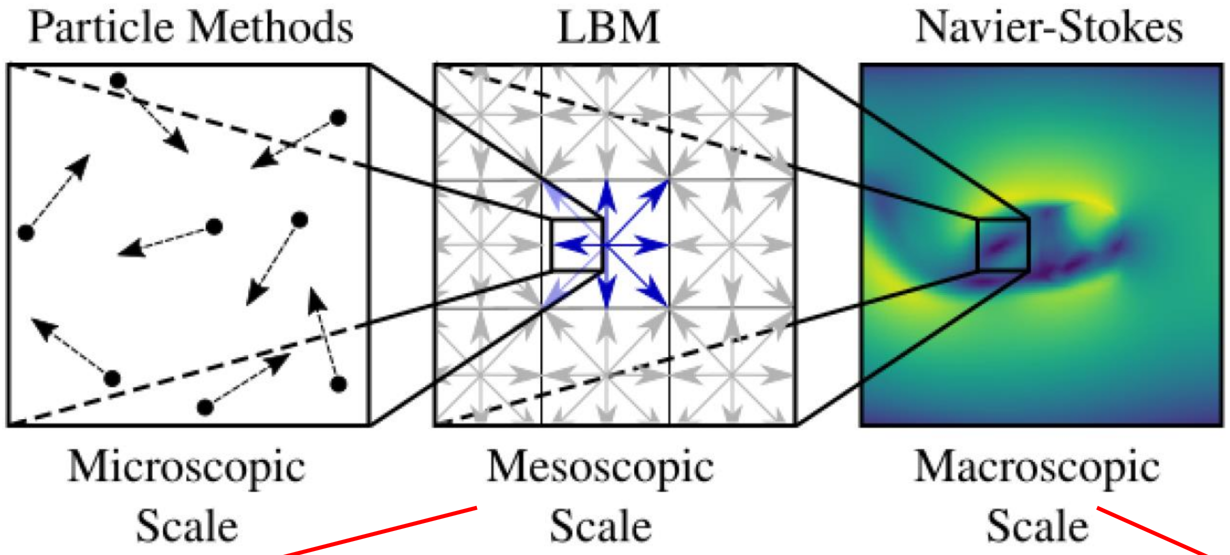
Waves trapped at the Earth's equator

- Spatial curvature causes peculiar flow phenomena
- Numerical simulations to characterize such phenomena
- Employ Lattice Boltzmann methods (LBM): highly-efficient, parallel structure → perfect HPC candidate!

Soap bubble

Introduction

What is LBM?



ρ: fluid density
$\boldsymbol{u}$: fluid macroscopic velocity
$\boldsymbol{v}$: particle velocity
f: particle distribution function (PDF)

Boltzmann Equation:

$$\frac{\partial f}{\partial t} + v^i \frac{\partial f}{\partial x^i} + \frac{F^i}{\rho}\frac{\partial f}{\partial v^i} = \mathcal{C}[f]$$

Continuity and NSE:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^i}\left(\rho u^i\right) = 0$$

$$\frac{\partial(\rho u^i)}{\partial t} + \frac{\partial(\rho u^i u^j)}{\partial x^j} = -\frac{\partial p}{\partial x_i} + \eta \frac{\partial^2 u^i}{\partial x_j \partial x^j} + F^i$$

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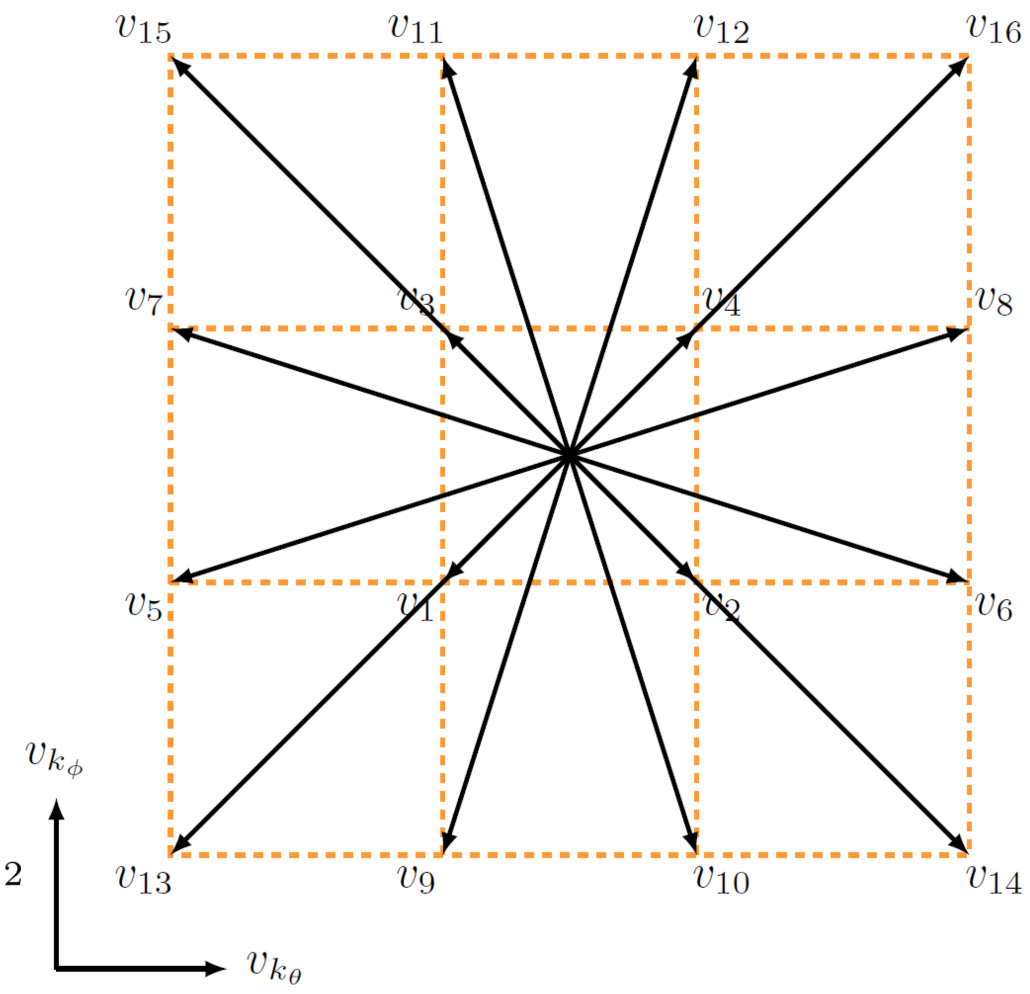
Mesoscopic to Macroscopic

Moments of the PDF are recovered exactly via a Gauss-Hermite quadrature procedure:

$$\rho(\mathbf{x},t) = m\int f(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v} = m\sum_{k=1}^{16} f_k$$

$$\rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t) = m\int f(\mathbf{x},\mathbf{v},t)\mathbf{v}\,\mathrm{d}\mathbf{v} = m\sum_{k=1}^{16} f_k\mathbf{v}_k$$

$$\rho(\mathbf{x},t)e(\mathbf{x},t) = \frac{m}{2}\int f(\mathbf{x},\mathbf{v},t)|\mathbf{v}-\mathbf{u}(\mathbf{x},t)|^2\,\mathrm{d}\mathbf{v} = \frac{m}{2}\sum_{k=1}^{16} f_k|\mathbf{v}_k-\mathbf{u}|^2$$



D2Q16 velocity stencil

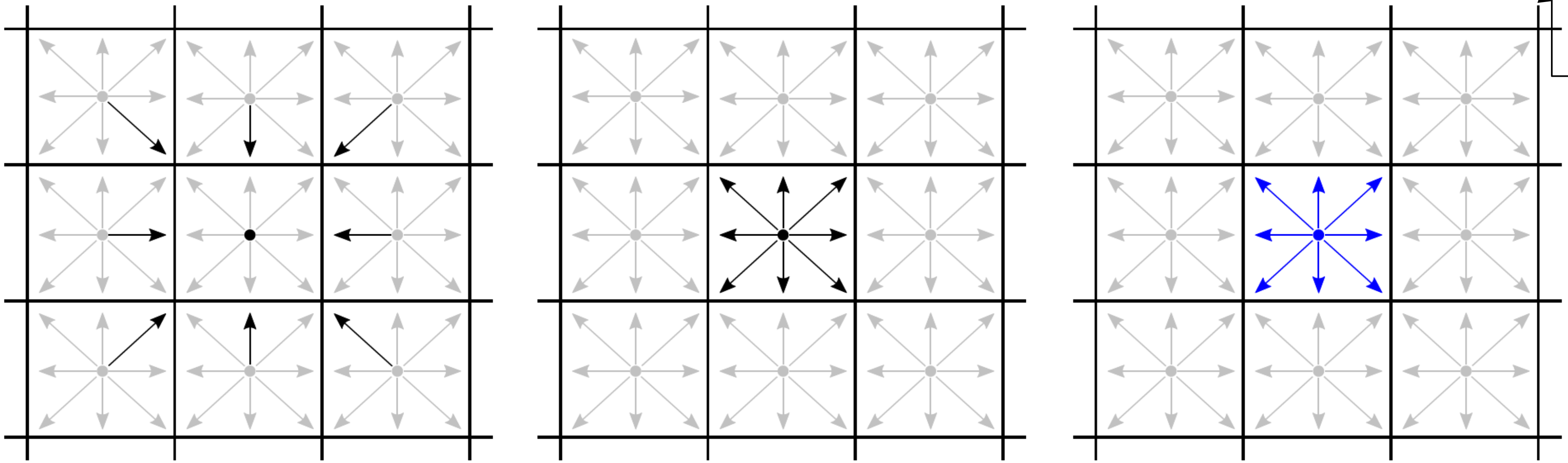
Lattice Boltzmann Method

Based on the Lattice Boltzmann Equation:

- **Streaming step:** $\widetilde{f}_k(\mathbf{x}, t) = f_k(\mathbf{x} - \mathbf{c}_k\, \delta t\,, t), \quad k = 1, \ldots, Q$
- **Collision step:** $f_k(\mathbf{x}, t + \delta t) = \widetilde{f}_k(\mathbf{x}, t) + \boxed{\frac{\delta t}{\tau}[f_k^{\mathrm{eq}}(\mathbf{x}, t) - f_k(\mathbf{x}, t)],} \quad k = 1, \ldots, Q$

BGK operator

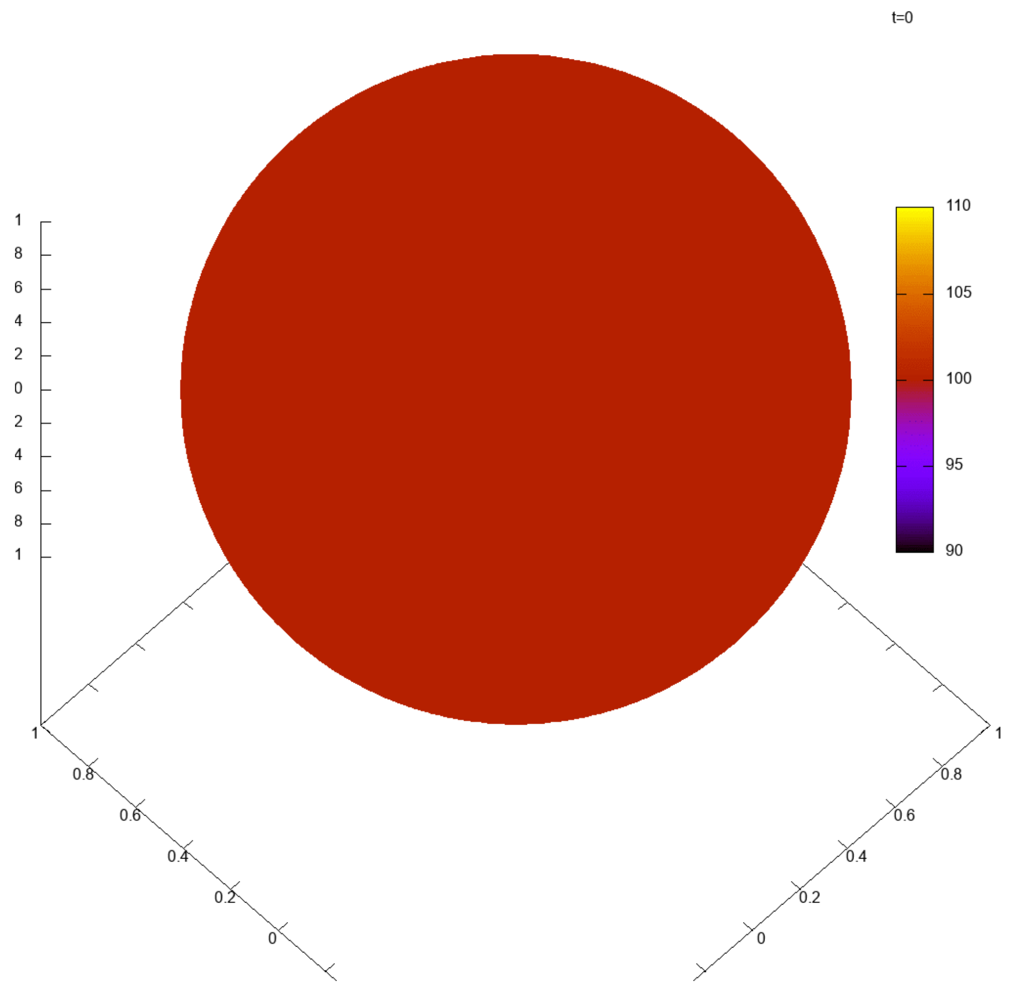
$$\mathbf{c}_k = \frac{\mathbf{v}_k}{c_s}$$



Method

Approach

We developed a Lattice Boltzmann method (LBM) suitable to deal with curvilinear coordinates for the spherical surface



Density profile of a 2D flow produced by our code

Outline

- Extension of LBM to spherical surface using vielbein formalism
- Formulation of (axisymmetric) benchmark problems for the spherical surface
- 2D flows simulations on the spherical surface

Vielbein Field

Construction of a vector field on the spherical surface as a non-coordinate basis (*vielbein* formalism)

$$e_{\hat a} = e^a_{\hat a}\,\partial_a, \qquad \mathbf{u} = u^{\hat a}\,\mathbf{e}_{\hat a}$$

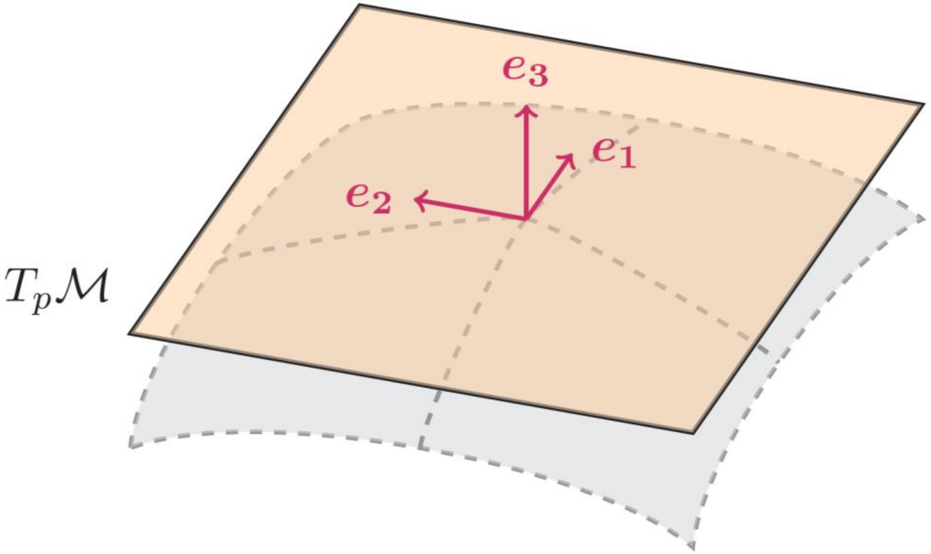
$$g_{ab}\, e^a_{\hat a} e^b_{\hat b} = \delta_{\hat a\hat b}, \qquad u^2 = \delta_{\hat a\hat b}\, u^{\hat a} u^{\hat b}$$

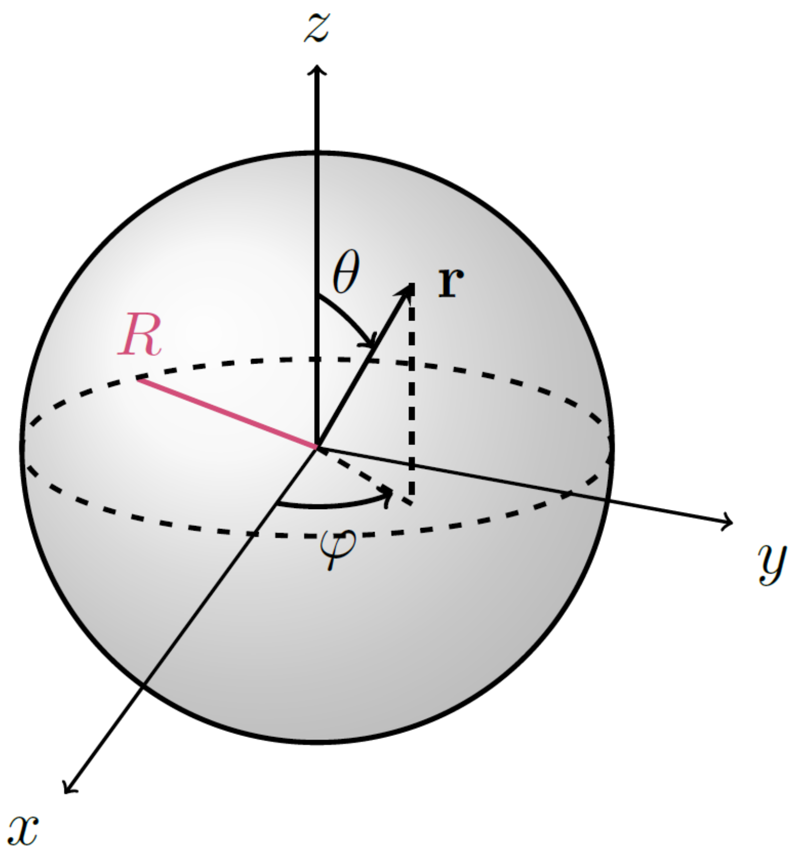
$$e^\theta_{\hat\theta} = \frac{1}{R}, \qquad e^\varphi_{\hat\varphi} = \frac{1}{R\sin\theta}$$

$$\omega^{\hat\theta}_\theta = R, \qquad \omega^{\hat\varphi}_\varphi = R\sin\theta$$

$$\omega^{\hat a}_a e^a_{\hat b} = \delta^{\hat a}_{\hat b}, \qquad \omega^{\hat a}_a e^b_{\hat a} = \delta^b_a$$

$$\Gamma^{\hat\theta}{}_{\hat\varphi\hat\varphi} = -\Gamma^{\hat\varphi}{}_{\hat\theta\hat\varphi} = -\frac{\cos\theta}{R\sin\theta}$$





Boltzmann Eq on the Spherical Surface

- Cartesian:

$$\frac{\partial f}{\partial t} + \underbrace{v^x \frac{\partial f}{\partial x} + v^y \frac{\partial f}{\partial y}}_{\text{Advection}} + \underbrace{\frac{F^x}{m}\frac{\partial f}{\partial v^x} + \frac{F^y}{m}\frac{\partial f}{\partial v^y}}_{\text{External forcing}} = \overbrace{-\frac{1}{\tau}(f - f^{\text{eq}})}^{\text{Collisions}}$$

- Covariant-vielbein:

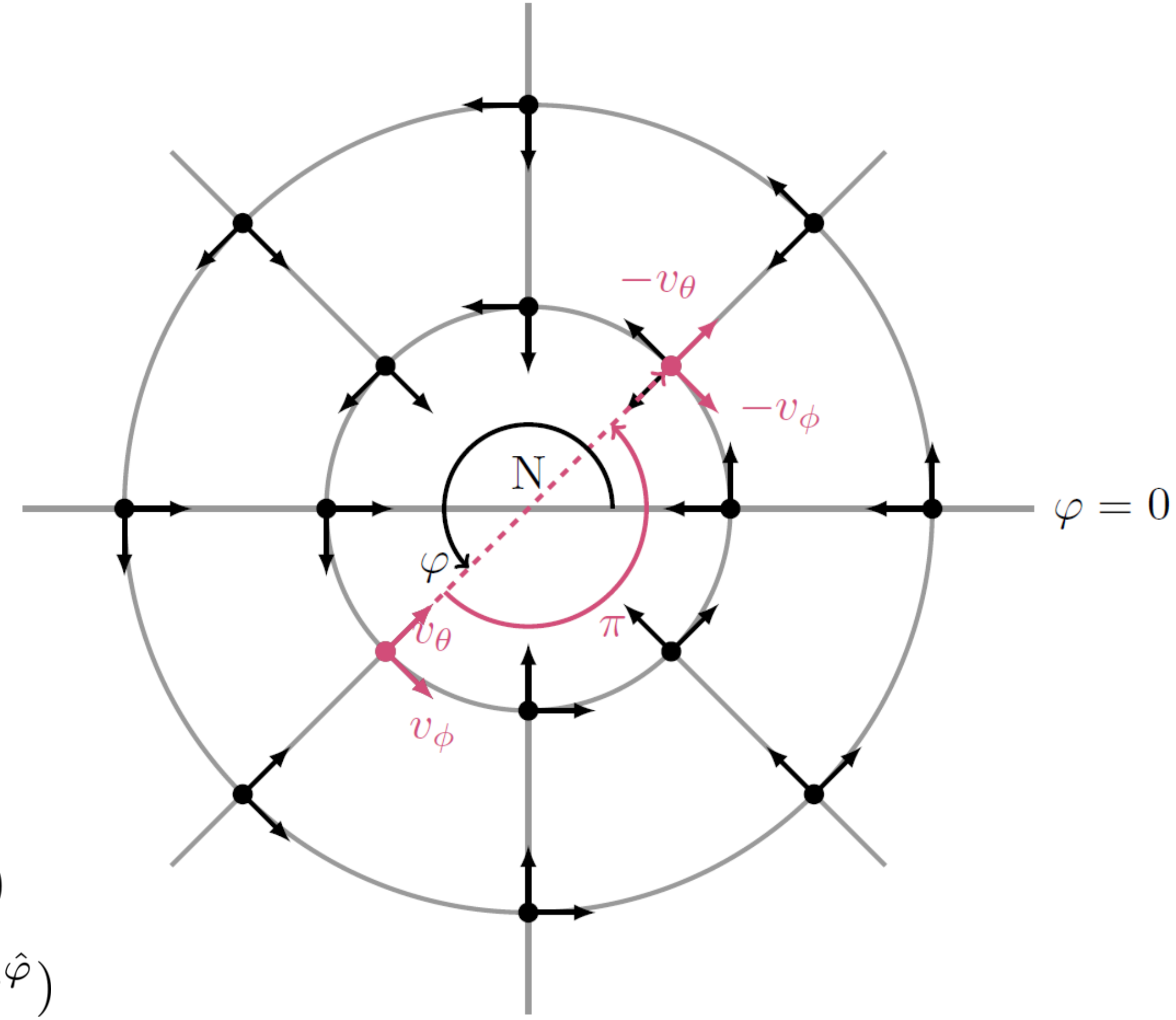
$$\frac{\partial f}{\partial t} + \underbrace{\frac{1}{R\sin\theta}\left[v^{\hat\theta}\frac{\partial(f\sin\theta)}{\partial\theta} + v^{\hat\varphi}\frac{\partial f}{\partial\varphi}\right]}_{\text{Advection}} + \underbrace{\frac{\cos\theta}{R\sin\theta}\left[v^{\hat\varphi}\frac{\partial(fv^{\hat\varphi})}{\partial v^{\hat\theta}} - v^{\hat\theta}\frac{\partial(fv^{\hat\varphi})}{\partial v^{\hat\varphi}}\right]}_{\text{Internal forcing}} + \underbrace{\frac{F^{\hat\theta}}{m}\frac{\partial f}{\partial v^{\hat\theta}} + \frac{F^{\hat\varphi}}{m}\frac{\partial f}{\partial v^{\hat\varphi}}}_{\text{External forcing}} = \overbrace{-\frac{1}{\tau}(f - f^{\text{eq}})}^{\text{Collisions}}$$

$$\Gamma^{\hat\theta}{}_{\hat\varphi\hat\varphi} = -\Gamma^{\hat\varphi}{}_{\hat\theta\hat\varphi} = -\frac{\cos\theta}{R\sin\theta}$$

Advection on the Spherical Surface

- We employ finite-difference schemes (Upwind 1, 2, 3; WENO-5)
- How to deal with the non-periodicity along theta?
- Populate the ghost nodes with:

$$f(-\delta\theta, \varphi; v^{\hat{\theta}}, v^{\hat{\varphi}}) = f(+\delta\theta, \varphi + \pi; -v^{\hat{\theta}}, -v^{\hat{\varphi}})$$
$$f(\pi + \delta\theta, \varphi; v^{\hat{\theta}}, v^{\hat{\varphi}}) = f(\pi - \delta\theta, \varphi + \pi; -v^{\hat{\theta}}, -v^{\hat{\varphi}})$$



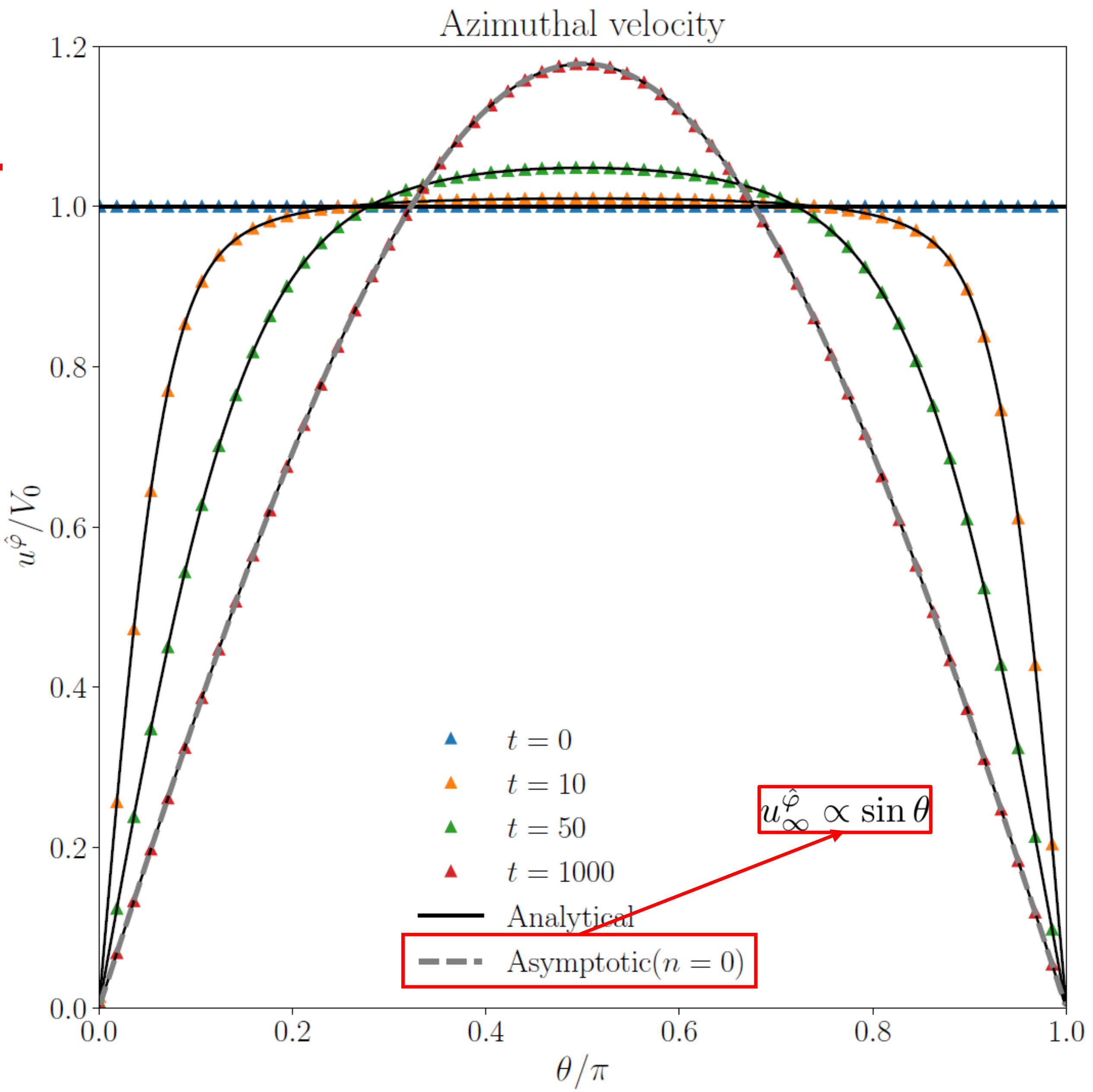
Numerical Results

Shear Wave Damping

Initial velocity profile: $u_0^{\hat{\varphi}}(\theta) = V_0$

$n = 0$ is not damped

1×256 grid, D2Q16,
$\delta t = 1 \times 10^{-4},\ \tau = 1 \times 10^{-3},\ R = 1$
(WENO-5)

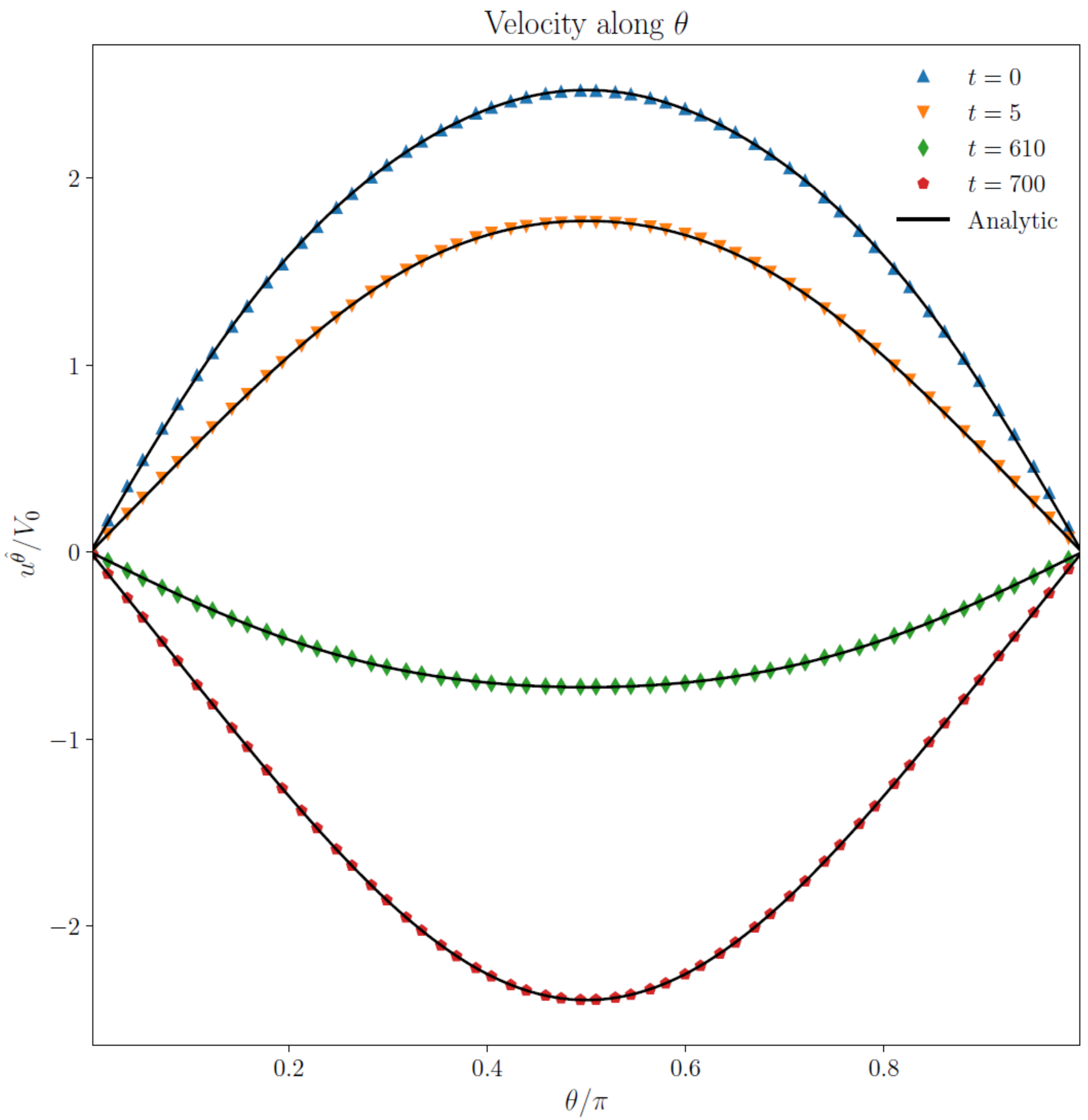


Sound Wave Propagation

The velocity profile of a sound wave in case of an axisymmetric flow is accurately recovered

$$u_0^{\hat{\theta}}(\theta) = U\theta(\pi - \theta)$$

1×256 grid, D2Q16
$\delta t = 2 \times 10^{-5},\ \tau = 2 \times 10^{-5},\ R = 1$
(WENO-5)



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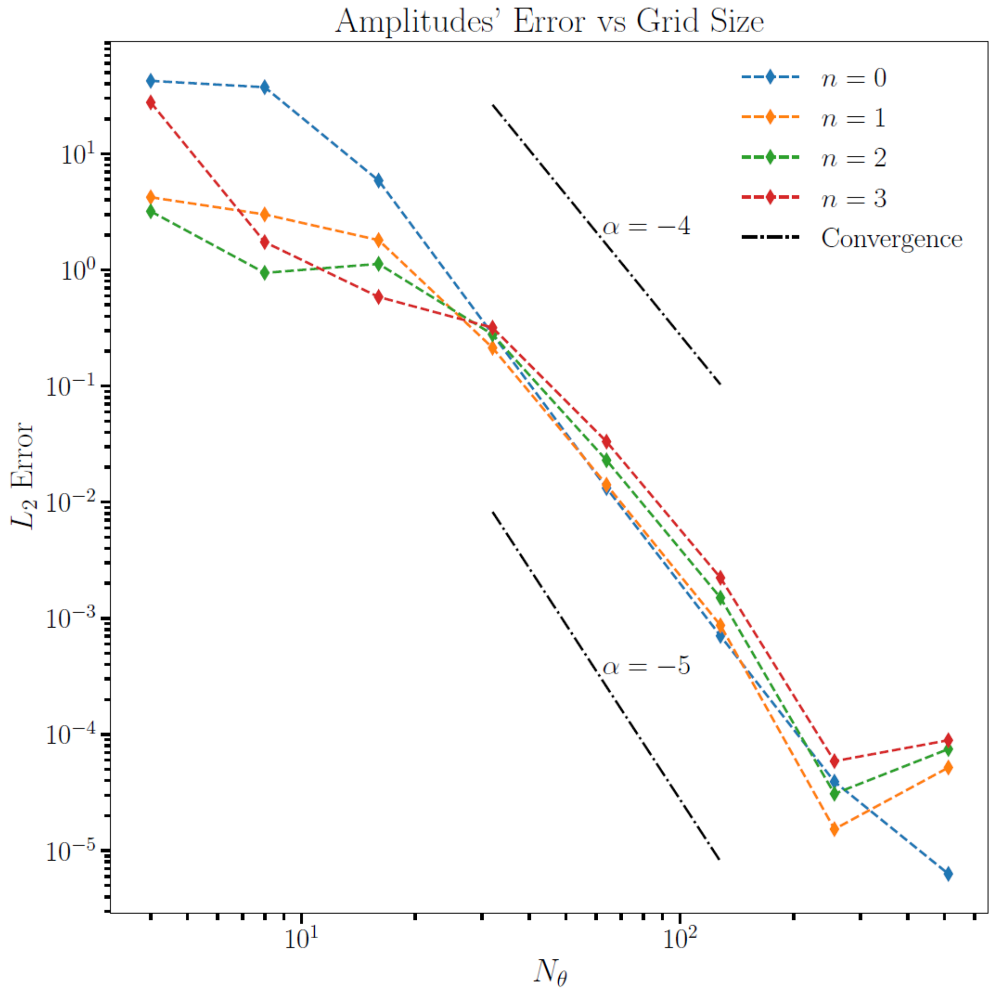
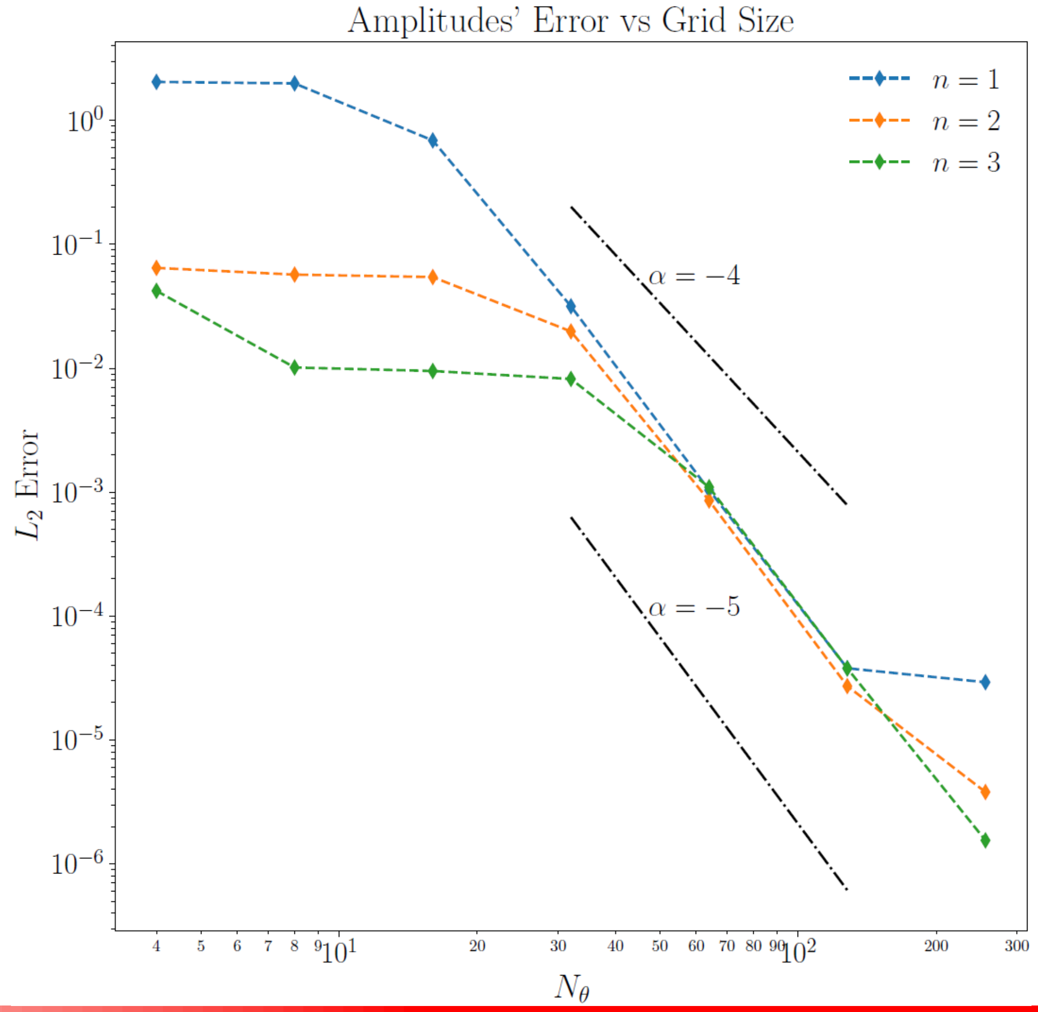
Convergence test

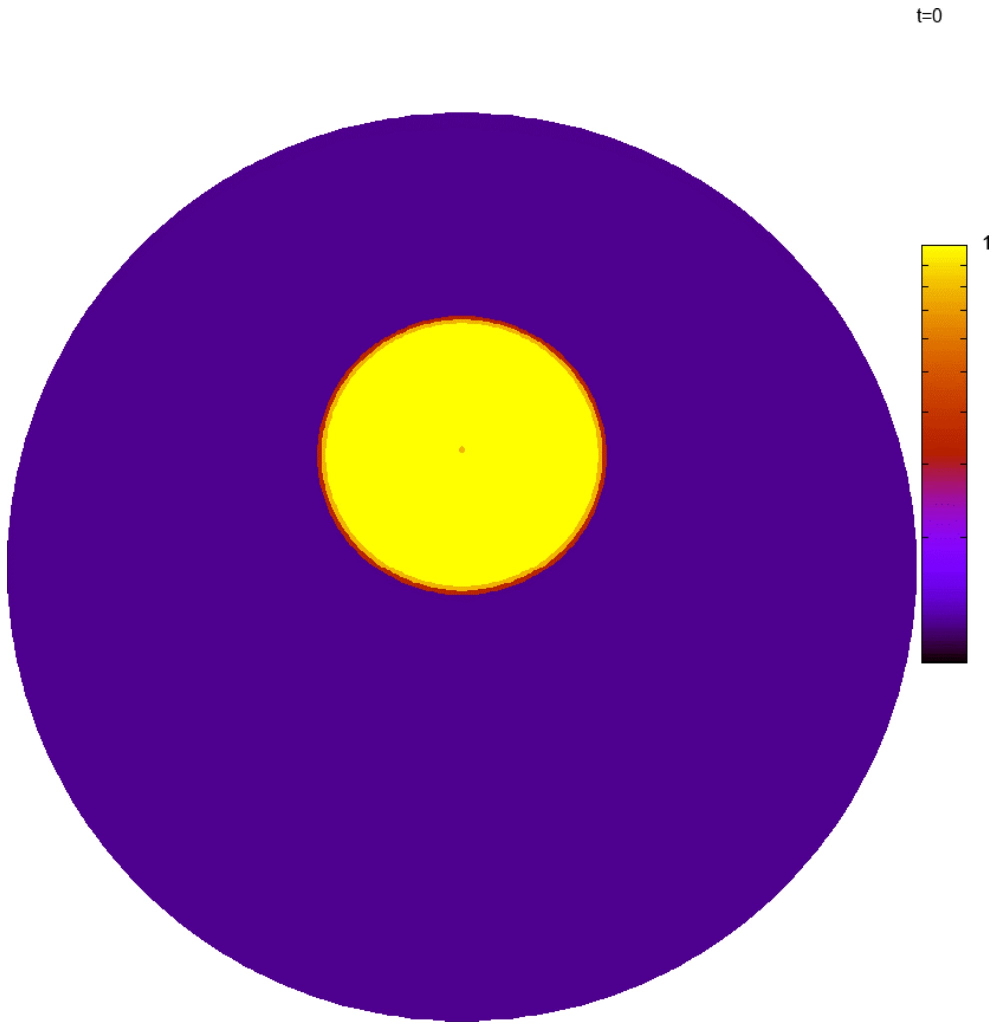
Shear Wave

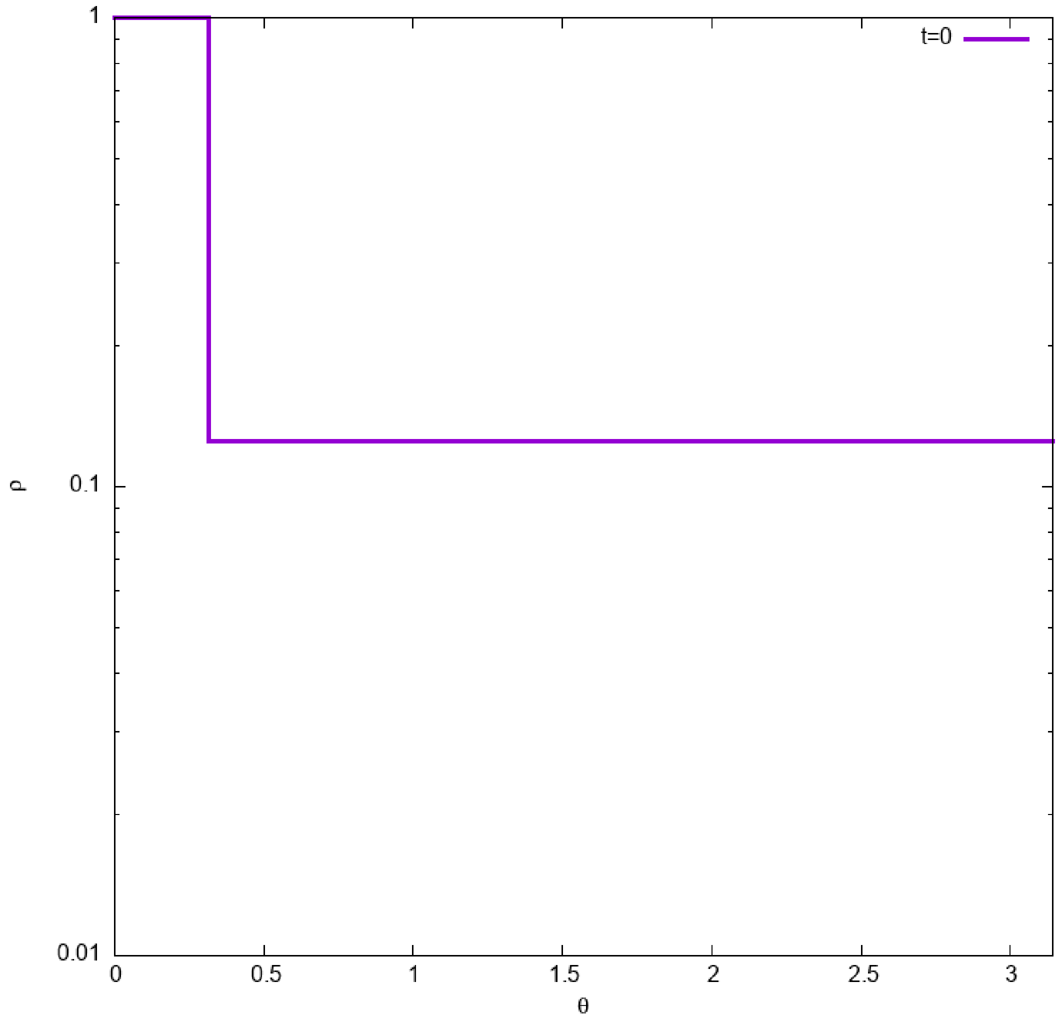


Sound Wave



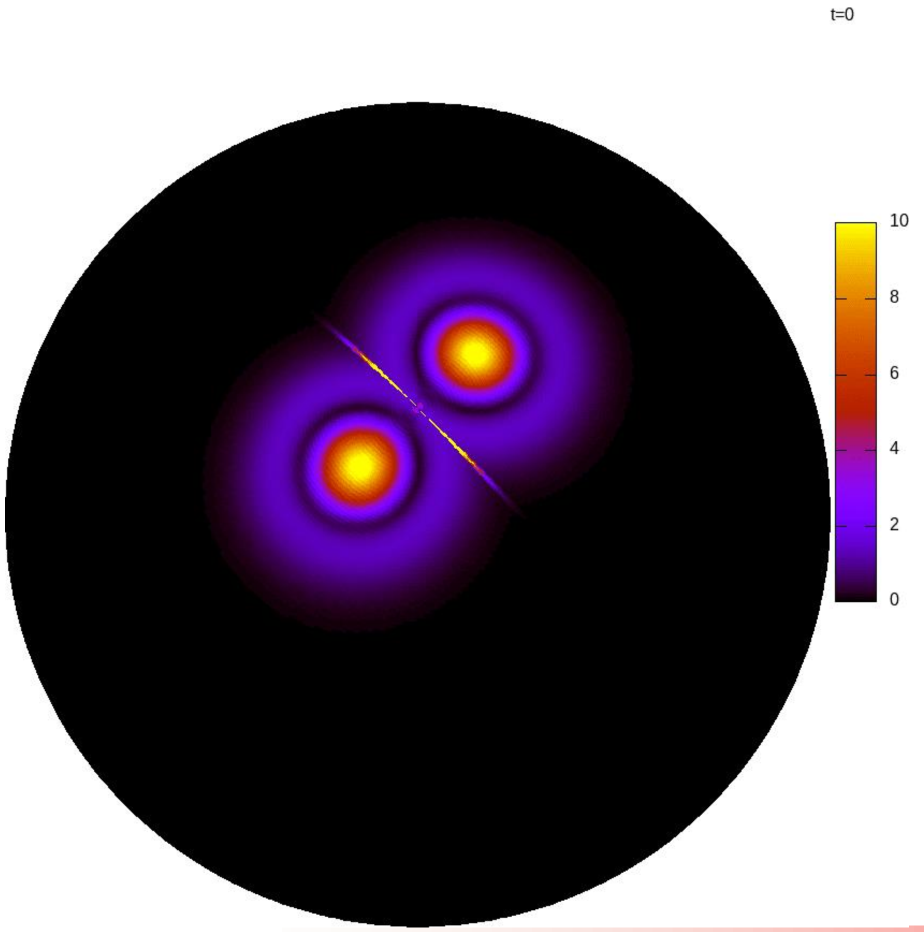
Shock Waves

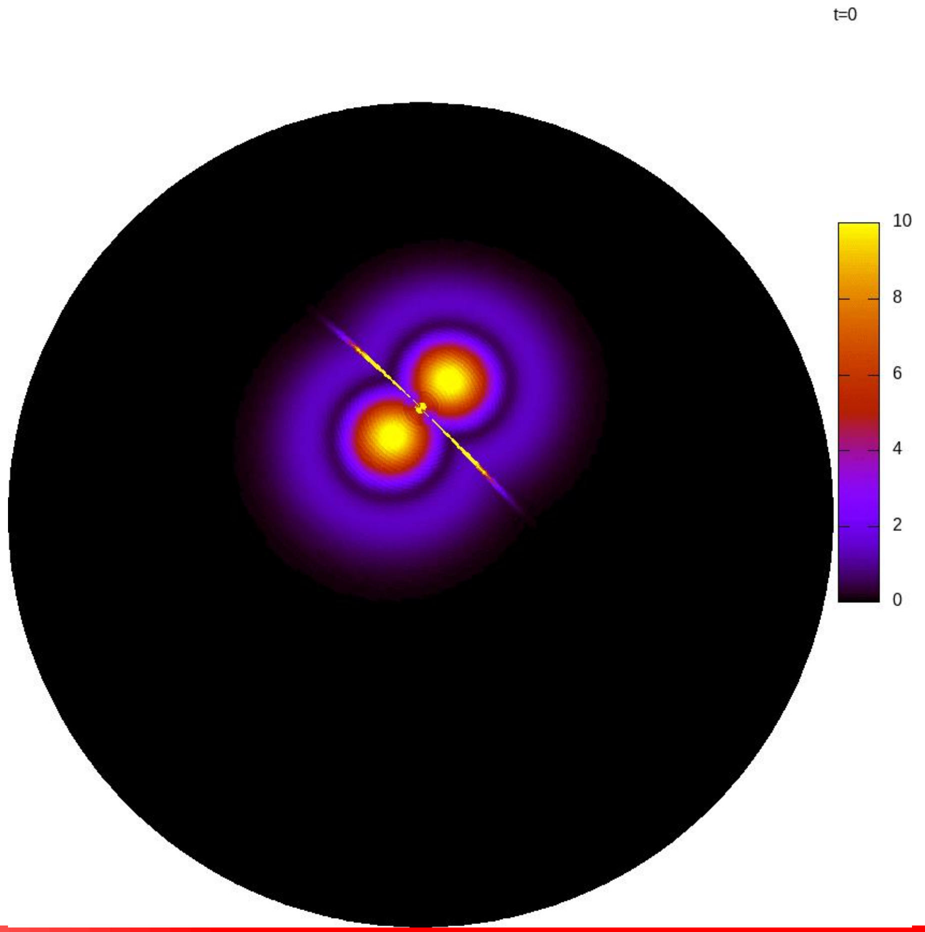




2D Flows

We reproduce the dynamics of two vortexes at the north pole:





Conclusions & Outlook

- Extension of LBM to spherical surfaces using vielbein formalism
- 1D benchmark problems and 2D flows
- Fully compressible solver (Riemann problem on the sphere)

Possible extensions:

- External forcing terms
- Rotating spherical surface

Thank you for your attention

- S. Busuioc and V. E. Ambrus, “Lattice boltzmann models based on the vielbein formalism for the simulation of flows in curvilinear geometries,” Phys. Rev. E, vol. 99
- V. E. Ambrus, S. Busuioc, A. J. Wagner, F. Paillusson, and H. Kusumaatmaja, “Multicomponent flow on curved surfaces: A vielbein lattice boltzmann approach,” Phys. Rev. E, vol. 100

Shear Wave Damping

- **Cartesian:** $u^y = u^y(x)$ $\qquad \dfrac{\partial u^y}{\partial t} = \nu \dfrac{\partial^2 u^y}{\partial x^2}$

- **Spherical surface:** $u^{\hat\varphi} = u^{\hat\varphi}(\theta)$

$$\frac{\partial}{\partial t}\left(\frac{u^{\hat\varphi}}{\sin\theta}\right) = \underbrace{\frac{\nu}{R^2\sin^3\theta}\frac{\partial}{\partial\theta}\left[\sin^3\theta\,\frac{\partial}{\partial\theta}\left(\frac{u^{\hat\varphi}}{\sin\theta}\right)\right]}_{\text{Eigenvalue Eq}}$$

$$u_0^{\hat\varphi}(\theta) = V_0$$

$$\boxed{u^{\hat\varphi}(t,\theta) = \sin\theta \sum_{n=0}^{\infty}\left[A_n(t)\ \underbrace{F_n(\theta)}_{\text{Eigenfunc}}\ \right]}$$

$$A_n(t) = a_n e^{-\alpha_{e;n} t}$$

$$\alpha_{e;n} = \frac{\nu}{R^2}\chi^2_{e;n} = \frac{\nu}{3}\underbrace{2n(2n+3)}_{\text{Eigenvalue}^2}$$

n	χ_n	$A_n(t)$	$F_n(\theta)$
0	0	$\pi\sqrt{3}/4$	$\sqrt{3/2}$
1	$\sqrt{10}$	$\pi\sqrt{21}e^{-\nu 10t/3R^2}/32\sqrt{2}$	$\sqrt{21/32}(3+5\cos 2\theta)$
2	$\sqrt{28}$	$\pi\sqrt{165}e^{-\nu 28tR^2}/256$	$\sqrt{165}(15+28\cos 2\theta+21\cos 4\theta)/128$

Sound wave propagation

We consider an initial velocity profile of the type: $u^{\hat{\theta}} = u^{\hat{\theta}}(\theta)$

Ideal fluid:

$$\frac{\partial^2 u^{\hat{\theta}}}{\partial t^2} = \underbrace{\frac{c_s^2}{R^2}\frac{\partial}{\partial \theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(u^{\hat{\theta}}\sin\theta\right)\right]}_{\text{Eigenvalue Eq}} \qquad \underbrace{\lambda_n^2 = \sqrt{2n(n-1)} - 1}_{\text{Eigenvalues}}$$

Dissipative fluid:

$$\left(-iR^2\omega_n + 2\nu\right) u_n^{\hat{\theta}} = \left(\frac{ic_s^2}{\omega_n} + 2\nu\right)\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial(u^{\hat{\theta}}\sin\theta)}{\partial\theta}\right], \qquad \omega_n = -i\zeta_n + \alpha_n(\nu)$$

$$A_n(t) = a_n e^{-\zeta_n t}\cos[\alpha_n(\nu)t]$$

$$u^{\hat{\theta}}(t,\theta) = \frac{1}{\sin\theta}\sum_n^{\infty}[A_n(t)F_n(\theta)]$$

$$\zeta_n = \frac{\nu}{2R^2}(\lambda_n^2 - 1)$$

$$\alpha_n(\nu) = \sqrt{c_s^2\lambda_n^2/R^2 - \zeta_n^2}$$

n	λ_n	a_n	$F_n(\theta)$
1	0	$\pi(\pi^2+3)/8\sqrt{3}$	$\sqrt{3/2}\sin\theta$
2	2	$-\pi\sqrt{7}\sqrt{6}(4\pi^2-33)/256$	$\sqrt{21/32}\sin^2\theta(1-3\cos^2\theta)$
3	$\sqrt{18}$	$\pi\sqrt{55}(4\pi^2-37)/2048\sqrt{3}$	$\sqrt{165}/16\sin^2\theta(1-14\cos^2\theta+21\cos^4\theta)$

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