Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Numerical explorations of non-linear electrodynamic theories

Gabriele Pierini

Work in progress with Prof. Dr. Marina Krstic Marinkovic and Dr. Verónica Errasti Díez

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Outline

Section 1

What?

Non-linear extensions of QED

The most well known example [Born & Infeld, 1934]

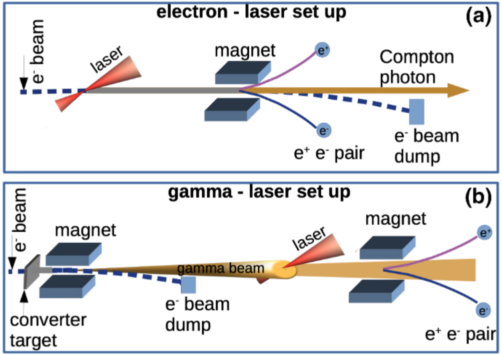
$$\mathcal{L} = b^2 \left[1 - \sqrt{1 + \frac{1}{2b^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{16b^4} \left(F_{\mu\nu} \tilde{F}^{\mu\nu} \right)^2} \right] \qquad \tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$$

$b \to \infty$ recovers QED

Section 2

Why?

Coming soon



LUXE @DESY

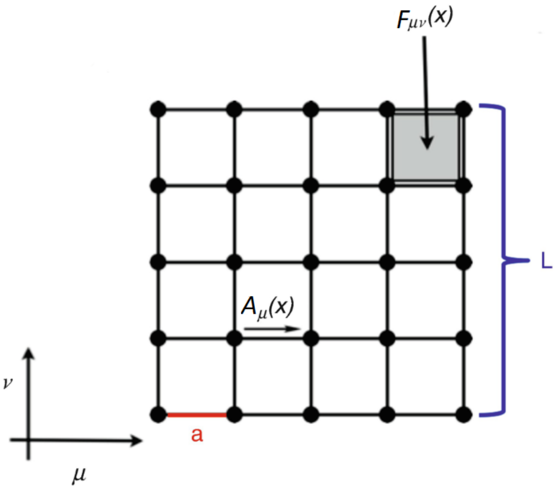
[Abramowicz et al. 2102.02032]

Section 3

Where?

Lattice approach

- $3 + 1$ dimensions
- L$=8$, a$=1$
- Periodic boundary conditions

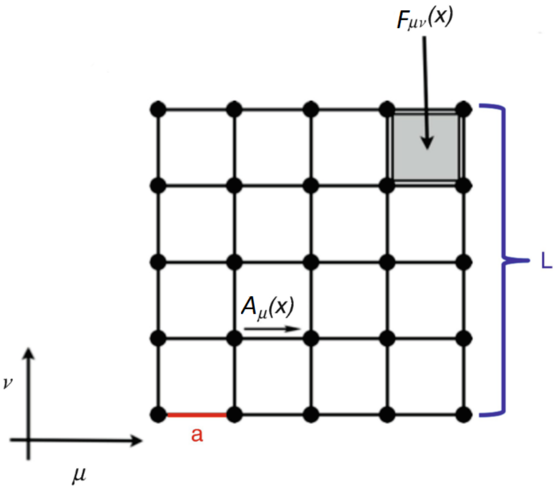


Lattice approach

- $3 + 1$ dimensions
- L=8, a=1
- Periodic boundary conditions

Non-compact description

- the field A_μ links two points instead of $U_\mu = \exp(iA_\mu)$
- $F_{\mu\nu}$ is the plaquette instead of $U_{\mu\nu}$



Born-Infeld on a lattice

Only one simulation so far [Kogut-Sinclaire, 0509097]

$$S = b^2 \int d^4x \left[\sqrt{1 + \frac{1}{2b^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{16b^4} F_{\mu\nu} \tilde{F}^{\mu\nu}} - 1 \right]$$

$\beta = b^2 a^4 = b^2$

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Continuum	**Discrete**
$\int d^4x$	$\sum_x$
$F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu$	$F_{\mu\nu}(x) = A_\nu(x+\hat{\mu}) - A_\nu(x) - A_\mu(x+\hat{\nu}) + A_\mu(x)$
$A_\mu(x) \to A_\mu(x) - \partial_\mu \chi$	$A_\mu(x) \to A_\mu(x) - \chi(x+\hat{\mu}) + \chi(x)$

Section 4

Gauge fixing

Subsection 1

Relaxation

Landau gauge

$\partial_\mu A^\mu = 0$ equivalent to extremal value of

$$F[A] = \sum_{\mu=1}^{4} \int \mathrm{d}^4 x \left[A_\mu(x)^2 \right]$$

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Compact description	Non-compact description
$F_{lat}[U] = \sum_{x} \sum_{\mu=1}^{4} [U_\mu(x) + U_\mu(x)^\dagger]$	$F_{lat}[A] = \sum_{x} \sum_{\mu=1}^{4} [A_\mu(x)]^2$

[Gattringer & Lang, Springer Berlin, 2010]
[Cardoso et al. 1206.0675]

[GP, 2023]

Relaxation: Non-compact description

The non-compact description improves convergence

Given the gauge transformation

$$A_\mu(x) \to A_\mu(x) - \chi(x+\hat{\mu}) + \chi(x)$$

we minimize

$$F_{\rm lat}^{\chi} = \sum_x \sum_\mu^4 \left[A_\mu(x) - \chi(x+\hat{\mu}) + \chi(x)\right]^2$$

with a local minimization algorithm

$$\chi(x) = -\frac{1}{8}\sum_\mu \left[A_\mu(x) - \chi(x+\hat{\mu}) - A_\mu(x-\hat{\mu}) + \chi(x-\hat{\mu})\right]$$

[GP, 2023]

Subsection 2

Exact approach

Exact gauge fixing 1/2: Main idea

Landau gauge $\partial_\mu A^\mu = 0$ discretized to

$$\sum_\mu [A_\mu(x+\hat{\mu}) - A_\mu(x)] = 0$$

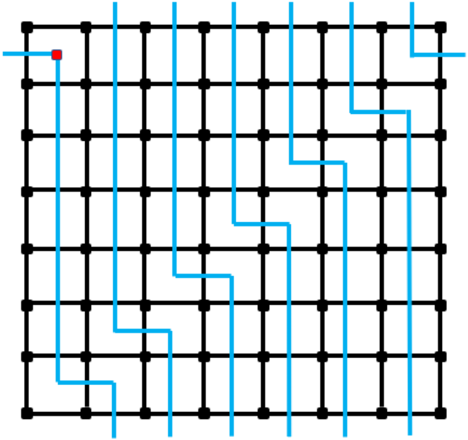
Must hold true in all points $\implies$ $N-1$ equations, because

$$\sum_x \sum_\mu [A_\mu(x+\hat{\mu}) - A_\mu(x)] \equiv 0$$

[GP, 2023]

Exact gauge fixing 2/2: Implementation

- Build a closed path
  visiting all the points once



[GP, 2023]

Exact gauge fixing 2/2: Implementation

- Build a closed path
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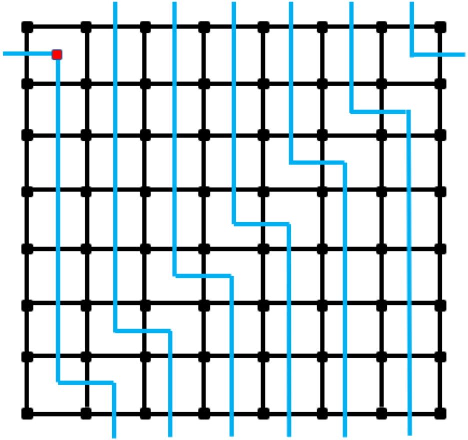
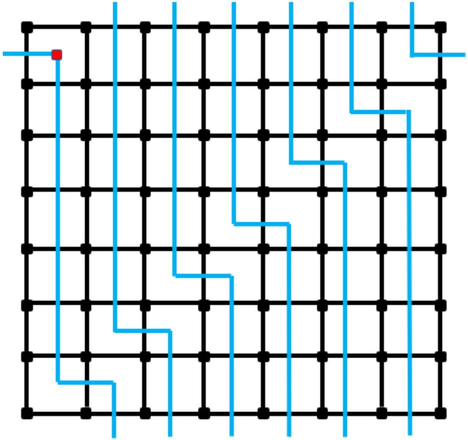
- Fix the link
  pointing towards the next site
  in all sites but one;



[GP, 2023]

Exact gauge fixing 2/2: Implementation

- Build a closed path
  visiting all the points once
- Fix the link
  pointing towards the next site
  in all sites but one;
- **Pros**: Speed
  $\mathcal{O}(N)$ versus $\mathcal{O}(N \log N)$
- **Cons**: Non-unitary algorithm



[GP, 2023]

Section 5

Observables

Multilevel algorithm 1/2: Main idea

For local actions (like Born-Infeld), multilevel algorithm improves the statistics

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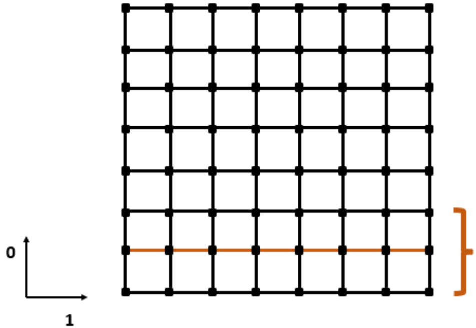
- Sublattices
  along the temporal direction



[Parisi, Petronzio & Rapuano, 1983] [Luscher & Weisz, 0108014]

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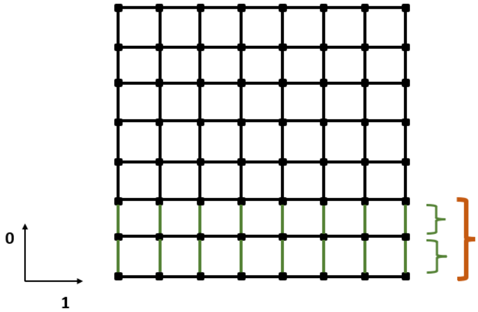

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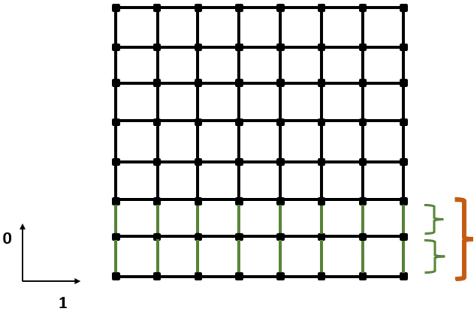
Multilevel algorithm 1/2: Main idea

For local actions (like Born-Infeld), multilevel algorithm improves the statistics

- Sublattices
  along the temporal direction
- Sub-sublattices
- Simultaneous update:
  inner links of sub-sublattices



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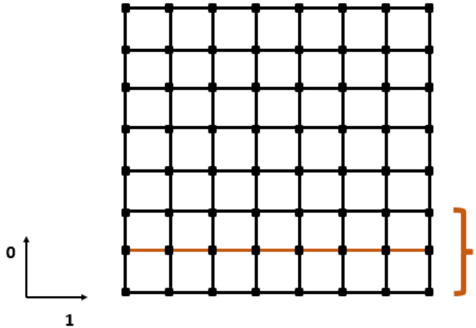

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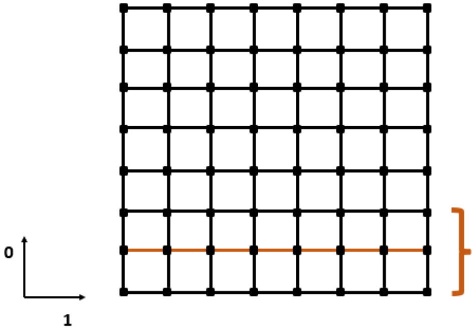

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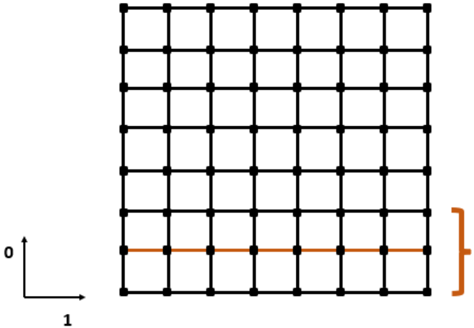
- Repeat



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Multilevel algorithm 2/2: Shortcomings & resolutions

Shortcoming 1:
Simultaneous updates interfere with RNG

Resolution 1:
One RNG for every time coordinate

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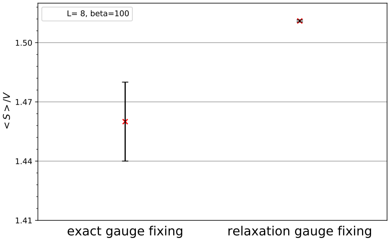
Shortcoming 2:
Sublattices are independent iff boundaries are fixed [Luscher & Weisz, 0108014]
But gauge fixing within multilevel changes the boundaries

Resolution 2:
Probably not if pure gauge → relaxation (not exact) gauge fixing algorithm

Observable 1: Average action density

Metropolis Monte-Carlo algorithm used to run the simulations

$\beta = 1$



$\beta = 100$

Observable 1: Average action density

Metropolis Monte-Carlo algorithm used to run the simulations

$\beta = 1$

$\beta = 100$

Exact gauge in agreement with [Kogut-Sinclaire, 0509097]

Relaxation in agreement only for $\beta = 100$

Wilson line

In continuum [Wilson, 1974]

$$W[\gamma] = \exp\left\{ i \oint_{\gamma} A_\mu(x) dx^\mu \right\}$$

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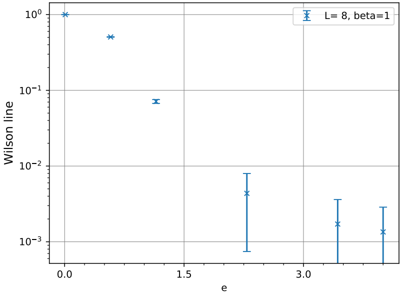
On the lattice [Kogut-Sinclaire, 0509097]

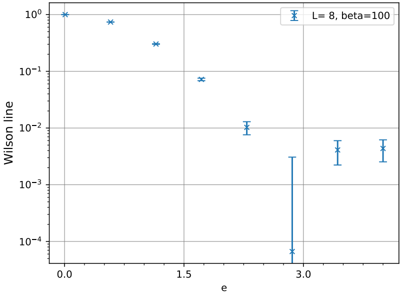
$$W[x] = \exp\left\{ ie \sum_t \left[A_4(\mathbf{x}, t) - \boxed{\frac{1}{L^3} \sum_{\mathbf{y}} A_4(\mathbf{y}, t)} \right] \right\}$$

Net charge $= 0$

Observable 2: Wilson line



$\beta = 1$, logarithmic scale



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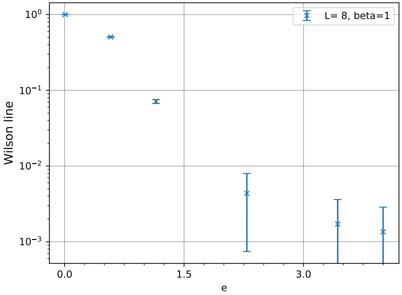

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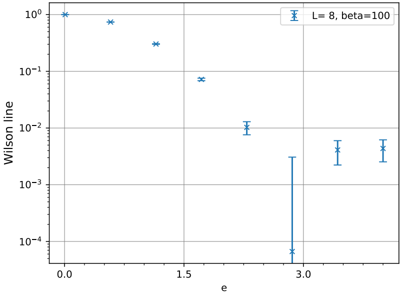

$\beta = 100$, logarithmic scale

All results ($\beta = 1, 100$) in agreement with [Kogut-Sinclaire, 0509097]

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Thanks a lot for your attention!

Take-homes:

- Non-perturbative QED soon within experimental reach: LUXE, PVLAS
- Lattice non-linear electrodynamics ideal theoretical approach – largely unexplored
- For Born-Infeld electrodynamics:
  - Improved gauge fixing algorithms
  - Partial agreement with [Kogut-Sinclaire, 0509097]
- We lay the ground for a pheno comparison of Non-Linear Electrodynamic theories

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Outlook:

- Improve on the code: speed, thermalization, statistics
- More challenging observables: e.g. Schwinger pair production [Liu, 2302.05143]
- Other theories: e.g. ModMax [Bandos, Lechner, Sorokin & Townsend, 2007.09092]

Section 6

Supplementary slides

Relaxation Approximation

$$\Delta(x) = \sum_{\mu} \left[U_\mu(x-\hat{\mu}) - U_\mu(x) - U_\mu^\dagger(x-\hat{\mu}) + U_\mu^\dagger(x) \right]$$

$$\theta = \frac{1}{N} \sum_x \left[\Delta(x) \Delta^\dagger(x) \right]$$

[Cardoso, Silva, Bicudo & Oliveira, 1206.0675]

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$$\theta = \frac{1}{N} \sum_x \left[\Delta(x) \Delta^\dagger(x) \right]$$

[Cardoso, Silva, Bicudo & Oliveira, 1206.0675]

$$\Delta'(x) = \sum_{\mu} \left[A_\mu(x) - A_\mu(x-\hat{\mu}) \right]$$

$$\theta' = \frac{1}{N} \sum_x \left[\Delta'(x)^2 \right]$$

[Schröck & Vogt, 1212.5221]

Metropolis algorithm and heat bath

Steps of metropolis algorithm:

1. Start with a configuration X with energy $E[X]$;
2. propose a new configuration X' with energy $E[X']$;
3. if $E[X'] < E[X]$ accept the new configuration;
4. if $E[X'] > E[X]$ accept the new configuration with a probability of $e^{-\Delta E}$;
5. repeat.

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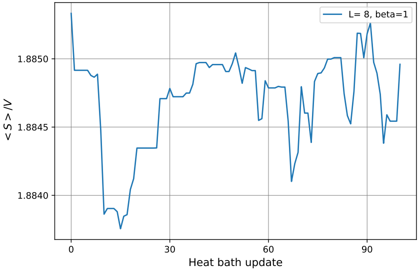
5. repeat.

Heat bath algorithm combines the steps of Metropolis: sample X directly according to the probability distribution

$$dP(X) = dX exp(-E[X])$$

Heat bath thermalization

Heat bath algorithm used to run the simulation



$\beta = 100$ and exact gauge

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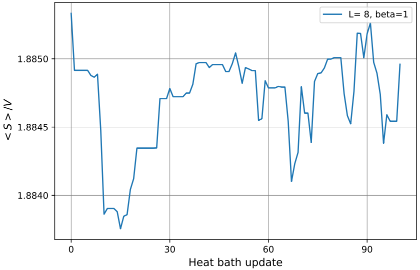

$\beta = 100$ and exact gauge

Heat bath is much slower than Metropolis.

Multihit algorithm I

The multihit algorithm is the starting point for the multilevel algorithm.

$$W(\mathbf{x}) = \exp\left\{ ie \sum_t A_0(t, \mathbf{x}) \right\}.$$

$$\left\langle \prod_t \exp\left\{ieA_0(t,\mathbf{x})\right\} \right\rangle = \frac{\int \mathrm{d}A_\mu \quad \left(\prod_t \exp\left\{ieA_0(t,\mathbf{x})\right\}\right) \exp(-\beta S)}{\int \mathrm{d}A_\mu \quad \exp(-\beta S)},$$

where we are integrating over all the links.

Multihit algorithm II

Same average with

$$\left\langle \prod_t \overline{\exp\left\{ieA_0(t,\mathbf{x})\right\}} \right\rangle$$

with

$$\overline{\exp\left\{ieA_0(t,\mathbf{x})\right\}} = \frac{\int \mathrm{d}A_0(t,\mathbf{x}) \quad \exp\left\{ieA_0(t,\mathbf{x})\right\}\exp(-\beta S)}{\int \mathrm{d}A_0(t,\mathbf{x}) \quad \exp(-\beta S)}.$$

but less fluctuation.

Jackknife resampling method

Jackknife reduces error bars in small data sets

Jackknife resampling method

Jackknife reduces error bars in small data sets

Given $\widehat{W} = f_N\left(x_1, ..., x_N\right)$, define $\overline{W}_j = f_{N-1}\left(x_1, ..., x_{j-1}, x_{j+1}, ..., x_N\right)$

The error is

$$\sigma_W = \sqrt{\frac{N-1}{N}\sum_{j=1}^{N}\left(\overline{W}_j - \widehat{W}\right)^2}$$

The unbiased estimator is

$$W_{nonb} = N\widehat{W} - (N-1)\left[\frac{1}{N}\sum_{j=1}^{N}\overline{W}_j\right]$$

[Gattringer & Lang, Springer Berlin, 2010]