“Data Assimilation Techniques for a Shell Model of Turbulence”

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AQTIVATE

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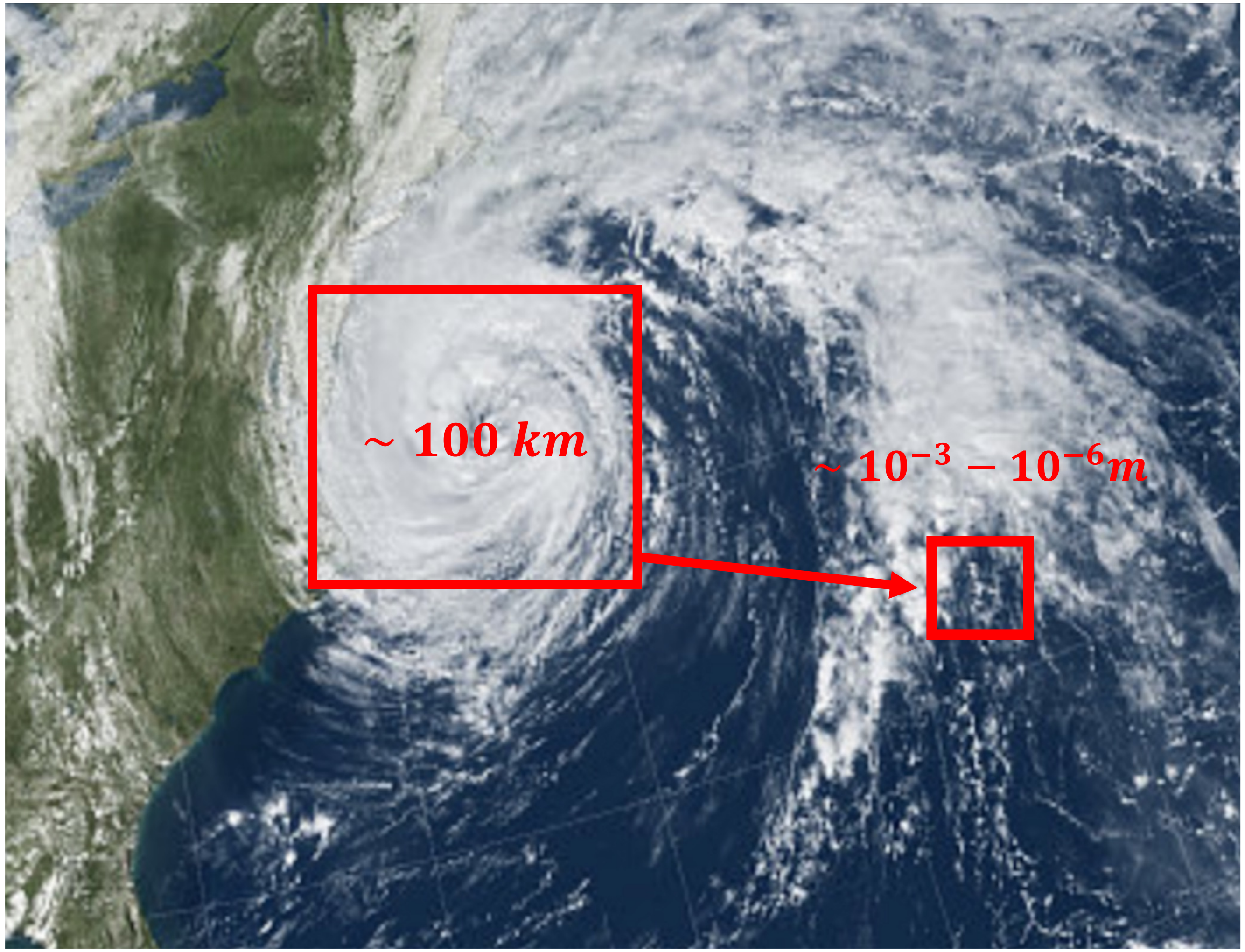
Overview on the thesis

- Fully Developed Turbulence and Richardson Cascade
- Dynamical system for Energy cascade : The Shell Model
- Data Assimilation Techniques:
  Nudging
  Ensemble Kalman Filter (EnKF)



Turbulence

Navier-Stokes equation for incompressible flow:

$$\partial_t \boldsymbol{v} + \nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{v}) = -\nabla P + \boldsymbol{f} + \frac{1}{R_e} \Delta \boldsymbol{v}$$

Reynolds Number

$$R_e = \frac{l_0 v_0}{\nu}$$

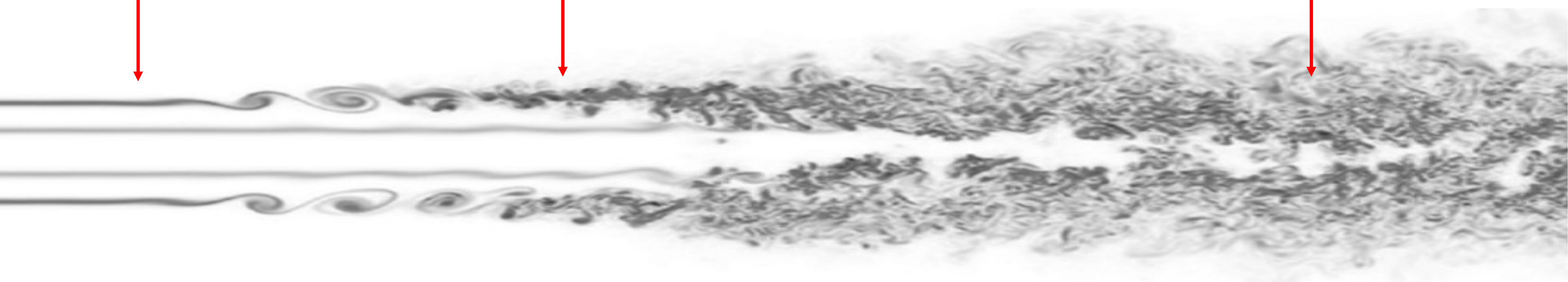
$R_e = 0$

Laminar Flow

Transition Regime

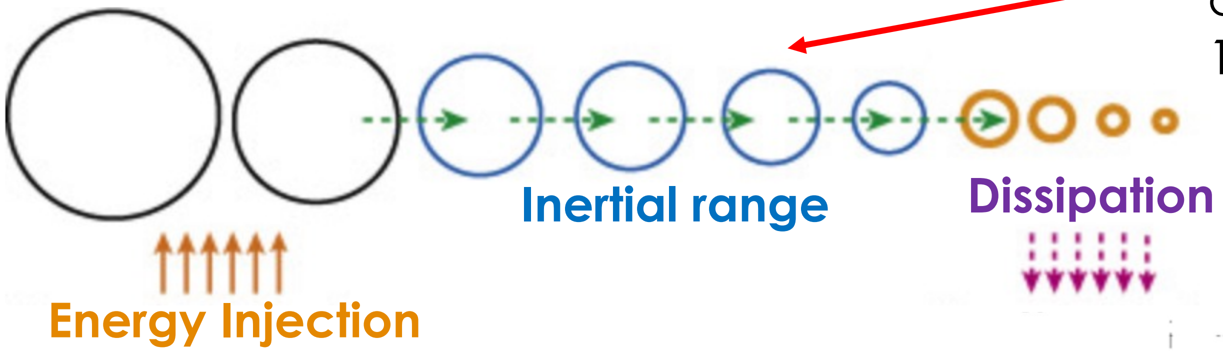
$R_e \to \infty$

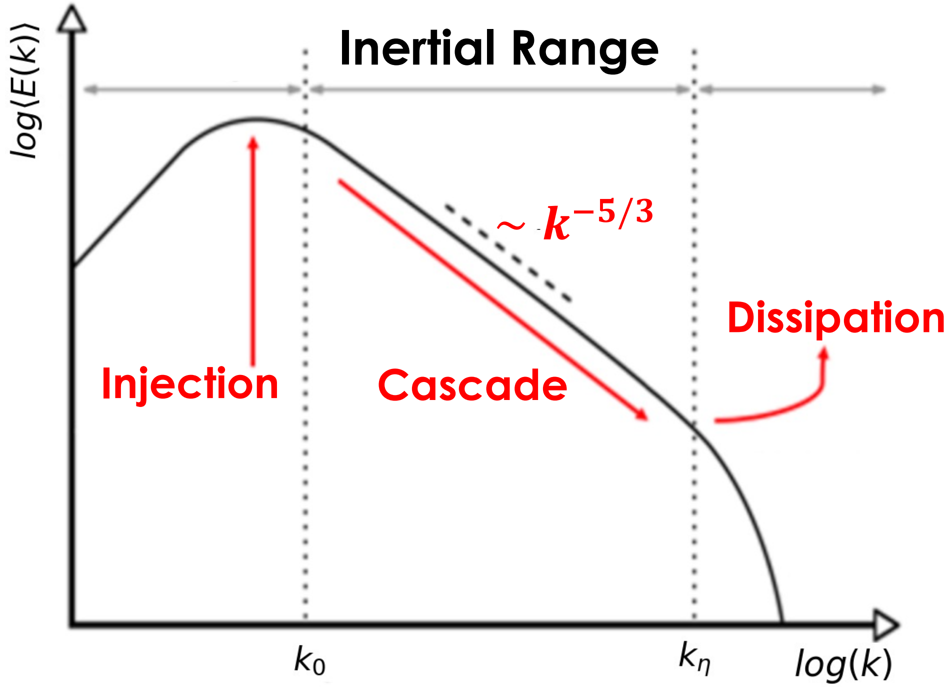
Fully Developed Turbulence



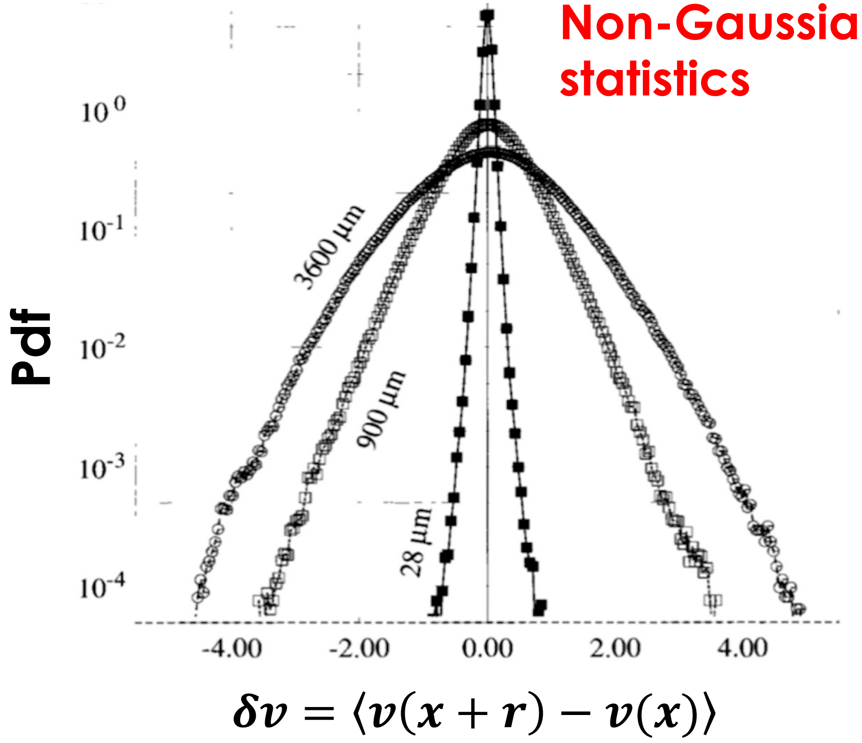
Phenomenology of Turbulence

Recovered **Homogeneity** and **Isotropy** (Kolmogorov 1941)





Non-Gaussian statistics



Shell Models

Sum over all wave vector triads

Navier–Stokes equation in Fourier Space:

$$\mathbf{k}+\mathbf{k}' = \mathbf{k}''$$

$$\left(\frac{d}{dt} + \nu|\mathbf{k}|^2\right)u_j(\mathbf{k},t) = -ik_l P_{j\alpha}(\mathbf{k})\sum_{\mathbf{k}'} u_\alpha(\mathbf{k}',t)u_l(\mathbf{k}-\mathbf{k}',t) + f_j(\mathbf{k},t)$$

To solve numerically the dissipative scale:

$$\sharp \sim R_e^{9/4}$$

Dynamical system mimics the energy cascade:

$$\{u_n \in \mathbb{C}, n = 1,2,..,N\} \qquad k_n = k_0 2^n$$

$$\frac{du_n}{dt} = i(ak_{n+1}u_{n+1}^* u_{n+2} + bk_n u_{n-1}^* u_{n+1} - ck_{n-1}u_{n-1}u_{n-2}) - \nu k_n^2 u_n + f_n$$

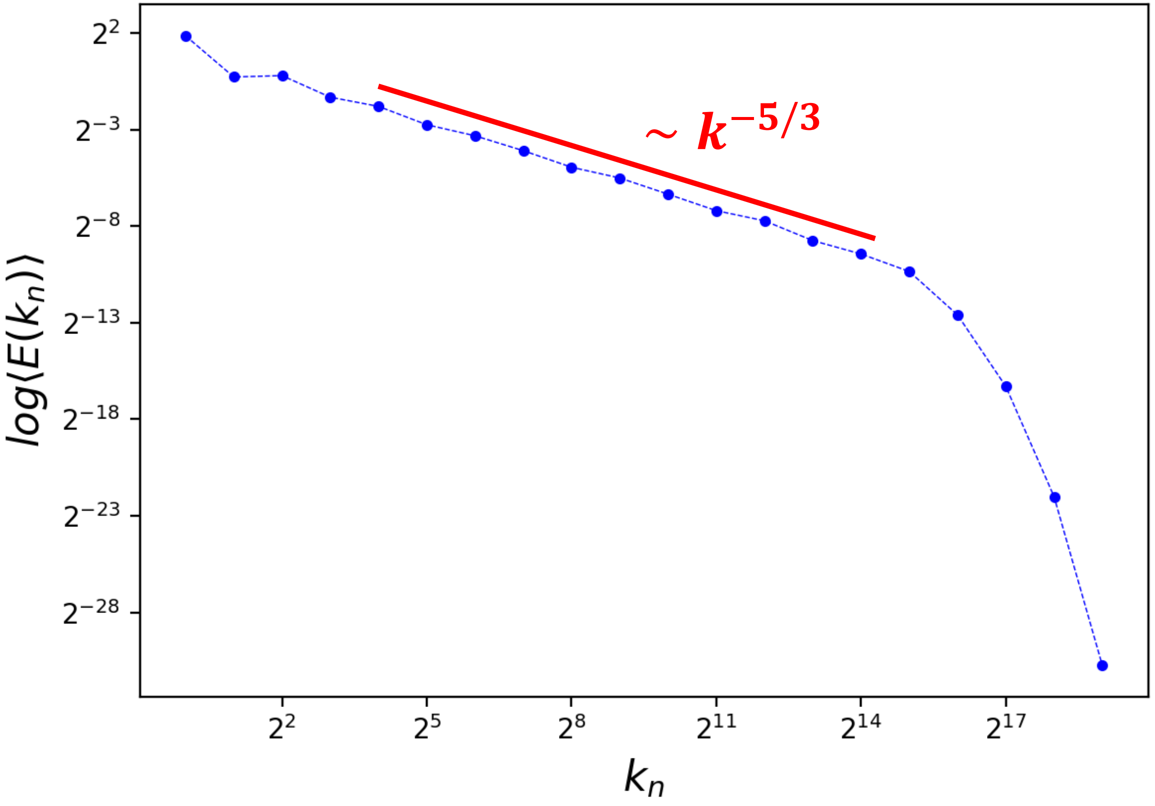
Sum over the first and second neighbors

To solve numerically the dissipative scale:

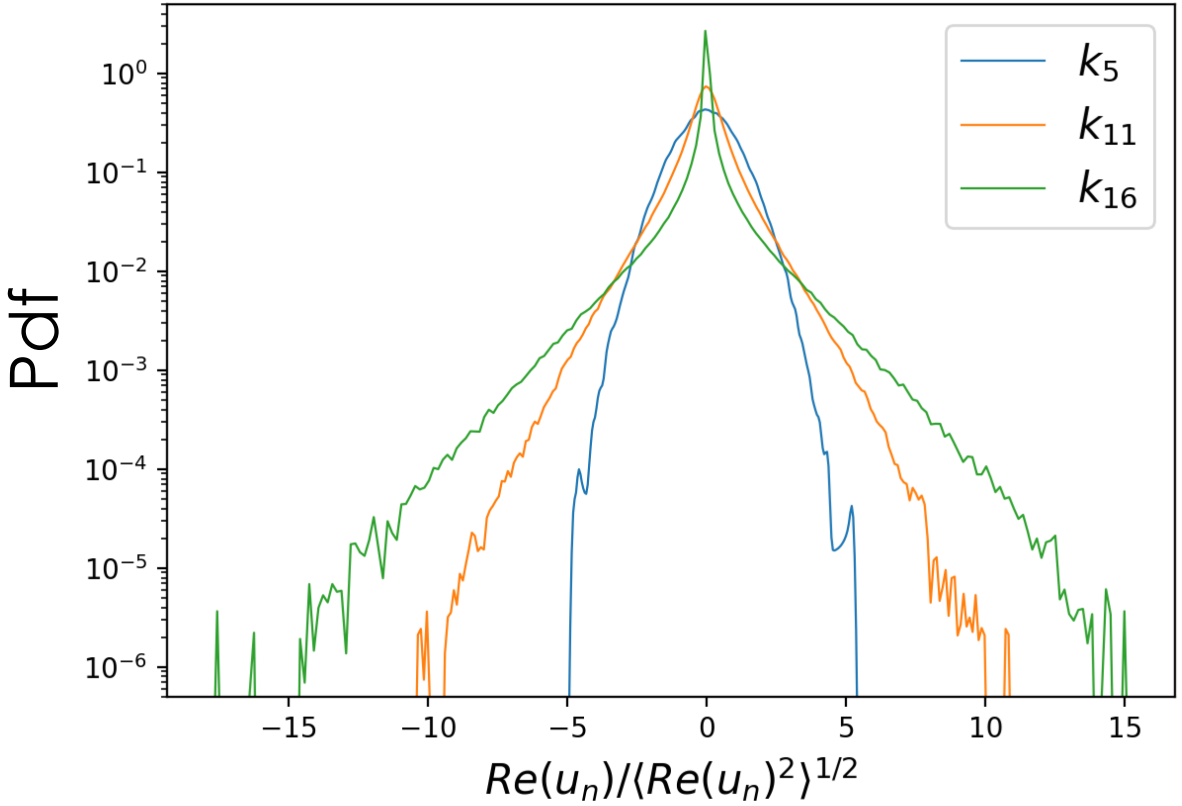
$$\sharp \sim \ln R_e$$

Shell Model Results

Energy Spectrum



Non-Gaussian statistics at the small scales



Ensemble Kalman Filter (EnKF)

Is a data Assimilation Technique optimized for system with :

- Linear dynamics
- Gaussian statistics

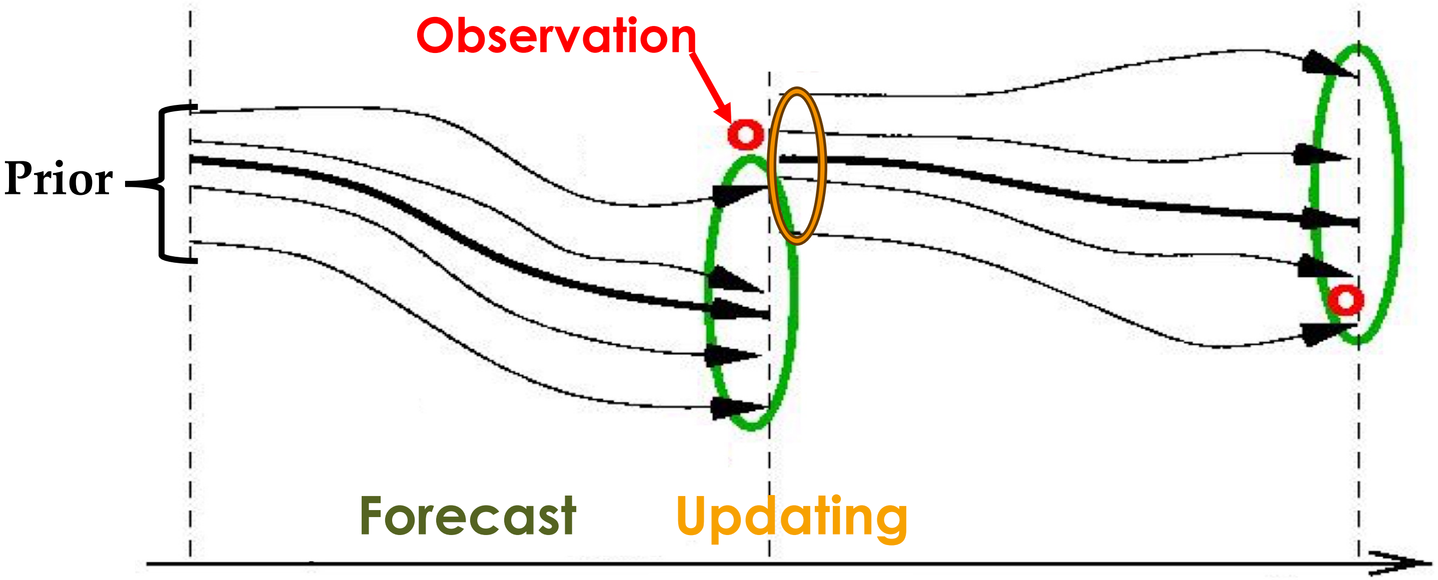
Prior Ensemble

$$\{\boldsymbol{u}_t^{(1)}, \boldsymbol{u}_t^{(2)}, \boldsymbol{u}_t^{(3)}, \dots, \boldsymbol{u}_t^{(L)}\}$$

Updating through the measurements →

Posterior Ensemble

$$\{\hat{\boldsymbol{u}}_t^{(1)}, \hat{\boldsymbol{u}}_t^{(2)}, \hat{\boldsymbol{u}}_t^{(3)}, \dots, \hat{\boldsymbol{u}}_t^{(L)}\}$$



$$\hat{\mathbf{u}}_t^{(j)} = \mathbf{u}_t^{(j)} + \mathbf{K}_t(\mathbf{z}_t^{(j)} - \mathbf{H}\mathbf{u}_t^{(j)})$$

$$\mathbf{K}_t = \hat{\boldsymbol{\Sigma}}_t\mathbf{H}^T(\mathbf{H}\hat{\boldsymbol{\Sigma}}_t\mathbf{H}^T + \mathbf{R}_t)^{-1}$$

Nudging (Newtonian Relaxing)

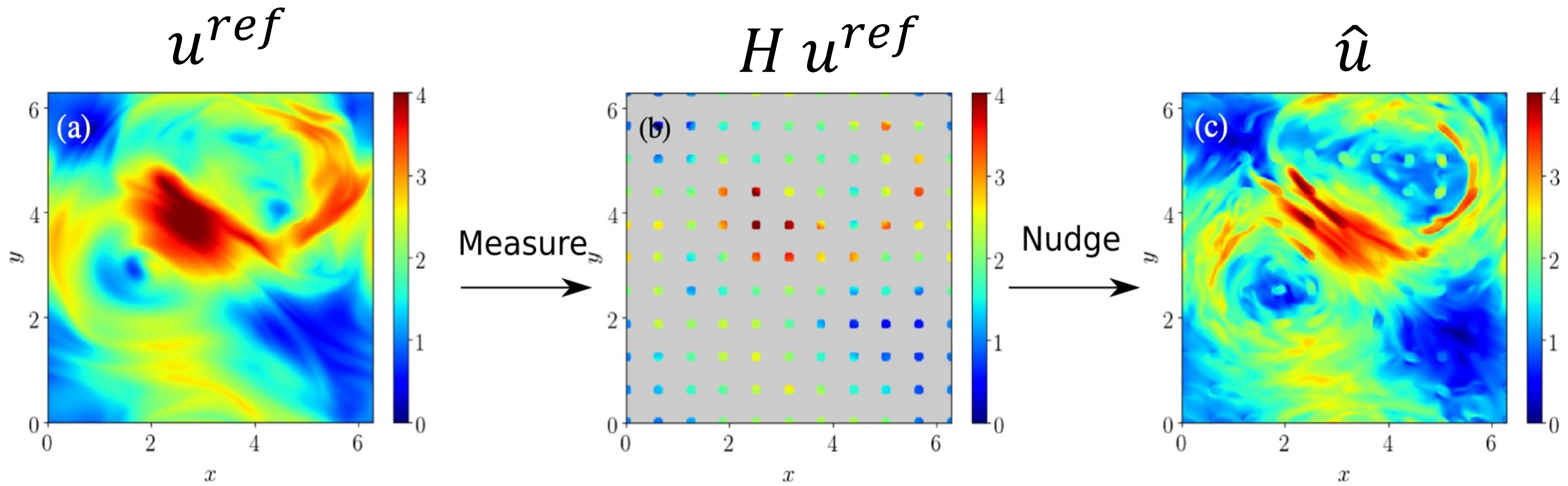
System state Estimation

Coupling coefficent

$$\frac{d\hat{u}_n(t)}{dt} = f_n\big(\hat{u}(t)\big) + \alpha\, H_{nm}(u_n^{ref} - \hat{u}_n)$$

Measurement operator

Reference solution + gaussian error



Simulation Setup

We want to test the assimilation techniques on a simulation time $T = 200\,T_0$, where $T_0 = 0.5$ is the characteristic time of the slowest mode.

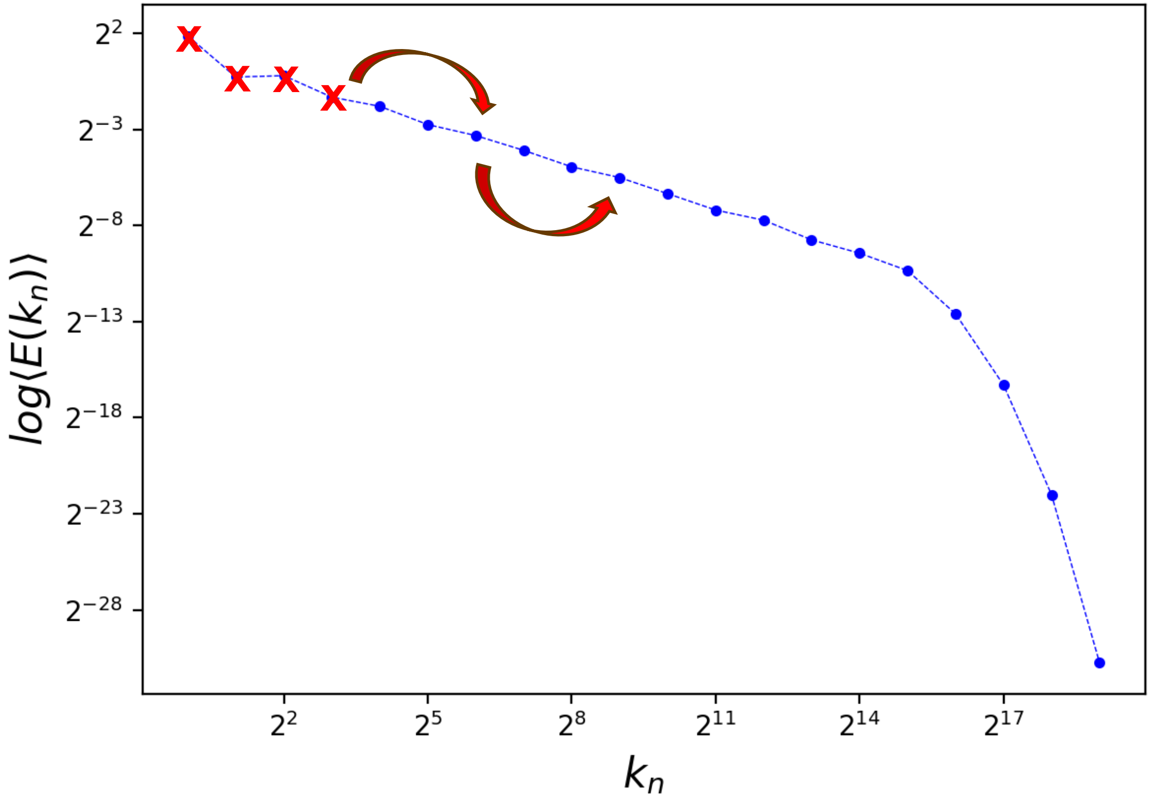
Different measurement frequencies:

- $\tau = 0.002\,T_0$ (measurement every 100 time steps)
- $\tau = 0.02\,T_0$ (every 1000 time steps)
- $\tau = 0.2\,T_0$ (every 10000 time step)

The measurement error of a mode is a percentage of its own energy.

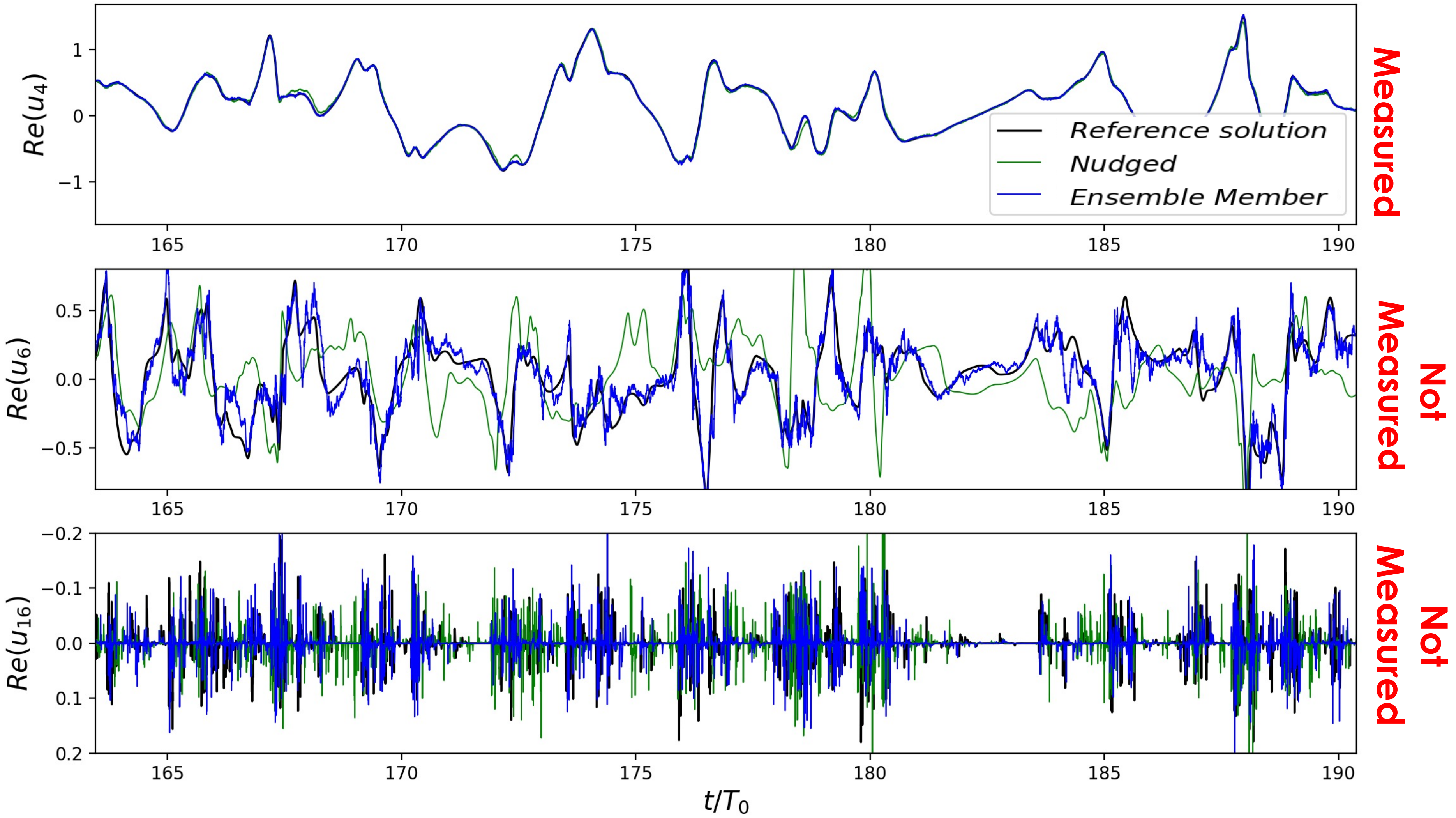
$$\mathcal{E}_n \sim N(0, \sigma_n^2) \qquad \sigma_n^2 = 0.05\,\langle |u_n|^2 \rangle$$

Different sets of measured shells:



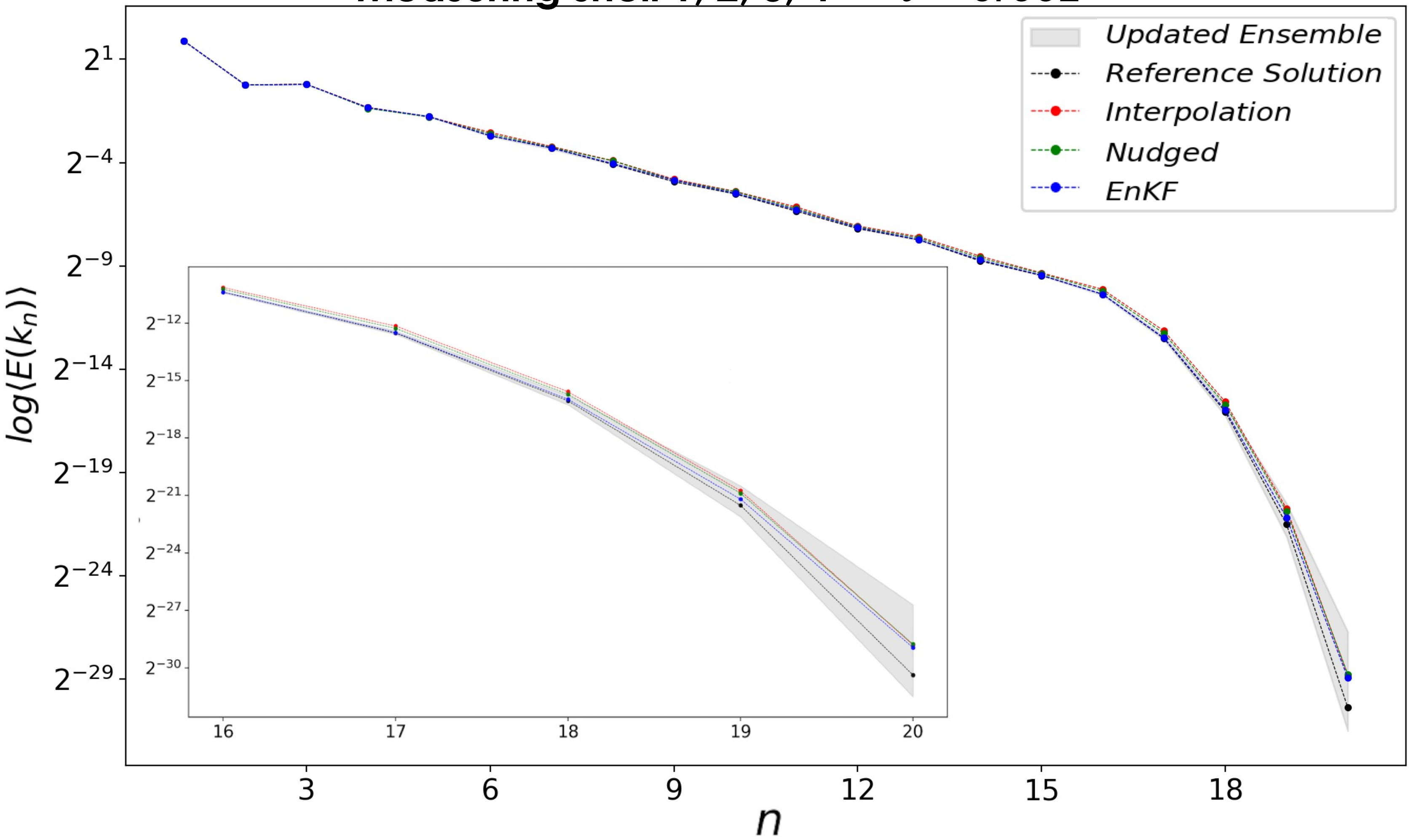
EnKF VS Nudging

Measuring shell 1,2,3,4 with $\tau = 0.002\ T_0$



Statistical Reconstruction

Measuring shell 1, 2, 3, 4 $\tau = 0.002$



Istantaneus Reconstruction: various measurement cadences

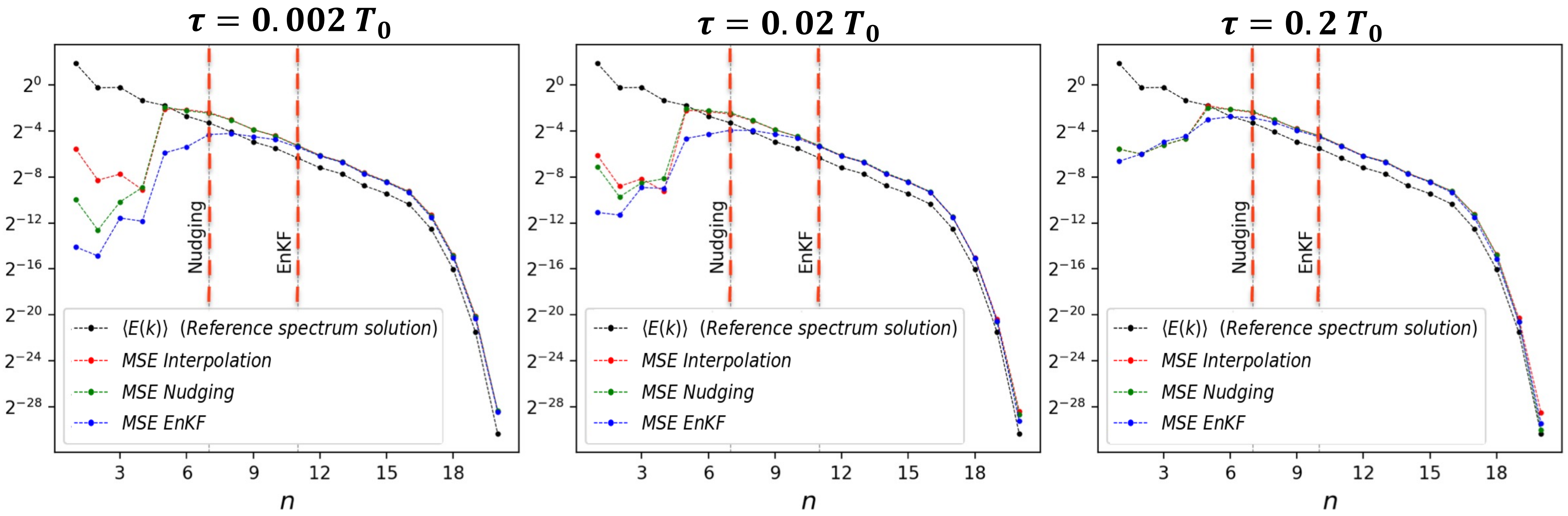
$$\left\langle \left|u^{ref} - \hat{u}\right|^2 \right\rangle = \langle |\hat{u}|^2 \rangle + \left\langle \left|u^{ref}\right|^2 \right\rangle - \left\langle u^{ref}\,\hat{u}^* + \hat{u}\,u^{ref^*} \right\rangle = E\left(u^{ref}\right) + E(\hat{u}) - C\left(\hat{u}, u^{ref}\right)$$

IF :

- $E\left(u^{ref}\right) = E(\hat{u})$ → $\left\langle \left|u^{ref} - \hat{u}\right|^2 \right\rangle = 2\,\mathrm{E}(\mathrm{u}^{\mathrm{ref}})$
- $C\left(\hat{u}, u^{ref}\right) = 0$

The presence of factor 2 means that we are assimilating but not altering the statistic

Measuring shells: 1, 2, 3, 4



Conclusions (**in red suggestions for improving the work**)

- ✓ We have optimized the Nudging process by selecting the average coupling coefficient.
- ✓ We could use different coupling coefficients for the different shells.

- ✓ The EnKF provides the best assimilation performance in the analyzed cases.
- ✓ We could do better by implementing inflation and localization techniques to improve the filtering performance.

- ✓ If we measure n shells through EnKF, we can assimilate up to n+6, while with Nudging, we can assimilate up to n+2. The results are limited to the measurement of the first shells; measuring faster modes, the code becomes unstable (presence of NaN).
- ✓ We need to understand the reason for it.

References

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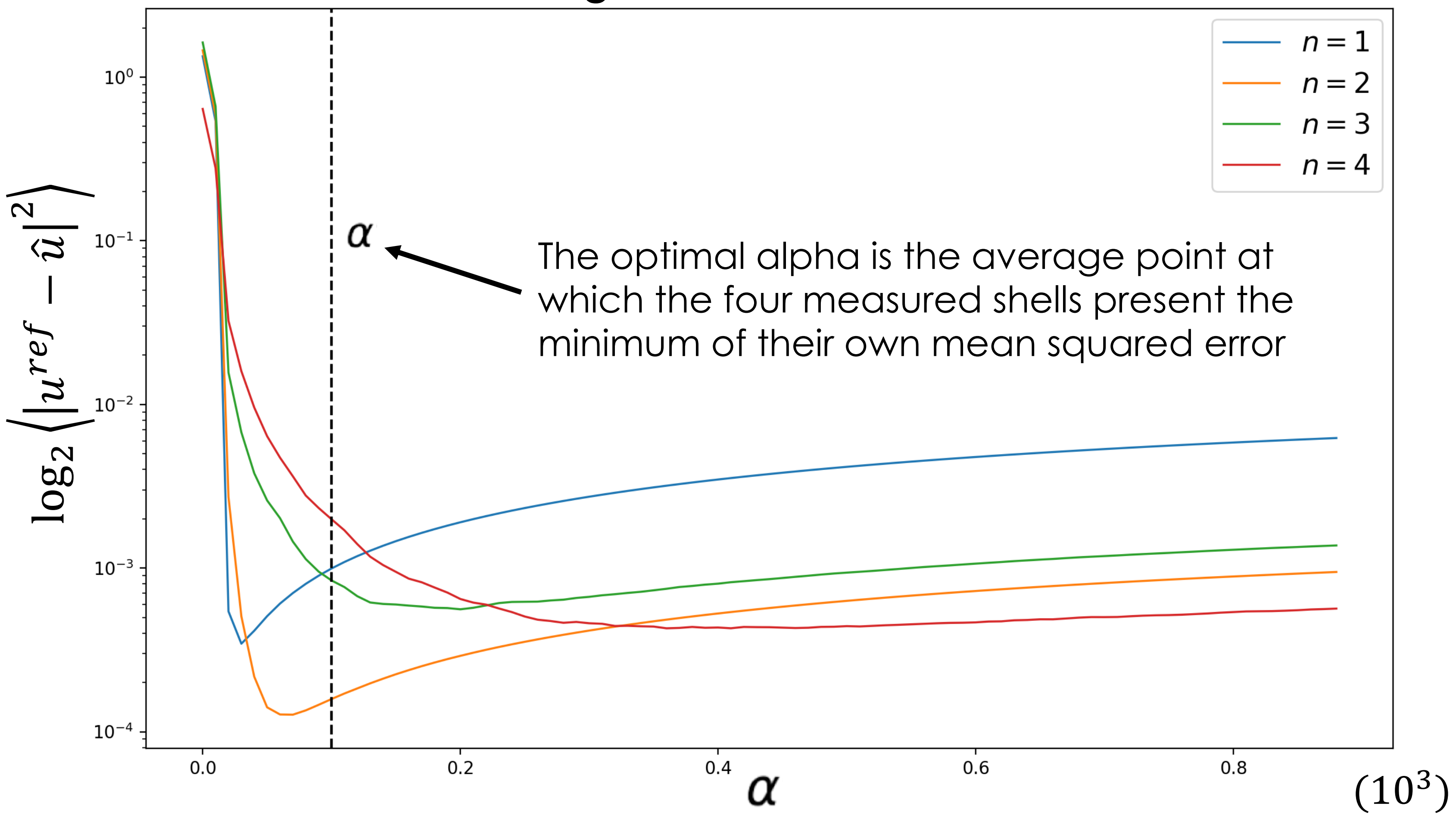
Optimizing Nudging Coefficient

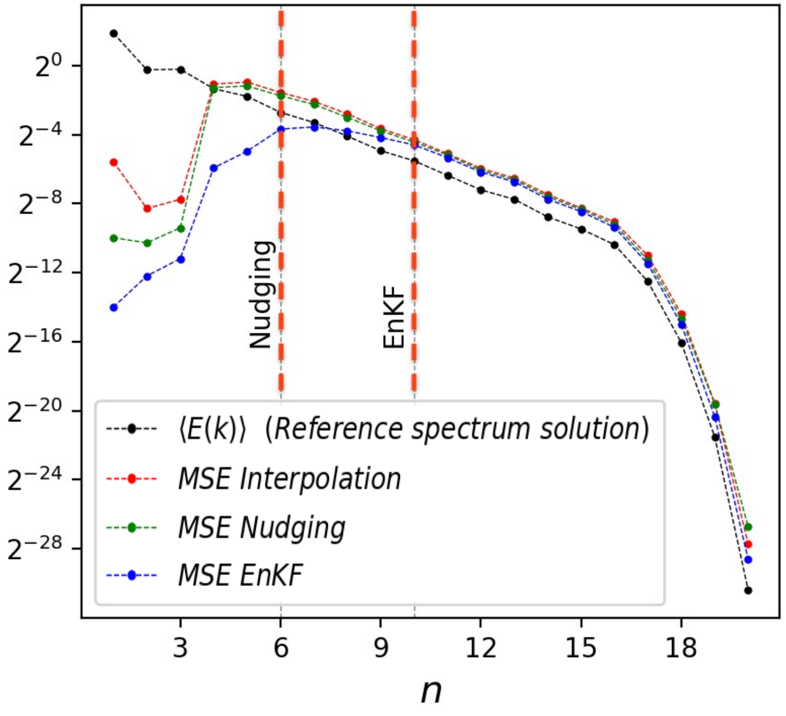
Measuring shell 1, 2, 3, 4 $\tau = 0.002$



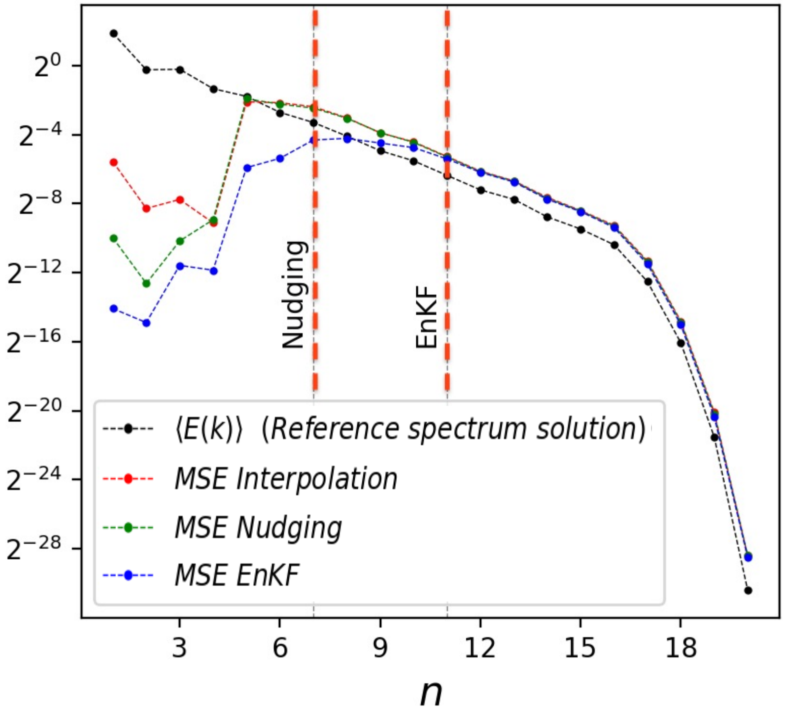
Istantaneus Reconstruction: various set of measured shells

The measurement cadence used is $\tau = 0.002\ T_0$

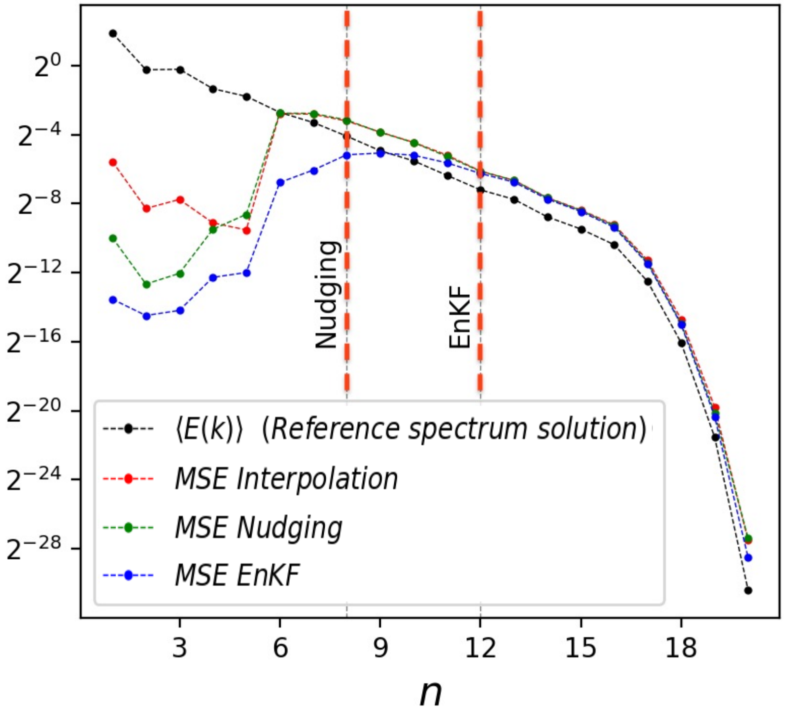
Measuring shells: 1, 2, 3



Measuring shells: 1, 2, 3, 4



Measuring shells: 1, 2, 3, 4, 5



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