Theory of reactor operation

Short lecture in reactor physics for the participants of the 2023 ARIEL Hands-on school on nuclear data from Research Reactors

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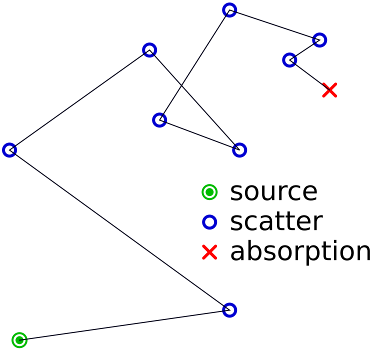
Content

Random walk

Physical model of neutron propagation

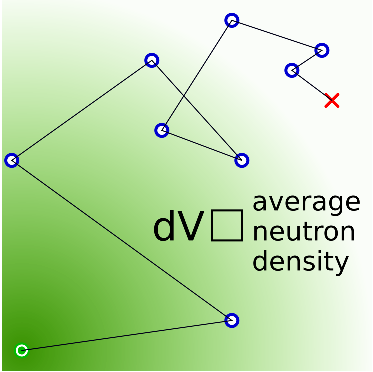
- point particle
- flying in space
- ocassionally colliding with nuclei
  - absorption
  - scatter
  - etc
- sequence of collisions to absorption
- subject of laws of probability
- random walk



Deterministic approach

Physical model of neutron propagation

- quantities of engineering interest
  - reaction rates for heat generation, burnup, etc
  - multiplication factor
- we don't care about fluctuations
- taking ensemble average
- instead of particles: fluid-like representation



Phase space

Fundamental concepts

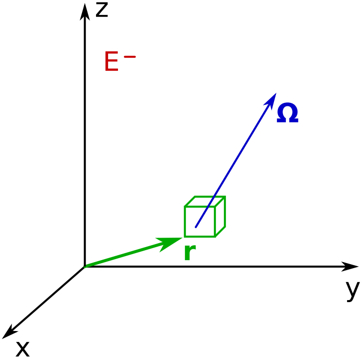
State of a neutron

It's traveling

- at location r,
- in direction Ω,
- and at energy *E*

Phase space

- Possible states of any neutron
- Quantities are functions of these independent variables (r, Ω, *E*)

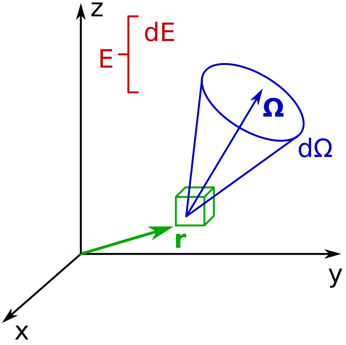


Neutron density

$$n(\mathrm{r}, \Omega, E) \quad \left[\frac{n}{\mathrm{cm}^3\ \mathrm{St\ eV}}\right]$$

Interpretation

Number of neutrons per unit volume of the phase space (unit volume, unit solid angle, and unit energy)



Flux and cross sections

Fundamental concepts

Reaction rates are proportional to

- number (density) of neutrons
- their velocity
- number (density) of nuclei

Proportionality constant:
microscopic cross section

$$R_x = \sigma_x(E) N(\mathrm{r}, E) v(E) n(\mathrm{r}, \Omega, E)$$

Flux and cross sections

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$$R_x = \sigma_x(E)N(r, E)\ v(E)n(r, \Omega, E)$$

$$R_x = \Sigma_x(r, E)\ \varphi(r, \Omega, E)$$

Flux and cross sections

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$$R_x = \Sigma_x(\mathrm{r},E)\, \varphi(\mathrm{r},\Omega,E)$$

Angular flux

$$\varphi(\mathrm{r},\Omega,E) = v(E)n(\mathrm{r},\Omega,E)$$

Macroscopic cross section

$$\Sigma_x(\mathrm{r},E) = \sigma_x(E)N(\mathrm{r},E)$$

Interpretation

Number of neutrons crossing at a small surface, in direction Ω

Interpretation

"Probability" of reaction per unit distance travelled

Scalar flux and current

Fundamental concepts

Scalar flux

$$\Phi(\mathrm{r}, E) = \int\limits_{4\pi} \varphi(\mathrm{r}, \Omega, E)\,\mathrm{d}\Omega$$

Flux of neutrons flying in any direction

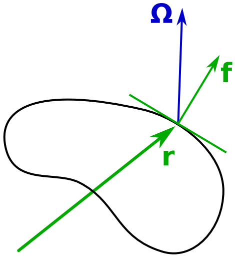
Net current

$$\mathrm{J}(\mathrm{r}, E) = \int\limits_{4\pi} \Omega\varphi(\mathrm{r}, \Omega, E)\,\mathrm{d}\Omega$$

Net number of neutrons crossing a surface

How many neutrons leave a small volume around r?

- Number of neutrons crossing surface f
  $\mathrm{f} \cdot \Omega\varphi(\mathrm{r}, \Omega, E)$
- Number of neutrons leaving V
  $L_V = \oint_F \mathrm{f}(\mathrm{r}) \cdot \Omega\varphi(\mathrm{r}, \Omega, E)\, \mathrm{dr}$
- Apply Gauss's theorem
  $L_V = \int_V \Omega \cdot \nabla\varphi(\mathrm{r}, \Omega, E)\, \mathrm{dr}$
- Streaming loss at r:
  $L = \lim_{V \to \mathrm{r}} \frac{L_V}{V} = \Omega \cdot \nabla\varphi(\mathrm{r}, \Omega, E)$



Gain and loss of neutrons

Balance of neutrons in a point of phase space

$$\frac{\partial n}{\partial t}(\mathrm{r}, \Omega, E, t) = \frac{1}{v}\frac{\partial \varphi}{\partial t}(\mathrm{r}, \Omega, E, t) = \text{source} - \text{loss}$$

Balance of neutrons in a point of phase space

$$\frac{\partial n}{\partial t}(\mathrm{r}, \Omega, E, t) = \frac{1}{v}\frac{\partial \varphi}{\partial t}(\mathrm{r}, \Omega, E, t) = \text{source} - \text{loss}$$

What's loss?

- streaming

$$\Omega \cdot \nabla \varphi(\mathrm{r}, \Omega, E, t)$$

- collision

$$\Sigma_t \varphi(\mathrm{r}, \Omega, E, t)$$

Balance of neutrons in a point of phase space

$$\frac{\partial n}{\partial t}(\mathrm{r}, \Omega, E, t) = \frac{1}{v}\frac{\partial \varphi}{\partial t}(\mathrm{r}, \Omega, E, t) = \text{source} - \text{loss}$$

What's source?

- scatter

$$\int_0^\infty \int_{4\pi} \Sigma_s \left(\mathrm{r}, \Omega' \to \Omega, E' \to E\right) \varphi(\mathrm{r}, \Omega', E', t) \,\mathrm{d}\Omega' \,\mathrm{d}E'$$

- fission

$$\frac{f(E)}{4\pi} \int_0^\infty \int_{4\pi} \nu\Sigma_f \left(\mathrm{r}, E'\right) \varphi(\mathrm{r}, \Omega', E', t) \,\mathrm{d}\Omega' \,\mathrm{d}E'$$

Transport equation

Balance of neutrons

The (first order differential form of the) transport equation

$$\frac{1}{v}\frac{\partial\varphi}{\partial t}(\mathrm{r},\Omega,E,t) = -\Omega\cdot\nabla\varphi(\mathrm{r},\Omega,E,t) - \Sigma_t\varphi(\mathrm{r},\Omega,E,t) + Q(\mathrm{r},\Omega,E,t)$$

Generalised source

$$\begin{aligned} Q(\mathrm{r},\Omega,E,t) = &\\ &\int_0^\infty\int_{4\pi}\Sigma_s\left(\mathrm{r},\Omega'\to\Omega,E'\to E\right)\varphi(\mathrm{r},\Omega',E',t)\,\mathrm{d}\Omega'\,\mathrm{d}E' + \\ &\quad\frac{f(E)}{4\pi}\int_0^\infty\int_{4\pi}\nu\Sigma_f\left(\mathrm{r},E'\right)\varphi(\mathrm{r},\Omega',E',t)\,\mathrm{d}\Omega'\,\mathrm{d}E' + \\ &\qquad S(\mathrm{r},\Omega,E,t)\end{aligned}$$

Solving the transport equation

Balance of neutrons

Very hard to solve because

- complicated dependence on energy E (resonances, thermalisation)
- lack of theoretical understanding (singular solutions)
- number of independent variables $\rightarrow$ combinatorial explosion

Possible routes out of the trouble

- simplification
- numerical methods needed
- but usually both of them

Solving the transport equation

Balance of neutrons

Ideas and approaches

- Point kinetics $\rightarrow$ focus on time dependence
- Diffusion $\rightarrow$ get rid of the angular variable Ω
- Discrete ordinates $\rightarrow$ discretise everything in (r, Ω, E)
- Collision probabilities $\rightarrow$ connect everything to everything
- Monte Carlo $\rightarrow$ simulate random walk
- Multigroup/fewgroup approximation
- Resonance shielding, cross section weighting

Idea

For ease of life, flux could be approximated with linear expansion:

$$\varphi(\mathrm{r}, \Omega, E) \approx \frac{1}{4\pi}\Phi(\mathrm{r}, E) + \frac{3\Omega}{4\pi}\mathrm{J}(\mathrm{r}, E) + \dots$$

Assumptions

Severe and restrictive conditions

- dominant reaction is scatter
- no strong absorbers
- spatial flux variation is weak
- angular dependence of flux is mild

Getting rid of Ω

Neutron diffusion

What you do

- Substitute the linear flux expansion into the transport equation
- Integrate the equation for Ω

What you get

$$\frac{1}{v}\frac{\partial \Phi}{\partial t}(\mathrm{r}, E, t) = -\nabla \mathrm{J}(\mathrm{r}, E, t) - \Sigma_t \Phi(\mathrm{r}, E, t) + Q_0(\mathrm{r}, E, t)$$

- $Q_0(\mathrm{r}, E, t)$ is the isotropic part of the generalised source
- Balance of neutrons, irrespective of their direction of flight
- One equation for two unknowns: $\Phi(\mathrm{r}, E, t)$, $\mathrm{J}(\mathrm{r}, E, t)$
- One more equation is needed

Fick's Law

Neutron diffusion

How to get connection between the flux $\Phi(\mathrm{r})$ and the $\mathrm{J}(\mathrm{r})$ current?

- multiply the transport equation with Ω, and integrate again
- neglect time derivative of the current $(\frac{\partial \mathrm{J}}{\partial t}(\mathrm{r}, E, t) \approx 0)$
- neglect energy change in the anisotropic part of the scatter

Fick's Law

$$\mathrm{J}(\mathrm{r}, E, t) = -D(\mathrm{r}, E)\nabla\Phi(\mathrm{r}, \mathrm{E}, \mathrm{t})$$

- diffusion coefficient $D(\mathrm{r}, E) = \frac{1}{3\Sigma_{tr}(\mathrm{r}, E)}$,
  $\Sigma_{\mathrm{re}}(\mathrm{r}, E) = \Sigma_{\mathrm{t}}(\mathrm{r}, E) - \mathrm{avg}(\cos\theta)\Sigma_{\mathrm{s}}(\mathrm{r}, E)$
- eliminate $\mathrm{J}(\mathrm{r}, E, t)$ from the previous equation

The diffusion equation

Neutron diffusion

Finally the diffusion equation

$$\frac{1}{v}\frac{\partial \Phi}{\partial t}(\mathrm{r}, E, t) = \nabla D(\mathrm{r}, E)\Phi(\mathrm{r}, \mathrm{E}, \mathrm{t}) - \Sigma_t \Phi(\mathrm{r}, E, t) + Q_0(\mathrm{r}, E, t)$$

- generalised source:

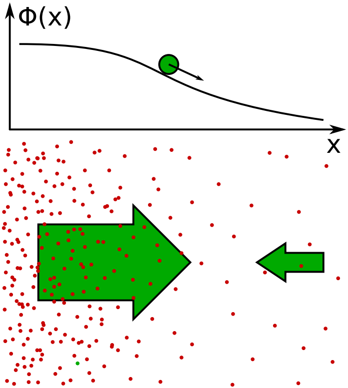
$$Q_0(\mathrm{r}, E, t) = \int_0^\infty \Sigma_{s0}\left(\mathrm{r}, E' \rightarrow E\right)\Phi(\mathrm{r}, E', t)\,\mathrm{d}E' + \\ f(E)\int_0^\infty \nu\Sigma_f\left(\mathrm{r}, E'\right)\Phi(\mathrm{r}, E', t)\,\mathrm{d}E' + S(\mathrm{r}, E, t)$$

- much-much easier to solve
- analytical and numerical solution methods are available

The physics behind Fick's law

Neutron diffusion

- Plenty of neutrons here, they will migrate to the neigbour region
- No neutrons in the neighbour, they can't migrate to here



Figuratively

Neutrons are rolling down from the flux hill.

Criticality and k_{eff}

Neutron diffusion

The diffusion equation

$$\boxed{0} = \frac{1}{v}\frac{\partial \Phi}{\partial t}(\mathrm{r}, E, t) = \nabla D(\mathrm{r}, E)\Phi(\mathrm{r}, \mathrm{E}, \mathrm{t}) - \Sigma_t \Phi(\mathrm{r}, E, t) + Q_0(\mathrm{r}, E, t)$$

- balancing the equation with the multiplication factor

$$Q_0(\mathrm{r}, E, t) = \int_0^\infty \Sigma_{s0}\left(\mathrm{r}, E' \to E\right)\Phi(\mathrm{r}, E', t)\,\mathrm{d}E' + \frac{f(E)}{\boxed{k_{\text{eff}}}}\int_0^\infty \nu\Sigma_f\left(\mathrm{r}, E'\right)\Phi(\mathrm{r}, E', t)\,\mathrm{d}E' + S(\mathrm{r}, E, t)$$

- eigenvalue equation instead of time dependence
- rather general, can be done to the transport equation as well

- Primarily, we would like to know the reactor power as a whole
- time dependent neutron transport/diffusion models are complicated
- too much detail $\rightarrow$ too much work
- seeking some simple approach
- trying to compute overall number of neutrons $N(t)$
- without considering phase space variables $(\mathrm{r}, \Omega, E, t)$

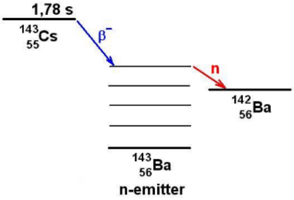
Point kinetic equation

$$\varphi(\mathrm{r}, \Omega, E, t) = N(t)\Phi(\mathrm{r}, \Omega, E)$$

separating time dependence and shape function

- fission neutron lifetime is short, $10 - 100\,\mu s$
- small deviation from $k_{\text{eff}} = 1.0$ results rapid increase reactor power
- fission products have excess neutrons
- strong β^- decay
- some fission neutrons are produced *after* one or more decay
- time scale $0.1\,s - 100\,s$, i.e. significantly later

example: ^{143}Cs



- divided into six groups
- λ_i decay constants
- β_i delayed n, fractions

Overall balance of neutrons

Reactor kinetics

How is the number of neutrons changing during the average life time τ_n of a neutron?

$$N(t+\tau_n) = k_{\text{eff}}(1-\beta)N(t) + \sum_{i=1}^{6} \lambda_i C_i(t)\,\tau_n + S(t)\,\tau_n$$

- $k_{\text{eff}}(1-\beta)N(t)$ increase from chain reaction
- $\sum_{i=1}^{6} \lambda_i C_i(t)\,\tau_n$ emitted by decaying delayed neutron precursors
- $S(t)\,\tau_n$ introduced by external source

Balance of delayed neutron precursors

Further equations are needed for the delayed precursors

$$C_i(t+\tau_n) = C_i(t) + \left[k_{\text{eff}}\beta_i N(t) - \lambda_i C_i(t) \right]\tau_n, \quad \text{where} \quad i = 1 \ldots 6$$

- $k_{\text{eff}}\beta_i N(t)$ precursors produced by fission
- $\lambda_i C_i(t)\tau_n$ precursors decaying to delayed neutrons

Finally using the Taylor expansions
$N(t+\tau_n) \approx N(t) + \tau_n \frac{\mathrm{d}N}{\mathrm{d}t}(t) + \ldots$ and
$C_i(t+\tau_n) \approx C_i(t) + \tau_n \frac{\mathrm{d}C_i}{\mathrm{d}t}(t) + \ldots$ to get $\cdots$

Point kinetic equations

$$\frac{\mathrm{d}N}{\mathrm{d}t}(t) = \frac{\rho - \beta}{\Lambda} N(t) + \sum_{i=1}^{6} \lambda_i C_i + S(t)$$

$$\frac{\mathrm{d}C_i}{\mathrm{d}t}(t) = \frac{\beta_i}{\Lambda} N(t) - \lambda_i C_i(t)$$

With

Reactivity

$$\rho = \frac{k_{\mathrm{eff}} - 1}{k_{\mathrm{eff}}}$$

Generation time

$$\Lambda = \frac{\tau_n}{k_{\mathrm{eff}}}$$

Assuming $S(t) = 0$, the solution can be written as

Ansatz

$$N(t) = N^0 \exp(\omega t)$$
$$C_i(t) = C_i^0 \exp(\omega t)$$

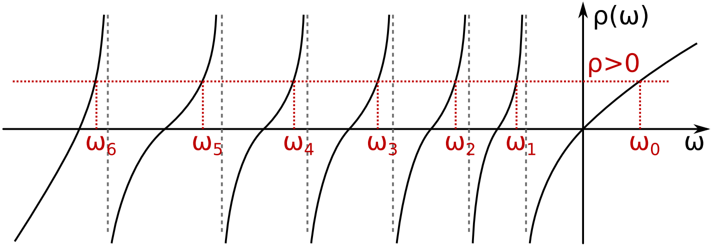
Substituting this into the point kinetic equations a formula for period ω can be obtained.

Charachteristic polynomial of the system

Relation between reactivity and reactor period

$$\frac{\rho}{\beta} = \frac{\Lambda}{\beta}\omega + \omega \sum_{i=1}^{6} \frac{\beta_i/\beta}{\lambda_i + \omega}$$

one eigenvalue can be positive or negative, depending on reactivity



Full solution of the point kinetic equations

The full solution can be written as the sum of 7 modes

$$N(t) = \sum_{j=0}^{6} N_j^0 \exp(\omega_j t)$$

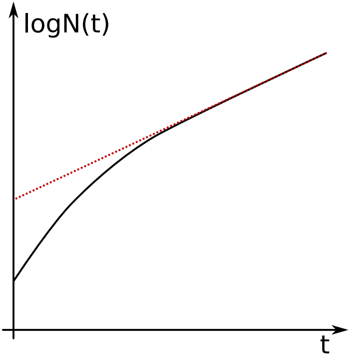
$$C_i(t) = \sum_{j=0}^{6} C_{i,j}^0 \exp(\omega_j t)$$

N_j^0 and $C_{i,j}^0$ can be tuned to fit initial condition.

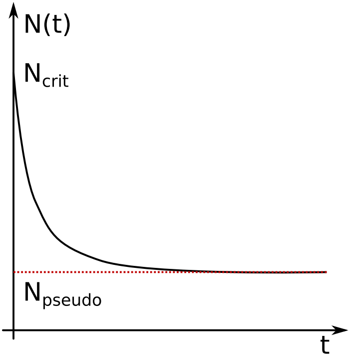
- critical reactor
- sudden increase in reactivity
- sum of 7 exponentials
- $\omega_0 > 0 > \omega_j$, where $j = 1 \dots 6$

Dominant mode

$\omega_0 > 0$ assymptotic behaviour



- critical reactor
- control rod dropped suddenly
- sum of 7 exponentials
- $\omega_6 \ll \omega_5 < \omega 4 < \ldots \omega 0 < 0$
- pseudostable flux after $t = 0.05\,\mathrm{s}$

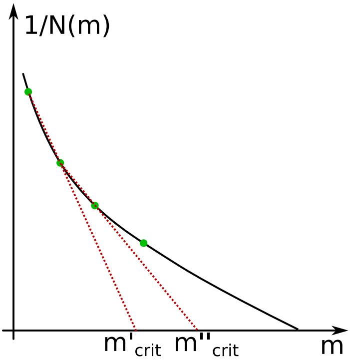


Reactivity estimation

$$\frac{\rho}{\beta} = 1 - \frac{N_{\text{crit}}}{N_{\text{pseudo}}}$$

Subcritical neutron multiplication

- subcritical reactor with source
- number of neutrons in the core $N = \tau_n \left(S_0 + S_0 k_{\text{eff}} + S_0 k_{\text{eff}}^2 + \dots\right)$
- geometric series



Inverse proportionality

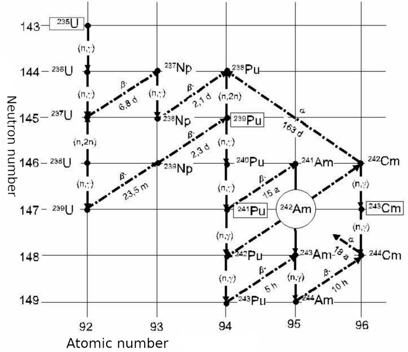
$$N = \frac{S_0 \tau_n}{1 - k_{\text{eff}}} = \frac{S_0 \Lambda}{-\rho}$$

estimation of critical mass, rod position, etc

Change of composition

Burnup

- nuclear reactions transmute isotopes from one into another
- composition of the reactor slowly changing
- main processes
  - fission into mid-size nuclei, e.g. $^{235}U + n \rightarrow$ fission products
  - build up of transuranium isotopes e.g. $^{238}U + n \rightarrow {}^{239}Np \rightarrow {}^{239}Pu + n \rightarrow {}^{240}Pu + n \rightarrow {}^{241}Pu$, etc
  - Activation of structural materials



Mathematical treatment of burnup

Equation for the number density (Bateman equations)

$$\frac{\mathrm{d}N_i}{\mathrm{d}t}(t) = -\left(\sigma_i^{\mathrm{a}}\Phi + \lambda_i\right) N_i(t) + \sigma_j^{\mathrm{c}}\Phi N_j(t) + \lambda_k N_k(t)$$

- balance equation of rates of production and loss
- system of ordinary differential equations
- a few dozen or hundred of coupled equations
- at afirst glance, this is an easy job

Reality kicks in

1. spatial dependence
2. cross sections and flux are changing in time
3. for real life reactors → bloody hard job → more in later lectures

The End

Thank you for your attention