Rapidity-only evolution of TMDs

Giovanni Antonio Chirilli

University of Salento - Lecce

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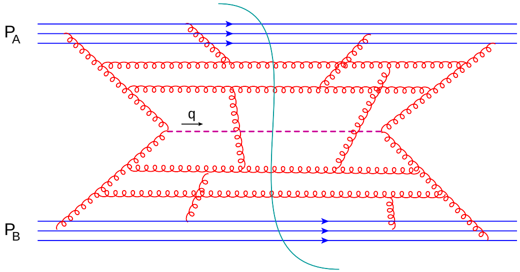
Outline

- Motivation.
- Rapidity factorization for particle production in hadron-hadron collisions.
- Conformal invariance of TMD Operators.
- TMD Rapidity-only evolution equation in Sudakov region with running coupling.
- Conclusions

Particle production in hadron-hadron collisions

e.g. production of Higgs particle



- typical TMD region: $s \sim q^2 = m_X^2 \gg q_\perp^2 \sim 1$Gev
- Sudakov region: $s \sim q^2 = m_X^2 \gg q_\perp^2 \gg 1$Gev
- small-x region : $s \gg q^2 \sim q_\perp^2 \gg m_N^2$

TMD factorization

$$\frac{d\sigma}{d\eta d^2 q_\perp} = \sum_f \int d^2 b_\perp e^{i(q,b)_\perp} \mathcal{D}_{f/A}(x_A, b_\perp, \eta) \mathcal{D}_{f/B}(x_B, b_\perp, \eta) \sigma(ff \to H)$$

$$+ \text{ power corrections} + \text{Y}-\text{terms}$$

- rapidity: η
- $\mathcal{D}_{f/A}$: TMD density of parton f in hadron A.
- $\sigma(ff \to H)$: cross section of production of particle H of invariant mass $m_H^2 = Q^2$ from two partons scattering.
- Power corrections: $\frac{q_\perp^2}{Q^2}$.
- Y-terms: for $q_\perp^2 \sim q^2$ allow transition to collinear factorization formula.

- TMD evolution equation are analyzed by different methods at moderate x_B, CSS and SCET, and at small-x_B resulting in different evolution equations.

- At the future Electron Ion Collider, TMD will be probed from low to high x_B. It is then necessary to develop a formalism which is valid in both limits.

- In the region of moderate x_B TMD analysis is performed with a combination of UV and rapidity cutoff which results in two evolution equations, in μ^2 and ζ (related to rapidity).
  - Such evolution equations are known at two and three-loop, but their relation to the conformal properties of TMD is not known.

Motivation

- I. Balitsky and A. Tarasov (2016): Evolution equation for gluon TMD valid for all $x_B = \beta_B$ and all $k_\perp (\geq 1\text{GeV})$.

$$
\begin{aligned}
\frac{d}{d\ln\sigma}\tilde{\mathcal{F}}^a_i(\beta_B,x_\perp)\mathcal{F}^a_j(\beta_B,y_\perp) \;=\; & -\alpha_s\mathrm{Tr}\Big\{\int d^2k_\perp (x_\perp|\Big\{U^\dagger\frac{1}{\sigma\beta_B s+p_\perp^2}(Uk_k+p_kU)\frac{\sigma\beta_B s g_{\mu i}-2k^\perp_\mu k_i}{\sigma\beta_B s+k_\perp^2} \\
& -2k^\perp_\mu g_{ik}U^\dagger\frac{1}{\sigma\beta_B s+p_\perp^2}U-2g_{\mu k}U^\dagger\frac{p_i}{\sigma\beta_B s+p_\perp^2}U+\frac{2k^\perp_\mu}{k_\perp^2}g_{ik}\Big\}\tilde{\mathcal{F}}^k\big(\beta_B+\frac{k_\perp^2}{\sigma s}\big)|k_\perp) \\
& \times (k_\perp|\mathcal{F}^l\big(\beta_B+\frac{k_\perp^2}{\sigma s}\big)\Big\{\frac{\sigma\beta_B s\delta^\mu_j-2k^\mu_\perp k_j}{\sigma\beta_B s+k_\perp^2}(k_lU^\dagger+U^\dagger p_l)\frac{1}{\sigma\beta_B s+p_\perp^2}U \\
& -2k^\mu_\perp g_{jl}U^\dagger\frac{1}{\sigma\beta_B s+p_\perp^2}U-2\delta^\mu_l U^\dagger\frac{p_j}{\sigma\beta_B s+p_\perp^2}U+2g_{jl}\frac{k^\mu_\perp}{k_\perp^2}\Big\}|y_\perp) \\
& +2\tilde{\mathcal{F}}_i(\beta_B,x_\perp)(y_\perp|\frac{p^m}{p_\perp^2}\mathcal{F}_k(\beta_B)(i\overleftarrow{\partial}_l+U_l)(2\delta^k_m\delta^l_j-g_{jm}g^{kl})U^\dagger\frac{1}{\sigma\beta_B s-p_\perp^2+i\epsilon}U \\
& \qquad +\mathcal{F}_j(\beta_B)\frac{\sigma\beta_B s}{p_\perp^2(\sigma\beta_B s-p_\perp^2+i\epsilon)}|y_\perp) \\
& +2(x_\perp|-U^\dagger\frac{1}{\sigma\beta_B s-p_\perp^2-i\epsilon}U(2\delta^k_i\delta^l_m-g_{im}g^{kl})(i\partial_k-U_k)\tilde{\mathcal{F}}_l(\beta_B)\frac{p^m}{p_\perp^2} \\
& \qquad +\tilde{\mathcal{F}}_i(\beta_B)\frac{\sigma\beta_B s}{p_\perp^2(\sigma\beta_B s-p_\perp^2-i\epsilon)}|x_\perp)\mathcal{F}_j(\beta_B,y_\perp)\Big\} \;+\; O(\alpha_s^2)
\end{aligned}
$$

Light-cone limit $\Rightarrow$ DGLAP

$$\frac{d}{d\eta}\alpha_s\,\mathcal{D}(x_B,0_\perp,\eta)=\frac{\alpha_s}{\pi}N_c\int_{x_B}^{1}\frac{dz'}{z'}\Big[\Big(\frac{1}{1-z'}\Big)_+ +\frac{1}{z'}-2+z'(1-z')\Big]\alpha_s\mathcal{D}(\frac{x_B}{z'},0_\perp,\eta)$$

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Low-x limit: $p_\perp \sim (x-y)_\perp^{-2} \ll s$ non-linear evolution equation

$$\frac{d}{d\eta}\tilde{U}_i^a(x)U_j^a(y)$$
$$= -\frac{\alpha}{2\pi^2}\mathrm{Tr}\{(-i\partial_i^x + \tilde{U}_i^x)\Big[\int d^2z(\tilde{U}_x\tilde{U}_z^\dagger - 1)\frac{(x-y)^2}{(x-z)^2(y-z)^2}(U_zU_y^\dagger - 1)\Big](i\overleftarrow{\partial}_j^y + U_j^y)\}$$

with $U_i = \partial_i U$ and $\frac{(x-y)^2}{(x-z)^2(y-z)^2}$ BK kernel

Light-cone limit $\Rightarrow$ DGLAP

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with $U_i=\partial_i U$ and $\frac{(x-y)^2}{(x-z)^2(y-z)^2}$ BK kernel

Double-Log: $1\gg\sigma\gg\frac{(x-y)_\perp^{-2}}{s}$ and $\sigma x_B s\gg p_\perp^2(x-y)_\perp^{-2}$ $\eta=\ln\sigma$

$$\frac{d}{d\eta}\mathcal{D}(x_B,z_\perp,\ln\sigma)=\frac{\alpha_sN_c}{\pi}\mathcal{D}(x_B,z_\perp,\ln\sigma)\int\frac{d^2p_\perp}{p_\perp^2}[e^{i(p,z)_\perp}-1]$$

Motivation

Result is complicated and not unique. Conformal Invariance may help.

$$
\begin{aligned}
\frac{d}{d\ln\sigma}\tilde{\mathcal{F}}_i^a(\beta_B,x_\perp)\mathcal{F}_j^a(\beta_B,y_\perp) \;=\; & -\alpha_s \mathrm{Tr}\Big\{\int d^2k_\perp (x_\perp|\Big\{U^\dagger\frac{1}{\sigma\beta_B s + p_\perp^2}(Uk_k + p_k U)\frac{\sigma\beta_B s g_{\mu i} - 2k_\mu^\perp k_i}{\sigma\beta_B s + k_\perp^2} \\
& - 2k_\mu^\perp g_{ik}U^\dagger\frac{1}{\sigma\beta_B s + p_\perp^2}U - 2g_{\mu k}U^\dagger\frac{p_i}{\sigma\beta_B s + p_\perp^2}U + \frac{2k_\mu^\perp}{k_\perp^2}g_{ik}\Big\}\tilde{\mathcal{F}}^k\big(\beta_B + \frac{k_\perp^2}{\sigma s}\big)|k_\perp) \\
& \times (k_\perp|\mathcal{F}^l\big(\beta_B + \frac{k_\perp^2}{\sigma s}\big)\Big\{\frac{\sigma\beta_B s\delta_j^\mu - 2k_\perp^\mu k_j}{\sigma\beta_B s + k_\perp^2}(k_l U^\dagger + U^\dagger p_l)\frac{1}{\sigma\beta_B s + p_\perp^2}U \\
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& + 2\tilde{\mathcal{F}}_i(\beta_B,x_\perp)(y_\perp|\frac{p^m}{p_\perp^2}\mathcal{F}_k(\beta_B)(i\overleftarrow{\partial}_l + U_l)(2\delta_m^k\delta_j^l - g_{jm}g^{kl})U^\dagger\frac{1}{\sigma\beta_B s - p_\perp^2 + i\epsilon}U \\
& + \mathcal{F}_j(\beta_B)\frac{\sigma\beta_B s}{p_\perp^2(\sigma\beta_B s - p_\perp^2 + i\epsilon)}|y_\perp) \\
& + 2(x_\perp| - U^\dagger\frac{1}{\sigma\beta_B s - p_\perp^2 - i\epsilon}U(2\delta_i^k\delta_m^l - g_{im}g^{kl})(i\partial_k - U_k)\tilde{\mathcal{F}}_l(\beta_B)\frac{p^m}{p_\perp^2} \\
& + \tilde{\mathcal{F}}_i(\beta_B)\frac{\sigma\beta_B s}{p_\perp^2(\sigma\beta_B s - p_\perp^2 - i\epsilon)}|x_\perp)\mathcal{F}_j(\beta_B,y_\perp)\Big\} \;+\; O(\alpha_s^2)
\end{aligned}
$$

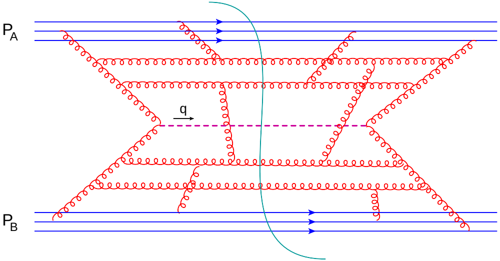
Conformal properties of 6-point functions in $\mathcal{N}$=4 SYM

For TMD we need to study 6-point correlation functions in $\mathcal{N}$=4 SYM theory

$\mathcal{S} \equiv \frac{4\pi^2\sqrt{2}}{\sqrt{N_c^2-1}}\mathrm{Tr}\{Z^2\}$ $\quad (Z = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2))$ renorm-invariant chiral primary operator

$$\langle \mathcal{S}(z_1)\mathcal{S}(z_2)\mathcal{S}(z_3)\mathcal{S}(z_4)\mathcal{S}(z_5)\mathcal{S}(z_6)\rangle$$

8 (out of 9) conformal ratios will contribute.



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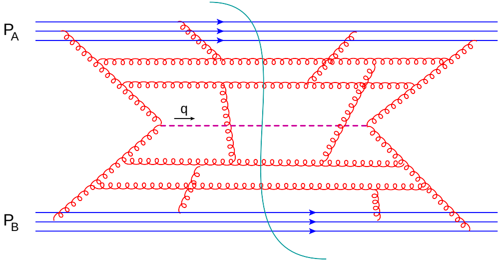
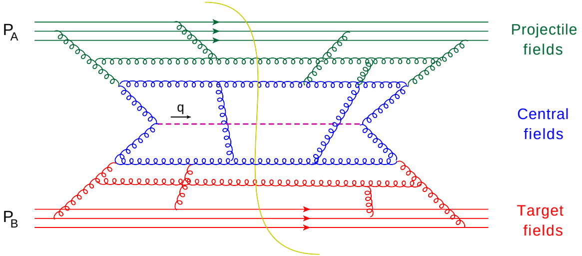


Simpler case has been studied: 4-point correlation functions in the Regge limit: only two conformal ratios.

Particle production in hadron-hadron collisions

Rapidity Factorization

I. Balitsky and G.A.C. (2008),
I. Balitsky and A. Tarasov (2015)



- Projectile fields: $k^- < \sigma_a \quad \Rightarrow$ Projectile TMD
- Central fields: coefficient functions
- Target fields: $k^+ < \sigma_b \quad \Rightarrow$ Target TMD

TMD factorization in coordinate space

$$\mathcal{F}^{i,a}(z_\perp, z^+) \equiv gF^{-i,m}(z)\left[\mathrm{P}e^{ig\int_{-\infty}^{z^+}dz^+ A^{-,\sigma}(up_1+x_\perp)}\right]^{ma}$$

$$\begin{aligned}&\langle \mathcal{S}(z_1)\mathcal{S}(z_2)\mathcal{S}(z_3)\mathcal{S}(z_4)\mathcal{F}^2(x)\mathcal{F}^2(y)\rangle\\ &= \langle \mathcal{S}(z_1)\mathcal{S}(z_2)\mathcal{F}_i(x^+, x_\perp)\mathcal{F}_j(y^+, y_\perp)\rangle\langle \mathcal{S}(z_3)\mathcal{S}(z_4)\mathcal{F}_i(x^-, x_\perp)\mathcal{F}_j(y^-, y_\perp)\rangle\end{aligned}$$

Sudakov region:

$$Q^2 \gg q_\perp^2 \quad \Leftrightarrow \quad (x-y)^+(x-y)^- \ll (x-y)_\perp^2$$

Small-x region:

$$s \gg Q^2 \sim q_\perp^2 \quad \Leftrightarrow \quad (x-y)^+(x-y)^- \sim (x-y)_\perp^2 \qquad z_1^-,\, z_3^+ \to \infty \ \text{ and } \ z_2^-,\, z_4^+ \to -\infty$$

Small-x and Sudakov region:

$$s \gg Q^2 \gg q_\perp^2 \quad \Leftrightarrow \quad (x-y)^+(x-y)^- \ll (x-y)_\perp^2 \qquad z_1^-,\, z_3^+ \to \infty \ \text{ and } \ z_2^-,\, z_4^+ \to -\infty$$

TMD factorization in coordinate space

$$\langle p_A, p_B | g^2 F^a_{\mu\nu} F^{a\mu\nu}(z_1) g^2 F^b_{\lambda\rho} F^{b\lambda\rho}(z_2) | p_A, p_B \rangle$$
$$= \frac{1}{N_c^2 - 1} \langle p_A | \tilde{\mathcal{O}}_{ij}(z_1^-, z_{1_\perp}; z_2^-, z_{2_\perp}) | p_A \rangle^{\sigma_A} \langle p_B | \mathcal{O}^{ij}(z_1^+, z_{1_\perp}; z_2^+, z_{2_\perp}) | p_B \rangle^{\sigma_B} \ + ...$$

Gluon TMD operator

$$\mathcal{O}_{ij}(z_1^+, z_{1_\perp}; z_2^+, z_{2_\perp}) = \ \mathcal{F}^a_i(z_1)[z_1 - \infty n, z_2 - \infty n]^{ab} \mathcal{F}^b_j(z_2) \Big|_{z_1^- = z_2^- = 0},$$

$$\tilde{\mathcal{O}}_{ij}(z_1^-, z_{1_\perp}; z_2^-, z_{2_\perp}) = \mathcal{F}^a_i(z_1)[z_1 - \infty n', z_2 - \infty n']^{ab} \mathcal{F}^b_j(z_2) \Big|_{z_1^+ = z_2^+ = 0},$$

$$\left(\mathcal{F}^{i,a}(z_\perp, z^+)\right)^\sigma \equiv g F^{-i,m}(z) \left[\mathrm{P} e^{ig \int_{-\infty}^{z^+} dz^+ A^{-,\sigma}(u p_1 + x_\perp)} \right]^{ma},$$

TMD factorization in coordinate space

Gluon TMD operator

$$\mathcal{O}_{ij}(z_1^+, z_{1_\perp}; z_2^+, z_{2_\perp}) = \mathcal{F}_i^a(z_1)[z_1 - \infty n, z_2 - \infty n]^{ab} \mathcal{F}_j^b(z_2)\Big|_{z_1^- = z_2^- = 0},$$

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$$\left(\mathcal{F}^{i,a}(z_\perp, z^+)\right)^\sigma \equiv g F^{-i,m}(z)\left[\mathrm{P} e^{ig\int_{-\infty}^{z^+} dz^+ A^{-,\sigma}(up_1 + x_\perp)}\right]^{ma},$$

$$A_\mu^\sigma(x) = \int \frac{d^4k}{16\pi^4}\, \theta\left(\frac{\sigma\sqrt{2}}{z_{12_\perp}} - |k^+|\right) e^{-ik\cdot x} A_\mu(k)$$

$\frac{\sigma\sqrt{2}}{z_{12_\perp}}$ cutoff preserving conformal invariance

$$[x, y] \equiv \mathrm{P} e^{ig\int du (x-y)^\mu A_\mu(ux + (1-u)y)}$$

TMD factorization in coordinate space

Gluon TMD operator

$$\mathcal{O}_{ij}(z_1^+, z_{1_\perp}; z_2^+, z_{2_\perp}) = \mathcal{F}_i^a(z_1)[z_1 - \infty n, z_2 - \infty n]^{ab} \mathcal{F}_j^b(z_2)\Big|_{z_1^- = z_2^- = 0},$$

$$\tilde{\mathcal{O}}_{ij}(z_1^-, z_{1_\perp}; z_2^-, z_{2_\perp}) = \mathcal{F}_i^a(z_1)[z_1 - \infty n', z_2 - \infty n']^{ab} \mathcal{F}_j^b(z_2)\Big|_{z_1^+ = z_2^+ = 0},$$

TMDs are invariant under the inversion

$$\begin{aligned}\mathcal{F}_i^m(z_\perp, z^+) = \quad & F^{-i,n}(z^+, z_\perp)[z^+, z^+ - \infty n]^{nm} \\ & \to F^{-i,n}\left(\frac{z^+}{z_\perp^2}, \frac{z_\perp}{z_\perp^2}\right)\left[\frac{z^+}{z_\perp}, \frac{z^+}{z_\perp^2} - \infty n\right]^{nm} = \mathcal{F}_i^m(z'_\perp, z'^+)\end{aligned}$$

We need the full conformal group of TMD operators.

Conformal SO(2,4) group

Conformal group has 15 generators: Poincare + Dilatation + Special conformal trasformation (inversion + shift + inversion)

$$
\begin{aligned}
&i[M_{\mu\nu}, M_{\alpha\beta}] = g_{\mu\alpha}M_{\nu\beta} + g_{\nu\beta}M_{\mu\alpha} - g_{\mu\beta}M_{\nu\alpha} - g_{\nu\alpha}M_{\mu\beta} \\
&i[M_{\alpha\beta}, P_\mu] = g_{\alpha\mu}P_\beta - g_{\beta\mu}P_\alpha \\
&i[M_{\alpha\beta}, K_\mu] = g_{\alpha\mu}K_\beta - g_{\beta\mu}K_\alpha \\
&i[D, P_\mu] = P_\mu, \quad i[D, K_\mu] = -K_\mu, \quad i[K_\mu, P_\nu] = 2(g_{\mu\nu}D + M_{\mu\nu})
\end{aligned}
$$

Action on scalar field of canonical dimension Δ;

$$
\begin{aligned}
&i[D, \Phi(x)] = (x^\alpha\partial_\alpha + \Delta)\Phi(x) \\
&i[K^\mu, \Phi(x)] = (2x^\mu x^\alpha \partial_\alpha - x^2\partial^\mu + 2\Delta x^\mu)\Phi(x)
\end{aligned}
$$

quantum correction: $\Delta \to \Delta + \text{anomalous}$

Conformal invariance of TMD operator

Conformal $SO(2,4)$ group has 15 generators

TMD operator transform covariantly under 11 generators:

TMD Conformal group

$$P^i, \;\; P^-, \;\; M^{12}, \;\; M^{-i}, \;\; D, \;\; K^i, \;\; K^-, \;\; M^{-+}$$

Generators P^+, K^+, M^{+i} do not preserve the form of $\mathcal{F}^{-j}$.

Conformal invariance of TMD operator

Action of conformal generators on TMD operators

$$-iP^i\mathcal{F}^{-j}(x^+,x_\perp) = \partial^i\mathcal{F}^{-j}(x^+,x_\perp)\,, \qquad -iP^-\mathcal{F}^{-j}(x^+,x_\perp) = \frac{\partial}{\partial x^+}\mathcal{F}^{-j}(x^+,x_\perp)\,,$$

$$-iM^{-i}\mathcal{F}^{-j}(x^+,x_\perp) = -x^i\frac{\partial}{\partial x^+}\mathcal{F}^{-j}(x^+,x_\perp)\,,$$

$$-iD\mathcal{F}^{-i}(x^+,x_\perp) = (x^+\frac{\partial}{\partial x^+} + x^k\frac{\partial}{\partial x^k} + 2)\mathcal{F}^{-i}(x^+,x_\perp)\,,$$

$$-iM^{ij}\mathcal{F}^{-k}(x^+,x_\perp) = (x^i\partial^j - x^j\partial^i)\mathcal{F}^{-k}(x^+,x_\perp) + g^{ik}\mathcal{F}^{-j}(x^+,x_\perp) - g^{jk}\mathcal{F}^{-i}(x^+,x_\perp)\,,$$

$$\begin{aligned}-K^i\mathcal{F}^{-j}(x^+,x_\perp) = {} & 2x^i(x^+\frac{\partial}{\partial x^+} + x^k\frac{\partial}{\partial x^k} + 2)\mathcal{F}^{-j}(x^+,x_\perp) + x_\perp^2\frac{\partial}{\partial x^i}\mathcal{F}^{-j}(x^+,x_\perp) \\ & -2x^j\mathcal{F}^{-i}(x^+,x_\perp) + 2g^{ij}x_i\mathcal{F}^{-i}(x^+,x_\perp)\,,\end{aligned}$$

$$-iK^-\mathcal{F}^{-j}(x^+,x_\perp) = x_\perp^2\frac{\partial}{\partial x^+}\mathcal{F}^{-j}(x^+,x_\perp)\,,$$

$$-iM^{+-}\mathcal{F}^{-i}(x^+,x_\perp) = (x^+\frac{\partial}{\partial x^+} + 1)\mathcal{F}^{-i}(x^+,x_\perp)$$

Conformal TMD group in the embedding formalism

Conformal TMD group in the 6-dim space; $g^{-2,-2} = 1, \; g^{-1,-1} = -1$

Conf. Transf. are Lorentz transf. of light-rays $\left(\frac{1-x^2}{2}, \frac{1+x^2}{2}, x_\mu\right)$ in 6-dim space with metric $(1, -1, 1, -1, -1, -1)$

$$L_{\mu\nu} \equiv M_{\mu\nu}\,, \quad L_{-2,\mu} \equiv \frac{1}{2}(P_\mu - K_\mu)\,, \quad L_{-1,\mu} \equiv \frac{1}{2}(P_\mu + K_\mu)\,, \quad L_{-2,-1} \equiv D$$

$$i[L_{ab}, L_{mn}] = g_{ma}L_{nb} + g_{nb}L_{ma} - g_{mb}L_{na} - g_{na}L_{mb}$$

Define

$$L_{mn} \equiv \mathbb{M}_{mn} \qquad L_{-n} \equiv \mathbb{P}_n \qquad L_{+-} \equiv \mathbb{D}$$

with $m, n, l = -2, -1, 1, 2$, get usual Poincare generators $\mathbb{M}_{mn}, \mathbb{P}_n$ and the "Dilatation" $\mathbb{D}$ in 4-dim sub-space orthogonal to the physical "$+$" and "$-$" directions

$$W_{\mu\nu}(q) = \sum_{\text{flavors}} e_f^2 \int d^2k_\perp \mathcal{D}^{(i)}_{f/A}(x_A, k_\perp)\mathcal{D}^{(i)}_{f/B}(x_B, q_\perp - k_\perp) C_{\mu\nu}(q, k_\perp)$$
$$+\text{power corrections} \ + \ \text{Y} - \text{terms}$$

$\mathcal{D}_{f/A}(x_A, k_\perp)$ is the TMD density of a quark f in hadron A with fraction of momentum x_A and transverse momentum $k_\perp$ (similar for hadron B)

TMD f_1 for total DY cross section for unpolarized hadrons

$$f_1^f(x_B, k_\perp) = \frac{1}{16\pi^3}\int dz_+ d^2z_\perp \ e^{-ix_B z^+\sqrt{\frac{s}{2}} + i(k,z)_\perp} \langle p_N|\bar{\psi}_f(z_+, z_\perp)[z, z - \infty n]\not{n}\psi_f(0)|p_N\rangle$$

unpolarized nucleon $|p_N\rangle$

$p_N \simeq p_N^-, \quad n = (\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}})$ light-like vector in "+" direction

Quark and Gluon TMD Operators

$$[x,y] \equiv \mathrm{P}e^{ig\int du\ (x-y)^\mu A_\mu(ux+(1-u)y)}$$

Quark TMDs

$$\bar\psi(x^+,x_\perp)[x,x\pm\infty n]\ [\pm\infty n+x_\perp,\pm\infty n+y_\perp]\Gamma[\pm\infty n+y,y]\psi(y^+,y_\perp)$$

Gluon TMDs

$$F^{-i}(x^+,x_\perp)[x,x\pm\infty n][\pm\infty n+x_\perp,\pm\infty n+y_\perp][\pm\infty n+y,y]F^{-j}(y^+,y_\perp)$$

single out “good” projections in the light-cone language

$$\Gamma \ \rightarrow\ \gamma^-\,,\ \ \gamma^-\gamma_5\,,\ \ \gamma^-\gamma_\perp$$

Rapidity-only cut-off

Rapidity divergences due to infinitely long gauge links

Rapidity-only cut-off: restrict the $+$ component of emitted gluons by Wilson lines

$$A^{\sigma}_{\mu}(x) \;=\; \int \frac{d^4k}{16\pi^4}\,\theta\big(\sigma\varrho - |k^+|\big)e^{-ik\cdot x}A_{\mu}(k)$$

$\varrho \equiv \sqrt{\frac{s}{2}}$

Goal: find evolution of TMDs with respect to σ in the Sudakov region $\sigma x_B s \gg k_\perp^2 \sim q_\perp^2$

Background field method

To find the evolution kernel:

1. Integrate over gluons with $\sigma > k^+/\varrho > \sigma'$
2. temporarily freeze the fields with $k^+/\varrho < \sigma'$

The result will be some kernel multiplied by TMD operators with rapidity cutoff σ'

$$\langle\bar{\psi}(x^+,x_\perp)[x^+,-\infty^+]_x[x_\perp-\infty^+,y_\perp-\infty^+][-\infty^+,y^+]_y\Gamma\psi(y^+,y_\perp)\rangle_{\Psi,A}$$

Ψ and A are quarks and gluons with small $k^+ < \sigma\varrho$

$$[x^+,y^+]_z \;\equiv\; [x^+n+z_\perp,y^+n+z_\perp]$$

Gauge choice

- Background field $\rightarrow \quad A^{-} = 0$
  - an extra background gluon line would mean an extra $F_{\mu\nu}$. This gives a higher-twist contribution which we neglect
- Quantum field $\rightarrow$ background-Feynman gauge
  - It reduces to the usual Feynman gauge in diagrams without background gluons.
  - in such a gauge the contribution of the gauge link at infinity $[x_{\perp} - \infty n, y_{\perp} - \infty n]$ does not contribute.

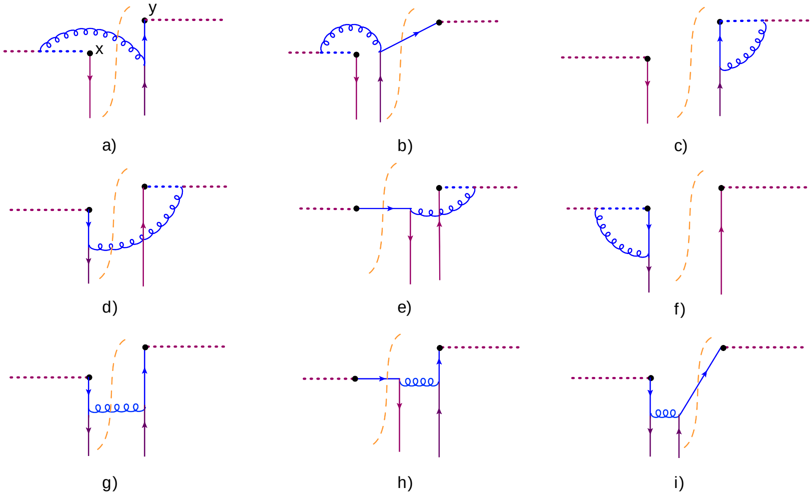
Fourier transform of background field

$$\Psi(\beta_B, p_{B_\perp}) = \varrho\int dz^+ dz_\perp\ \Psi(z^+, z_\perp) e^{i\varrho\beta_B z^+ - i(p_B, z)_\perp}$$

$$\bar{\Psi}(\beta'_B, p'_{B_\perp}) = \varrho\int dz^+ dz_\perp\ \bar{\Psi}(z^+, z_\perp) e^{i\varrho\beta'_B z^+ - i(p'_B, z)_\perp}$$

Background field not on the mass shell

Diagrams with no time ordering



Cancellation of IR divergences

IR divergences in diagrams (a), (b) and (c) should cancel

$p_\perp \to 0 \quad \Leftrightarrow \quad x \to y$

$$\langle [x^+, -\infty^+]_x [-\infty^+, y^+]_y \Gamma \psi(y^+, y_\perp) \rangle_\Psi \stackrel{x\to y}{=} \langle [x^+, y^+]_y \Gamma \psi(y^+, y_\perp) \rangle_\Psi$$

The use of naive rigid cut-off

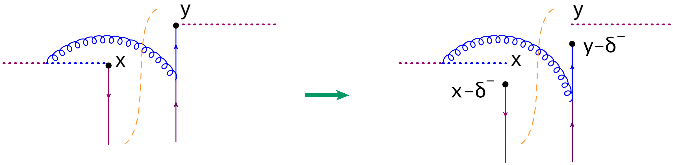
$$\int_0^\sigma \frac{d\alpha}{\alpha} \qquad \sigma \equiv \frac{1}{\varrho \delta^-} > 0$$

The use of naive rigid cut-off does not allow IR cancellation

Rigid cut-off with point splitting regulator

⇒ Use rigid cut-off with point splitting

$$\int_0^{\sigma} \frac{d\alpha}{\alpha} \;\rightarrow\; \int_0^{\infty} \frac{d\alpha}{\alpha}\, e^{-i\frac{\alpha}{\sigma}}$$



Rapidity-regularized operators

$$\bar{\psi}^{\sigma}(x^+, x_\perp) \;\equiv\; \bar{\psi}\Big(x^+, x_\perp, -\frac{1}{\rho\sigma}\Big)[x^+, -\infty]_x$$

$$\psi^{\sigma}(y^+, y_\perp) \;\equiv\; [-\infty, y^+]_y \psi\Big(y^+, y_\perp, -\frac{1}{\rho\sigma}\Big)$$

Evolution equations for quark TMDs in momentum space

$$\Big(\sigma\frac{d}{d\sigma}+\sigma'\frac{d}{d\sigma'}\Big)\bar\psi^{\sigma'}(\beta'_B,x_\perp)\Gamma\psi^{\sigma}(\beta_B,y_\perp)$$
$$=-\frac{\alpha_s}{2\pi}c_F\bar\psi^{\sigma'}(\beta'_B,x_\perp)\Gamma\psi^{\sigma}(\beta_B,y_\perp)\Big[\ln\Big(-\frac{i}{4}(\beta'_B+i\epsilon)\sigma' s\Delta_\perp^2e^\gamma\Big)$$
$$+\ln\Big(-\frac{i}{4}(\beta_B+i\epsilon)\sigma s\Delta_\perp^2e^\gamma\Big)\Big]+\mathcal{O}\Big(\frac{m_\perp^2}{\beta_B\sigma s},\frac{m_\perp^2}{\beta'_B\sigma' s}\Big)$$

$m_\perp^2\sim\Delta_\perp^{-2}\sim p_{B\perp}^2$

The solution of the evolution equation reads

$$\bar\psi^{\sigma'}(\beta'_B,x_\perp)\Gamma\psi^{\sigma}(\beta_B,y_\perp)$$
$$=\ e^{-\frac{\alpha_s c_F}{4\pi}\ln\frac{\sigma'}{\sigma'_0}\Big[\ln\sigma'\sigma'_0+2\ln\Big(-\frac{i}{4}(\beta'_B+i\epsilon)s\Delta_\perp^2e^\gamma\Big)\Big]}\bar\psi^{\sigma'_0}(\beta'_B,x_\perp)$$
$$\times\Gamma\psi^{\sigma_0}(\beta_B,y_\perp)e^{-\frac{\alpha_s c_F}{4\pi}\ln\frac{\sigma}{\sigma_0}\Big[\ln\sigma\sigma_0+2\ln\Big(-\frac{i}{4}(\beta'_B+i\epsilon)s\Delta_\perp^2e^\gamma\Big)\Big]}$$

In the Sudakov approximation each operator evolve independently.
Beyond Sudakov this is no longer true.

Evolution equations for quark TMDs in coordinate space

$$
\Big(\sigma\frac{d}{d\sigma}+\sigma'\frac{d}{d\sigma'}\Big)\bar\psi^{\sigma'}(x^+,x_\perp)\Gamma\psi^{\sigma}(y^+,y_\perp)
$$

$$
=\ \frac{\alpha_s}{4\pi^2}c_F\Big\{\int dz^+\Big[i\frac{\ln\varrho(-x^++z^++i\epsilon)-\ln\frac{\sigma b_\perp^2 s}{4}e^\gamma}{-x^++z^++i\epsilon}+\text{c.c.}\Big]\bar\psi^{\sigma'}(z^+,x_\perp)\Gamma\psi^{\sigma}(y^+,y_\perp)
$$

$$
+\int dw^+\Big[i\frac{\ln\varrho(-y^++w^++i\epsilon)-\ln\frac{\sigma' b_\perp^2 s}{4}e^\gamma}{-y^++w^++i\epsilon}+\text{c.c.}\Big]\bar\psi^{\sigma'}(x^+,x_\perp)\Gamma\psi^{\sigma}(w^+,y_\perp)\Big\}
$$

Note “causality”: $z^+\leq x^+$ and $w^+\leq y^+$:

The evolved $\bar\psi$, ψ operators lag behind the original ones,

Evolution equations for quark TMDs in coordinate space

Solution

$$\bar{\psi}^{\sigma'}(x^+,x_\perp)\Gamma\psi^{\sigma}(y^+,y_\perp) \;=\; e^{-\frac{\alpha_s c_F}{4\pi}\left(\ln\frac{\sigma'}{\sigma'_0}\ln\sigma'\sigma'_0+\ln\frac{\sigma}{\sigma_0}\ln\sigma\sigma_0\right)}$$

$$\times\int dz^+\left[\frac{i\Gamma\left(1-\frac{\alpha_s c_F}{2\pi}\ln\frac{\sigma'}{\sigma'_0}\right)}{(z^+-x^++i\epsilon)^{1-\frac{\alpha_s c_F}{2\pi}\ln\frac{\sigma'}{\sigma'_0}}}+\text{c.c.}\right]\int dw^+\left[\frac{i\Gamma\left(1-\frac{\alpha_s c_F}{2\pi}\ln\frac{\sigma}{\sigma_0}\right)}{(w^+-y^++i\epsilon)^{1-\frac{\alpha_s c_F}{2\pi}\ln\frac{\sigma}{\sigma_0}}}+\text{c.c.}\right]$$

$$\times\frac{1}{4\pi^2}\left(b_\perp^2 e^{\gamma}\sqrt{s/8}\right)^{-\frac{\alpha_s c_F}{2\pi}\left(\ln\frac{\sigma'}{\sigma'_0}+\ln\frac{\sigma}{\sigma_0}\right)}\bar{\psi}^{\sigma'_0}(z^+,x_\perp)\Gamma\psi^{\sigma_0}(w^+,y_\perp)$$

Conformal invariance $SO(2,4)$: choose $\sigma=\sigma'=\frac{\varsigma\sqrt{2}}{\varrho|\Delta_\perp|}$

ς is an evolution parameter.

Sudakov evolution: transverse separation between the gluon operators ψ and $\bar{\psi}$ does not change while the longitudinal one increases.

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Running coupling

BLM procedure:

1. Calculate quark loop diagrams
2. Promote $-\frac{1}{6\pi}n_f$ to full $b_0 = \frac{11}{12\pi}N_c - \frac{1}{6\pi}n_f$

$$
\begin{aligned}
\frac{1}{p^2+i\epsilon} &\rightarrow \frac{1}{p^2+i\epsilon}\Big(1+b_0\alpha_s(\mu)\ln\frac{\tilde{\mu}^2}{-p^2-i\epsilon}\Big)\\
\frac{1}{p^2-i\epsilon} &\rightarrow \frac{1}{p^2-i\epsilon}\Big(1+b_0\alpha_s(\mu)\ln\frac{\tilde{\mu}^2}{-p^2+i\epsilon}\Big)\\
2\pi\delta(p^2)\theta(p_0) &\rightarrow \frac{i\theta(p_0)}{p^2+i\epsilon}\Big(1+b_0\alpha_s(\mu)\ln\frac{\tilde{\mu}^2}{-p^2-i\epsilon}\Big)-\frac{i\theta(p_0)}{p^2-i\epsilon}\Big(1+b_0\alpha_s(\mu)\ln\frac{\tilde{\mu}^2}{-p^2+i\epsilon}\Big)
\end{aligned}
$$

where $\tilde{\mu}^2 \equiv \bar{\mu}^2_{\rm MS} e^{5/3}$.

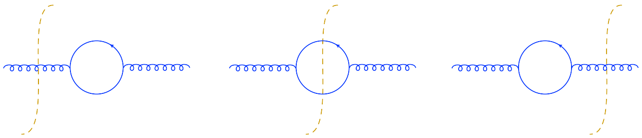

Figure : Quark loop correction to cut gluon propagator.

Evolution with running coupling

$$\left(\sigma\frac{d}{d\sigma}+\sigma'\frac{d}{d\sigma'}\right)\bar\psi^{\sigma'}(\beta'_B,x_\perp)\Gamma\psi^{\sigma}(\beta_B,y_\perp)$$
$$= -\frac{c_F}{2\pi}\bar\psi^{\sigma'}(\beta'_B,x_\perp)\Gamma\psi^{\sigma}(\beta_B,y_\perp)$$
$$\times\Big[\alpha_s(\mu_{\sigma'})\ln\Big(-\frac{i}{4}(\beta'_B+i\epsilon)\sigma' s b_\perp^2 e^{\gamma}\Big)+\alpha_s(\mu_\sigma)\ln\Big(-\frac{i}{4}(\beta_B+i\epsilon)\sigma s b_\perp^2 e^{\gamma}\Big)\Big]$$

$\mu^2_{\sigma'}\equiv\sqrt{\frac{\sigma'|\beta'_B|s}{\Delta^2_\perp}}$ $\qquad$ $\mu^2_{\sigma}\equiv\sqrt{\frac{\sigma|\beta_B|s}{\Delta^2_\perp}}$

$b_\perp\equiv\Delta_\perp=(x-y)_\perp$

The effective argument of a coupling constant is halfway in the logarithmic scale between the transverse momentum and energy of TMD distribution.

Solving evolution equation with running coupling

$b_0 = \frac{1}{4\pi}\left(\frac{11}{3}N_c - \frac{2}{3}n_f\right)$

$$\sigma\frac{d}{d\sigma} = -\frac{b_0}{2}\alpha^2(\mu_\sigma)\frac{d}{d\alpha(\mu_\sigma)}$$

(and similarly for $\sigma'\frac{d}{d\sigma'}$)

$$\begin{aligned}&\left(\alpha^2(\mu_\sigma)\frac{d}{d\alpha(\mu_\sigma)} + \alpha^2(\mu_{\sigma'})\frac{d}{d\alpha(\mu_{\sigma'})}\right)\bar{\psi}^{\sigma'}(\beta'_B, x_\perp)\Gamma\psi^\sigma(\beta_B, y_\perp)\\ &= -\frac{2c_F}{\pi b_0^2}\left\{\frac{\alpha_s(\mu_{\sigma'})}{\alpha_s(\tilde{b}_\perp^{-1})} + \frac{\alpha_s(\mu_\sigma)}{\alpha_s(\tilde{b}_\perp^{-1})} - 2\right.\\ &\quad\left. -\frac{b_0\alpha_s(\mu_{\sigma'})}{2}\ln[-i(\tau'_B + i\epsilon)] - \frac{b_0\alpha_s(\mu_\sigma)}{2}\ln[-i(\tau_B + i\epsilon)]\right\}\bar{\psi}^{\sigma'}(\beta'_B, x_\perp)\Gamma\psi^\sigma(\beta_B, y_\perp)\end{aligned}$$

$\tilde{b}_\perp^2 = \frac{b_\perp^2}{2}e^{\gamma/2}$ and $\tau_B = \frac{\beta_B}{|\beta_B|}\,,\ \ \tau'_B = \frac{\beta'_B}{|\beta'_B|}$

Solution with running coupling

$$
\begin{aligned}
&\bar\psi^{\sigma'}(\beta'_B, x_\perp)\Gamma\psi^{\sigma}(\beta_B, y_\perp)\\
&= \exp\Big\{-\frac{2c_F}{\pi b_0^2}\big[\big(\frac{1}{\alpha_s(\tilde b_\perp^{-1})} - \frac{b_0}{2}\ln[-i(\tau'_B + i\epsilon)]\big)\ln\frac{\alpha_s(\mu_{\sigma'})}{\alpha_s(\mu_{\sigma'_0})} + \frac{1}{\alpha_s(\mu_{\sigma'})} - \frac{1}{\alpha_s(\mu_{\sigma'_0})}\big]\Big\}\\
&\times \exp\Big\{-\frac{2c_F}{\pi b_0^2}\big[\big(\frac{1}{\alpha_s(\tilde b_\perp^{-1})} - \frac{b_0}{2}\ln[-i(\tau_B + i\epsilon)]\big)\ln\frac{\alpha_s(\mu_{\sigma})}{\alpha_s(\mu_{\sigma_0})} + \frac{1}{\alpha_s(\mu_{\sigma})} - \frac{1}{\alpha_s(\mu_{\sigma_0})}\big]\Big\}\\
&\times\bar\psi^{\sigma'_0}(\beta'_B, x_\perp)\Gamma\psi^{\sigma_0}(\beta_B, y_\perp)
\end{aligned}
$$

Like in LO, Sudakov evolution looks like two independent exponential factors which describe two independent evolutions of operators

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Quark loop contribution from light-cone expansion

Background field on the mass shell

$$\Psi(z^+) \;=\; \int đ\,\beta_B\; e^{-i\varrho\beta_B z^+}\,\Psi(\beta_B)$$

the x^+ dependence is power-suppressed so we can take $x^+ = y^+$ from the beginning:

$$\sigma\frac{d}{d\sigma}\langle[y^+,-\infty]_x[-\infty,y^+]_y\Gamma\psi(y^+,y_\perp,-\delta^-)\rangle_\Psi$$
$$= -\delta^-\frac{d}{d\delta^-}\langle[y^+,-\infty]_x[-\infty,y^+]_y\Gamma\psi(y^+,y_\perp,-\delta^-)\rangle_\Psi^{\rm Fig1a-c\ loop}$$

In this case, all relevant distances are space-like so we can replace product of operators in the matrix element in the l.h.s. by T-product.

$$\frac{1}{b\alpha_s^2c_F}\delta^-\frac{d}{d\delta^-}\langle {\rm T}\{[0^+,-\infty]_x[-\infty,0^+]_0\Gamma\psi(0^+,0_\perp,-\delta^-)\}\rangle_\Psi^{\rm Fig1a-c\ loop}$$
$$=\;2\delta^-\frac{d}{d\delta^-}\Big[\int_{-\infty}^{0}dz^+(z^+,x_\perp|\frac{\ln\frac{\tilde\mu^2}{-p^2}}{p^2}\Gamma\Psi\frac{p^-}{p^2}|0^+,-\delta^-,0_\perp)\;-\;(x_\perp\to 0)\Big]$$

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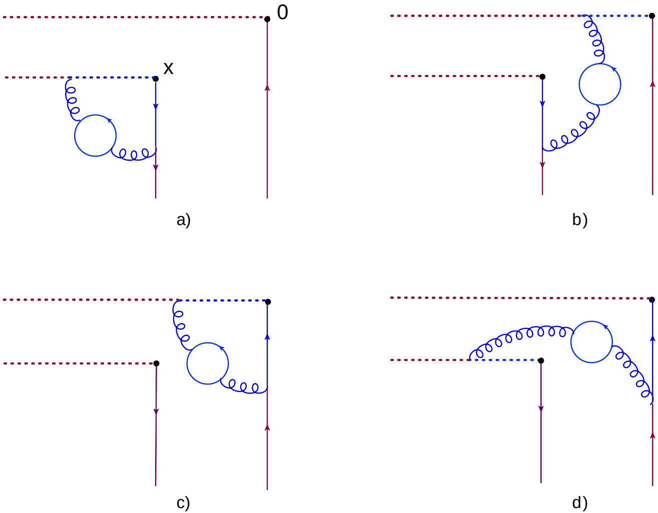
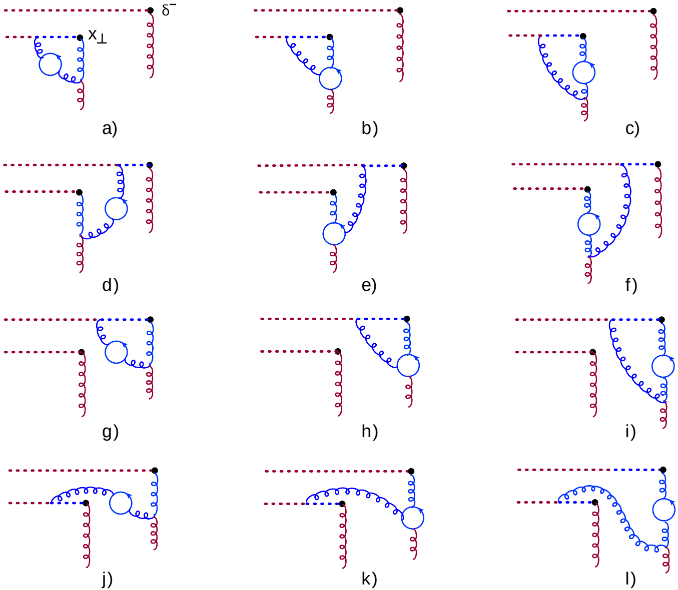
Time-ordered product diagrams for quark TMD evolution



Time-ordered product diagrams for gluon TMD evolution



$$\left(\sigma\frac{d}{d\sigma}+\sigma'\frac{d}{d\sigma'}\right)\mathcal{F}^{i,a;\sigma'}(\beta'_B,x_\perp)\mathcal{F}^{a;\sigma}_{\ i}(\beta_B,y_\perp)$$
$$=-\frac{N_c}{2\pi}\mathcal{F}^{i,a;\sigma'}(\beta'_B,x_\perp)\mathcal{F}^{a;\sigma}_{\ i}(\beta_B,y_\perp)$$
$$\times\Big[\alpha_s(\mu_{\sigma'})\ln\Big(-\frac{i}{4}(\beta'_B+i\epsilon)\sigma' s b_\perp^2 e^\gamma\Big)+\alpha_s(\mu_\sigma)\ln\Big(-\frac{i}{4}(\beta_B+i\epsilon)\sigma s b_\perp^2 e^\gamma\Big)\Big]$$

The solution of this equation is the same as for quark with $c_F\to N_c$ replacement

$$\mathcal{F}^{i,a;\sigma'}(\beta'_B,x_\perp)\mathcal{F}^{a;\sigma}_{\ i}(\beta_B,y_\perp)$$
$$\times = e^{-\frac{2N_c}{\pi b_0^2}\left[\ln\frac{\alpha_s(\mu_{\sigma'})}{\alpha_s(\mu_{\sigma'_0})}\left(\frac{1}{\alpha_s(\tilde b_\perp^{-1})}+\ln[-i(\tau'_B+i\epsilon)]\right)+\frac{1}{\alpha_s(\mu_{\sigma'})}-\frac{1}{\alpha_s(\mu_{\sigma'_0})}\right]}$$
$$\times\, e^{-\frac{2N_c}{\pi b_0^2}\left[\ln\frac{\alpha_s(\mu_\sigma)}{\alpha_s(\mu_{\sigma_0})}\left(\frac{1}{\alpha_s(\tilde b_\perp^{-1})}+\ln[-i(\tau_B+i\epsilon)]\right)+\frac{1}{\alpha_s(\mu_\sigma)}-\frac{1}{\alpha_s(\mu_{\sigma_0})}\right]}\mathcal{F}^{i,a;\sigma'_0}(\beta'_B,x_\perp)\mathcal{F}^{a;\sigma_0}_{\ i}(\beta_B,y_\perp)$$

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Conclusions

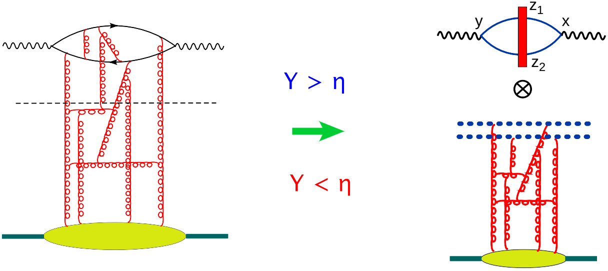
- The conformal TMD group has been obtained: it is made out of 11 generators of the full conformal group.
- We obtained conformal evolution of gluon and quark TMDs in the Sudakov region.
- quark and gluon TMD evolution equation with running coupling
  - The effective argument of a coupling constant is halfway in the logarithmic scale between the transverse momentum and energy of TMD distribution.

Outlook

- Conformal properties of TMD evolution in the small-x region.
- Conformal evolution for all x_B.
- The plan is to perform the calculation for 6 point function similarly to the one done for the 4 point function in $\mathcal{N}$=4 and in in QCD.

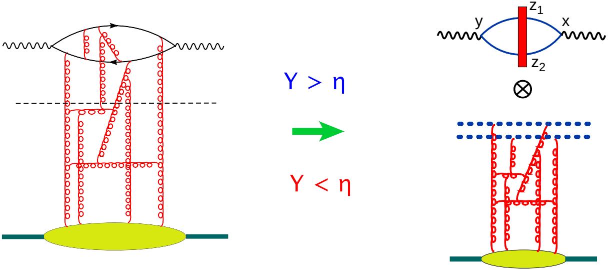
High-Energy Operator Product Expansion

DIS amplitude is factorized in rapidity: η



$|B\rangle$ is the target state.

$$\langle B|T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}|B\rangle = \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\langle B|{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}|B\rangle + \dots$$



$$\langle B|T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}|B\rangle = \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\langle B|{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}|B\rangle + \dots$$

Rapidity Regularization:

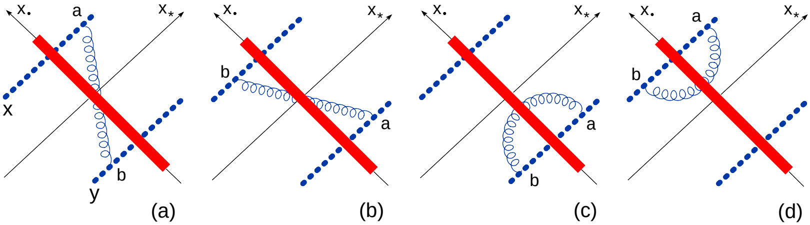
$$A^\eta_\mu(x) = \int d\!\!{}^-{}^4k\,\theta(e^\eta - |k^+|)e^{-ik\cdot x}A_\mu(k)$$

$d\!\!{}^-{}^n k \equiv \frac{d^nk}{(2\pi)^n}$

Leading order: BK equation

$$\frac{d}{d\eta}\mathrm{tr}\{\hat{U}_x\hat{U}^\dagger_y\} = K_{\mathrm{LO}}\mathrm{tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}}$$



$x_\bullet = \sqrt{\frac{s}{2}}x^-$ $\qquad x_* = \sqrt{\frac{s}{2}}x^+$

Balitsky-Kovchegov equation

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) = \frac{\alpha_s N_c}{2\pi^2}\int\frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z)+\hat{\mathcal{U}}(z,y)-\hat{\mathcal{U}}(x,y)-\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

- LLA for DIS in pQCD $\Rightarrow$ BFKL
  - (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$): describes proliferation of gluons.

- LLA for DIS in semi-classical-QCD $\Rightarrow$ BK eqn
  - background field method: describes recombination process.

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;=\; \mathrm{Pexp}\left\{ ig\int_{-\infty}^{\infty} dx^+ \, A_+(x^+, x_\perp)\right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

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Indeed,
$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \to x_\perp / x_\perp^2$ and $x^+ \to x^+ / x_\perp^2$

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$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;\to\; \mathrm{Pexp}\left\{ ig\int_{-\infty}^{\infty} d\frac{x^+}{x_\perp^2}\, A_+\left(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2}\right)\right\} \;=\; [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}]$$

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$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;\to\; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} d\frac{x^+}{x_\perp^2}\, A_+(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2})\right\} \;=\; [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}]$$

$\Rightarrow$The dipole kernel is invariant under the inversion $V(x_\perp) = U(x_\perp/x_\perp^2)$

$$\frac{d}{d\eta}\mathrm{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int \frac{d^2z}{\not{z}^4}\frac{(x-y)^2 \;\; \not{z}^4}{(x-z)^2(z-y)^2}[\mathrm{Tr}\{V_x V_z^\dagger\}\mathrm{Tr}\{V_z V_y^\dagger\} - N_c \mathrm{Tr}\{V_x V_y^\dagger\}]$$

Conformal invariance of the BK equation

SL(2,C) for Wilson lines

$$\hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2)$$

$$[\hat{S}_0, \hat{S}_\pm] = \pm\hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0,$$

$$[\hat{S}_-, \hat{U}(z,\bar{z})] = z^2\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_0, \hat{U}(z,\bar{z})] = z\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_+, \hat{U}(z,\bar{z})] = -\partial_z\hat{U}(z,\bar{z})$$

$z \equiv z^1 + iz^2, \quad \bar{z} \equiv z^1 - iz^2, \quad U(z_\perp) = U(z,\bar{z})$

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$$[\hat{S}_0, \hat{S}_\pm] = \pm\hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0,$$

$$[\hat{S}_-, \hat{U}(z,\bar{z})] = z^2\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_0, \hat{U}(z,\bar{z})] = z\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_+, \hat{U}(z,\bar{z})] = -\partial_z\hat{U}(z,\bar{z})$$

$z \equiv z^1 + iz^2, \qquad \bar{z} \equiv z^1 - iz^2, \qquad U(z_\perp) = U(z,\bar{z})$

Conformal invariance of the evolution kernel

$$\frac{d}{d\eta}[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}] = \frac{\alpha_s N_c}{2\pi^2}\int dz\, K(x,y,z)[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}\mathrm{Tr}\{U_x U_y^\dagger\}]$$

$$\Rightarrow \left[x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z}\right]K(x,y,z) = 0$$

Conformal 4-point function in $\mathcal{N}$=4 SYM in the Regge limit

In a conformal theory the amplitude of 4-point functions depends on two conformal ratios which can be chosen as

$$R = \frac{(x-x')^2(y-y')^2}{(x-y)^2(x'-y')^2} \quad \xrightarrow{\text{Regge limit}} \quad 0$$

$$r = R\Big[1 - \frac{(x-y')^2(y-x')^2}{(x-x')^2(y-y')^2} + \frac{1}{R}\Big]^2 \quad \xrightarrow{\text{Regge limit}} \quad \text{fixed}$$

Conformal 4-point function in $\mathcal{N}$=4 SYM in the Regge limit

Cornalba (2007)
Provides general structure of the 4-point function in the regge limit as an integral over one real variable ν

$$\langle\{\mathcal{S}(z_1^-, z_{1\perp})\mathcal{S}(z_2^-, z_{2\perp})\mathcal{S}(x^+, x_\perp)\mathcal{S}(y^+, y_\perp)\}\rangle = \int d\nu\ \Phi(r,\nu)F(\nu)R^{\frac{1}{2}\omega(\nu)}$$

$\mathcal{S} \equiv \frac{4\pi^2\sqrt{2}}{\sqrt{N_c^2-1}}\mathrm{Tr}\{\mathrm{Z}^2\}$ $\quad (Z = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2))$ renorm-invariant operator

- $\omega(\nu) \equiv \omega(0,\nu)$ is the pomeron intercept.
- $F(\nu)$ is the "pomeron residue".
- $\Phi(r,\nu)$ some function of ν and the conformal ratio r.

Explicit calculation of the 4-point function in the Regge limit at NLLO

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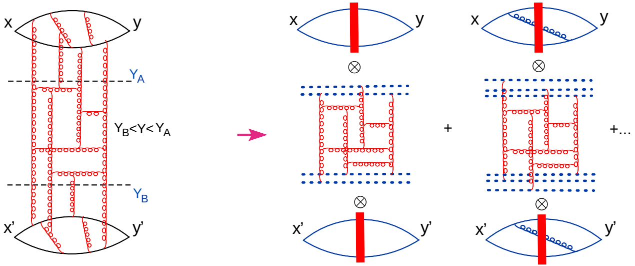
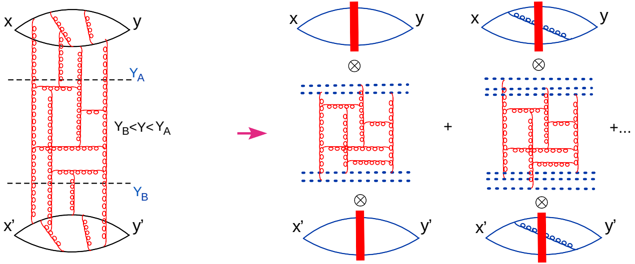
Factorization in rapidity



4-point function at NLO in the Regge limit using conformal composite dipole in Wilson line.

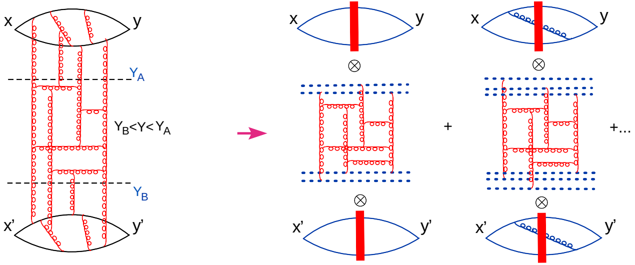
Factorization in rapidity



- $F(\nu)$ Pomeron residue (Impact factor) at NLO in $\mathcal{N}$=4 SYM and QCD
  I. Balitsky and G.A.C. (2009)

- NLO Pomeron intercept $\omega(\nu)$
  - QCD: Fadin-Lipatov (1998) and I.Balitsky and G.A.C (2007)
  - $\mathcal{N}$=4: Lipatov-Kotikov (2000) and I. Balitsky and G.A.C. (2008)

Factorization in rapidity

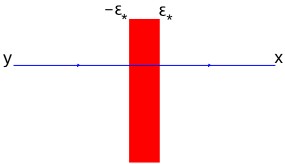


$$\langle T\{\mathcal{S}(x)\mathcal{S}^\dagger(y)\mathcal{S}(x')\mathcal{S}^\dagger(y')\}\rangle$$
$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\,\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

in QCD: NLO $\gamma^*\gamma^*$ cross section G.A.C. and Yu. Kovchegov (2015)

Getting the interpolating evolution equation

Shock-wave with finite width



$$A^-(x_\bullet, x^+, x_\perp) \rightarrow \lambda A^-(\lambda^{-1} x_\bullet, \lambda x^+, x_\perp)$$
$$A^+(x_\bullet, x^+, x_\perp) \rightarrow \lambda^{-1} A^+(\lambda^{-1} x_\bullet, \lambda x^+, x_\perp)$$
$$A_\perp(x_\bullet, x^+, x_\perp) \rightarrow A_\perp(\lambda^{-1} x_\bullet, \lambda x^+, x_\perp)$$

λ is the boost parameter

- small k^+ gluons are classical fields   large k^+ gluons are quantum fields.
- Long. size classical fields: $\epsilon^+ \sim \frac{k^+}{l_\perp^2}$ with $l_\perp$ trans. mom. of classical fileds
- Distance traveled by quantum fields: $z^+ \sim \frac{k^+}{k_\perp^2}$ with $k_\perp$ trans. mom. of quantum fields
- Shock wave: $l_\perp \sim k_\perp$
- Light-cone expansion: $l_\perp \ll k_\perp$

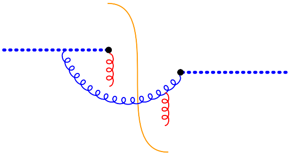
Rapidity evolution of Gluon TMD operator

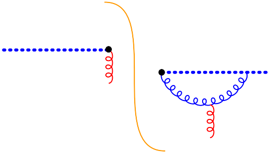
Gluon TMD $\tilde{\mathcal{F}}_i^{a\eta}(z_{1\perp}, x_B)\mathcal{F}^{a\eta}(z_{2\perp}, x_B)$

$$\left(\mathcal{F}^{i,a}(z_\perp, z^+)\right)^\sigma \equiv gF^{-i,m}(z)\left[\mathrm{P}e^{ig\int_{-\infty}^{z^+} dz^+ A^{-,\sigma}(up_1+x_\perp)}\right]^{ma},$$

$$A_\mu^\sigma(x) \;=\; \int \frac{d^4k}{16\pi^4}\,\theta\Big(\frac{\sigma\sqrt{2}}{z_{12_\perp}} - |k^+|\Big)e^{-ik\cdot x}A_\mu(k)$$

Typical LO diagrams





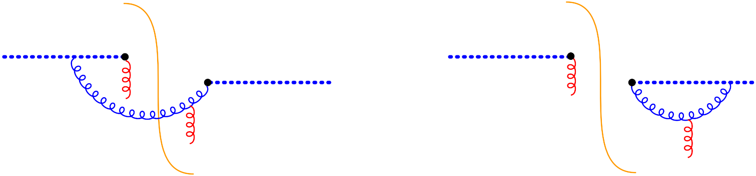
- Shock wave: $l_\perp \sim k_\perp$
- Light-cone expansion: $l_\perp \ll k_\perp$

Rapidity evolution of Gluon TMD operator

$m_X^2 = q^2 \equiv Q^2$ Sudakov region: $Q \gg q_\perp \gg 1$Gev

Sudakov region in coord. space: $z_{12\parallel}^2 \equiv 2z_{12}^- z_{12}^+ \ll z_{12\perp}^2$

Typical LO diagrams



To calculate these diagrams the approximation is: $k^+ \gg \frac{z_{12}^+}{z_{12\perp}^2}$

Sudakov evolution:transverse separation between the gluon operators $\mathcal{F}_i$ and $\mathcal{F}_j$ does not change while the longitudinal one increases.

Rapidity evolution at leading in Sudakov region

$$\mathcal{O}^{\sigma_2}(z_1^+, z_2^+) \;=\; \frac{\alpha_s N_c}{2\pi} \int_{\frac{\sigma_1\sqrt{2}}{z_{12\perp}}}^{\frac{\sigma_2\sqrt{2}}{z_{12\perp}}} \frac{dk^+}{k^+}\, K\mathcal{O}^{\sigma_1}(z_1^+, z_2^+)$$

where the kernel K is given by

$$K\mathcal{O}(z_1^+, z_2^+) \;=\; \mathcal{O}(z_1^+, z_2^+)\int_{-\infty}^{z_1^+}\frac{dz^+}{z_2^+ - z^+} e^{-i\frac{z_{12\perp}\sigma}{\sqrt{2}(z_2-z)^+}} + \mathcal{O}(z_1^+, z_2^+)\int_{-\infty}^{z_2^+}\frac{dz^+}{z_1^+ - z^+} e^{i\frac{z_{12\perp}\sigma}{\sqrt{2}(z_1-z)^+}}$$
$$-\int_{-\infty}^{z_1^+} dz^+ \frac{\mathcal{O}(z_1^+, z_2^+) - \mathcal{O}(z^+, z_2^+)}{z_1^+ - z^+} - \int_{-\infty}^{z_2^+} dz^+ \frac{\mathcal{O}(z_1^+, z_2^+) - \mathcal{O}(z_1^+, z^+)}{z_2^+ - z^+}$$

Solution of the evolution equation

To solve the evolution equation, perform Fourier transform

$$Ke^{-ik^- z_1^+ + ik'^- z_2^+} \;=\; \Big[-2\ln \sigma z_{12_\perp} - \ln(ik^-)$$
$$-\; \ln(-ik'^-) + \ln 2 - 4\gamma_E \;+\; O\Big(\frac{z_{12}^+}{z_{12_\perp}\sigma}\Big)\Big] e^{-ik^- z_1^+ + ik'^- z_2^+}$$

Result ($\bar{\alpha}_s = \frac{\alpha_s N_c}{4\pi}$)

I. Balitsky and G.A.C (2019)

$$\mathcal{O}^{\sigma_2}(z_1^+, z_2^+) \;=\; e^{-2\bar{\alpha}_s \ln\frac{\sigma_2}{\sigma_1}\left[\ln \sigma_1\sigma_2 + 4\gamma_E - \ln 2\right]} \int dz_1'^+ dz_2'^+ \; \mathcal{O}^{\sigma_1}(z_1'^+, z_2'^+)\; z_{12_\perp}^{-2\bar{\alpha}_s \ln\frac{\sigma_2}{\sigma_1}}$$
$$\times\; \frac{1}{4\pi^2}\left[\frac{i\Gamma\big(1 - 2\bar{\alpha}_s \ln\frac{\sigma_2}{\sigma_1}\big)}{(z_1^+ - z_1'^+ + i\epsilon)^{1 - 2\bar{\alpha}_s \ln\frac{\sigma_2}{\sigma_1}}} + c.c.\right]\left[\frac{i\Gamma\big(1 - 2\bar{\alpha}_s \ln\frac{\sigma_2}{\sigma_1}\big)}{(z_2^+ - z_2'^+ + i\epsilon)^{1 - 2\bar{\alpha}_s \ln\frac{\sigma_2}{\sigma_1}}} + c.c.\right]$$

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$$\times\; \frac{1}{4\pi^2}\left[\frac{i\Gamma\left(1 - 2\bar{\alpha}_s \ln\frac{\sigma_2}{\sigma_1}\right)}{(z_1^+ - z_1'^+ + i\epsilon)^{1-2\bar{\alpha}_s \ln\frac{\sigma_2}{\sigma_1}}} + c.c.\right]\left[\frac{i\Gamma\left(1 - 2\bar{\alpha}_s \ln\frac{\sigma_2}{\sigma_1}\right)}{(z_2^+ - z_2'^+ + i\epsilon)^{1-2\bar{\alpha}_s \ln\frac{\sigma_2}{\sigma_1}}} + c.c.\right]$$

Result transform covariantly under the TMD-conformal group generators except the Lorentz boost M^{+-} which is the generator of the evolution equation: The Lorentz boost in z direction changes the cutoffs for the evolution.

Conformal invariance of the TMD matrix element

Sudakov-region result is applicable in the region between:

$$\sigma_2 = \sigma_B = \frac{z_{12\perp}}{z_{12}^- \sqrt{2}} \quad \text{and} \quad \sigma_1 = \frac{z_{12}^+ \sqrt{2}}{z_{12\perp}}$$

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Lorentz boost: $z^+ \to \lambda z^+, \qquad z^- \to \frac{1}{\lambda} z^-$

- $\langle p_B|\mathcal{O}|p_B\rangle \to \langle p_B|\mathcal{O}|p_B\rangle \exp\{4\lambda\bar{\alpha}_s \ln \frac{z^2_{12\parallel}}{z^2_{12_\perp}}\}$ Target
- $\langle p_A|\tilde{\mathcal{O}}|p_A\rangle \to \langle p_A|\tilde{\mathcal{O}}|p_A\rangle \exp\{-4\lambda\bar{\alpha}_s \ln \frac{z^2_{12\parallel}}{z^2_{12_\perp}}\}$ Projectile

So the amplitude is invariant:

$$\langle p_A, p_B| g^2 F^a_{\mu\nu} F^{a\mu\nu}(z_1) g^2 F^b_{\lambda\rho} F^{b\lambda\rho}(z_2) |p_A, p_B\rangle$$
$$= \frac{1}{N_c^2 - 1} \langle p_A|\tilde{\mathcal{O}}_{ij}(z_1^-, z_{1_\perp}; z_2^-, z_{2_\perp})|p_A\rangle^{\sigma_A} \langle p_B|\mathcal{O}^{ij}(z_1^+, z_{1_\perp}; z_2^+, z_{2_\perp})|p_B\rangle^{\sigma_B}$$

Comparing with conventional TMD analysis

Evolution for generalized TMD $\xi = \frac{p_B - p'_B}{\sqrt{2s}}$

$$D^\sigma(x_B, \xi) = \int dz^+ \, e^{-ix_B\sqrt{\frac{s}{2}}z^+} \langle p'_B|\mathcal{O}^\sigma\big(-\frac{z^+}{2}, \frac{z^+}{2}\big)|p_B\rangle$$

From our result we get

$$\frac{D^{\sigma_2}(x,\xi)}{D^{\sigma_1}(x,\xi)} = e^{-2\bar{\alpha}_s \ln\frac{\sigma_2}{\sigma_1}[\ln\sigma_2\sigma_1(x^2-\xi^2)sz^2_{12_\perp} + 4\gamma_E - 2\ln 2]}$$

For usual TMD at $\xi = 0$ with the limits of Sudakov evolution one obtains

$$\frac{D^{\sigma_2}(x,q_\perp)}{D^{\sigma_1}(x,q_\perp)} = e^{-2\bar{\alpha}_s \ln\frac{Q^2}{q^2_\perp}\left[\ln\frac{Q^2}{q^2_\perp} + 4\gamma_E - 2\ln 2\right]}$$

- Coincides with usual one-loop evolution of TMDs up to replacement $4\gamma_E - 2\ln 2 \to 4\gamma_E - 4\ln 2$.
- constant depends on the way of cutting k^+-integration which should be coordinated with the cutoffs in the “coefficient function” $\sigma(ff \to H)$
  - The discrepancy is just like using two different schemes for usual renormalization.

Going beyond the Sudakov approximation

Sudakov: $Q \gg q_\perp$; in coord. space $(x-y)^2_\perp \gg (x-y)^2_\parallel$

- $x_B \sim 1$ and $q_\perp \sim m_N$
  - The relative energy between Wilson-line operators $\mathcal{F}$ and target nucleon at the final point of the evolution is $\sim m_N^2$ so one should use phenomenological models of TMDs with this low rapidity cutoff as a starting point of the evolution.
- $x_B \ll 1$
  - This relative energy is within $\frac{q_\perp^2}{x_B s} > \sigma > \frac{m_N^2}{s}$ beyond Sudakov region into the low-x region: the TMD operator, known as Weiczsäcker-Williams distribution, will produce a of color dipoles as a result of the non-linear evolution. The transition between Sudakov region and small-x region is described by rather complicated interpolation formula. In coordinate space this means the study of the operator $\mathcal{O}$ at $z^2_\parallel \sim z^2_\perp$. Conformal invariance may help us obtain the TMD evolution in that region.