Lipatov's EFT approach as a computational tool

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Workshop on overlap between QCD resummations
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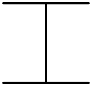
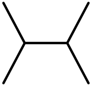
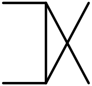
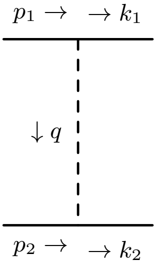
I. Introduction to the EFT approach to High-Energy QCD

Warm-up: ϕ^3 amplitudes at high energy

In the limit $s \sim -u \gg -t$ (**Regge limit**) the t-channel diagram dominates in the tree-level $2 \to 2$ amplitude:

 $\sim \frac{1}{t}$,  $\sim \frac{1}{s}$,  $\sim \frac{1}{u} \sim -\frac{1}{s}$

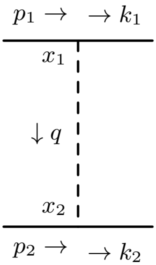


Sudakov (light-cone) basis: $p_1^\mu = \frac{p_1^+}{2} n_-^\mu$, $p_2^\mu = \frac{p_2^-}{2} n_+^\mu$, $n_\pm^2 = 0$, $n_+ n_- = 2$:

$$k^\mu = \frac{1}{2}(k^+ n_-^\mu + k^- n_+^\mu) + k_T^\mu, \quad k^2 = k^+ k^- - \mathbf{k}_T^2.$$

(Multi-)Regge Kinematics ($s = p_1^+ p_2^- \to \infty$, $|\mathbf{q}_T|$-fixed):

$$p_1^+ \simeq k_1^+ \ll k_1^-, \quad p_2^- \simeq k_2^- \ll k_2^+, \quad q^2 \simeq -\mathbf{q}_T^2.$$



Matrix element of the T-matrix (modulo 2π's):

$$T \simeq \frac{i\lambda^2}{\mathbf{q}_T^2}\delta(p_1^+ - k_1^+)\delta(p_2^- - k_2^-)\delta(\mathbf{k}_{T1} + \mathbf{k}_{T2})$$

$$= \frac{i\lambda^2}{\mathbf{q}_T^2}\int dq_+ dq_- d^2\mathbf{q}_T \; \delta(p_1^+ - k_1^+)\delta(k_1^- + q_-)\delta(\mathbf{k}_{T1} + \mathbf{q}_T)$$

$$\times\delta(p_2^- - k_2^-)\delta(k_2^+ - q_-)\delta(\mathbf{k}_{T2} - \mathbf{q}_T)$$

$$= \lambda^2 \int d^4x_1 d^4x_2 e^{ix_1(p_1-k_1)} e^{ix_2(p_2-k_2)} G(x_1, x_2),$$

Where

$$G(x_1, x_2) = \delta(x_1^+)\delta(x_2^-) \int \frac{d^2\mathbf{q}_T}{\mathbf{q}_T^2} e^{i\mathbf{q}_T(\mathbf{x}_{T1} - \mathbf{x}_{T2})}.$$

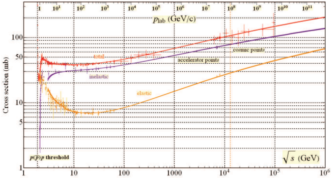
So fields of the target $\phi(x) \sim \delta(x_+)$ and independent of x_- from the point of view of the projectile and vice versa (shockwave approximation). Consequence of the Lorentz contraction (rapidity $y \to \infty$):

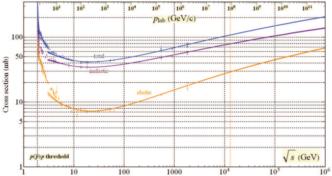
$$\int dk_+ dk_- \exp[i(k_+ x_- + k_- x_+)]\phi(k_+, k_-) \underset{\phi(k_+,k_-)\to\phi(k_+e^y, k_-e^{-y})}{\longrightarrow}$$

$$\int dk_+ dk_- \exp[i(k_+(x_- e^{-y}) + k_-(x_+ e^y))]\phi(k_+, k_-)$$

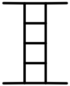
ϕ^3 amplitudes at high energy, LLA

Experimental plots of pp and $p\bar{p}$ scattering cross sections as function of energy:





The LLA terms $\propto (\lambda^2 \ln s)^n$ come from ladder diagrams and exponentiate [Eden, Landshoff, Olive, Polkinghorne "The Analytic S-matrix"] :

$$\sum$$  $$\sim \frac{i\lambda^2}{\mathbf{q}_T^2} \exp[\omega_s(\mathbf{q}_T^2) \ln s],$$

where

$$\omega_s(\mathbf{q}_T^2) = \frac{-\lambda^2}{(4\pi)^2} \int \frac{d^2\mathbf{l}_T}{[\mathbf{l}_T^2 + m^2][(\mathbf{q}_T - \mathbf{l}_T)^2 + m^2]}$$

is the one-loop *Regge trajectory* of a scalar in this theory.

This *Regge(-pole) behaviour* of the amplitude is remeniscent of the actual behaviour of the pp and $p\bar{p}$ elastic cross sections at high energy, due to the *Pomeron* exchange.
But asymptotics $\sigma \sim s^{\omega_P}$ with $\omega_P > 0$ contradicts Froissart bound, stating that $\sigma \leq \frac{1}{m_0^2} \ln^2(s/s_0)$, what is the *unitarization mechanism* at high energy?

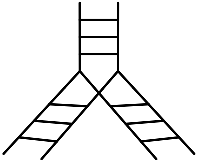
Reggeon Field Theory

The idea of RFT had been proposed by Gribov [Gribov, '68] . We introduce Reggeon fields which depend on *rapidity* y ($\sim \ln s$) and *transverse coordinates* $\mathbf{x}_T$: $R_\pm(y, \mathbf{x}_T)$. Then the "Reggeized" t-channel exchange follows from the Lagrangian:

$$L_{\rm RFT}^{(\rm kin.)} = R_+(y,\mathbf{x}_T)\left(\frac{\partial}{\partial y} - \omega_s(\mathbf{x}_T^2)\right)\partial_T^2 R_-(y,\mathbf{x}_T)$$

$$\Rightarrow \langle R_-(y_1,\mathbf{q}_T^2) R_+(y_2,\mathbf{q}_T^2)\rangle = \frac{i}{\mathbf{q}_T^2}\theta(y_1-y_2)\exp[(y_1-y_2)\omega_s(\mathbf{q}_T^2)].$$

Reggeons also can interact:



In phenomenological RFT the local interactions of Pomerons, Odderons etc is assumed, e.g.:

$$L_{\rm RFT}^{(\rm int.)} = g[R_+(y,\mathbf{x}_T)R_-(y,\mathbf{x}_T)R_-(y,\mathbf{x}_T) + (R_+ \leftrightarrow R_-)] + \dots,$$

which is probably a crude approximation.

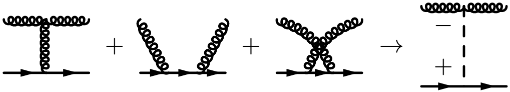
Our goal is to construct RFT from QCD and use it to do perturbative resummations for various observables.

Reggeized gluon

Studying tree-level amplitudes one quickly finds that **t-channel gluon exchanges dominate at the leading power at $s \to \infty$.** Often (e.g. for $qq' \to qq'$), just the replacement of the t-channel gluon propagator (Gribov's trick) extracts the LP contribution:

$$g^{\mu\nu} \to \frac{1}{2}(n_-^\mu n_+^\nu + n_+^\mu n_-^\nu).$$

But in a generic gauge, all 3 diagrams contribute to $qg \to qg$ amplitude:



The $R_\pm gg$ vertex reads:

$$\Gamma^{abc}_{\mu_1\mu_2 -} = f^{abc}\Big[(k_1^+ - k_2^+)g_{\mu_1\mu_2} + n^+_{\mu_1}(2k_1+k_2)_{\mu_2} + n^+_{\mu_2}(2k_2+k_1)_{\mu_1} - \frac{q^2}{k_1^+}n^+_{\mu_1}n^+_{\mu_2}\Big].$$

- The vertex satisfies Slavnov-Taylor identity: $\varepsilon^{\mu_1}(k_1)k_2^{\mu_2}\Gamma^{abc}_{\mu_1\mu_2 -} = 0 = k_1^{\mu_1}\varepsilon^{*\mu_2}(k_2)\Gamma^{abc}_{\mu_1\mu_2 -}$
- It contains nonlocal "*induced*" term
- Terms in the second row are zero in the gauge $A_+ = 0$

Action of the Rg interaction

$$S_{\text{int.}} \supset \int dy \int d^4x\ \delta(x_+) 2\,\text{tr}\left\{R_-(y,\mathbf{x}_T) j_+[A_\mu^{[y,y+\eta]}](x)\right\}$$

where $1 \ll \eta \ll Y$. The j_+ in the gauge $\overline{A}_+ = 0$ is given by the

$$\Gamma^{abc}_{\mu_1\mu_2 -} = f^{abc}(k_1^+ - k_2^+) g_{\mu_1\mu_2} \leftrightarrow j_+[A_\mu] = ig_s\left[\overline{A}_\mu, \partial_+ \overline{A}^\mu\right]$$
$$= -\left[\overline{D}_\mu, \overline{G}_{\mu+}\right] - \partial_\mu \partial_+ \overline{A}^\mu,$$

where the first term can be dropped (at tree level) due to equations of motion $\left[\overline{D}_\mu, \overline{G}_{\mu+}\right] = 0$.

The field in the gauge $\overline{A}_+ = 0$ can be obtained from the field in arbitrary gauge A_μ by the following gauge transformation:

$$\overline{A}_\mu = \frac{-i}{g_s} W^\dagger[A_+](x) D_\mu W[A_+](x),$$

where $D_\mu = \partial_\mu + ig_s A_\mu$ and

$$W[A_\pm](x) = P\exp\left[\frac{-ig_s}{2}\int\limits_{-\infty}^{x_\mp} dx'_\mp A_\pm(x_\pm, x'_\mp, \mathbf{x}_T)\right]$$
$$= \left(\hat{1} + ig_s \partial_\pm^{-1} A_\pm\right)^{-1}\hat{1} = \hat{1} - ig_s(\partial_\pm^{-1}A_\pm) - (ig_s)^2(\partial_\pm^{-1}A_\pm\partial_\pm^{-1}A_\pm) + \dots,$$

Action of the Rg interaction

The interaction term can be further simplified:

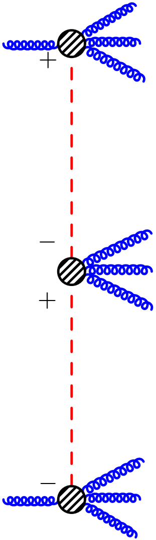
$$
\begin{aligned}
j_+[A_\mu](x) \quad \to \quad & -\partial_\mu \partial_+ \overline{A}^\mu = \frac{i}{g_s} \partial_\mu \partial_+ W^\dagger[A_+](x) D_\mu W[A_+](x) \\
= \quad & \frac{i}{g_s} \partial_\mu \partial_+ \left(\hat{1} + i g_s (\partial_+^{-1} A_\pm) + \ldots \right) \left[\partial^\mu + i g_s A^\mu \right] W[A_+](x) \\
= \quad & \frac{i}{g_s} \partial^2 \partial_+ W[A_+](x) - \partial_+ \partial_\mu \left((\partial_+^{-1} A_\pm) + \ldots \right) D^\mu W[A_+](x),
\end{aligned}
$$

the last term gives the vanishing contribution due to the conservation of the k_+-momentum component and $\partial^2 \to \partial_T^2$. Finally the interaction term takes the form

$$
S_{\text{int.}} \supset \frac{i}{g_s} \int dy \int d^4x \; \delta(x_+) 2 \, \text{tr} \left\{ R_-(y, \mathbf{x}_T) \partial_T^2 \partial_+ W[A_+^{[y, y+\eta]}] \right\},
$$

clearly it is *non-Hermitian*, which does not cause problems at tree level. Beyond tree level, the simplest Hermitian form, compatible with *negative signature* of R-exchange is [Lipatov '97; Bondarenko, Zubkov, '18] :

$$
S_{\text{int.}} \supset \frac{i}{g_s} \int dy \int d^4x \; \delta(x_+) \, \text{tr} \left\{ R_-(y, \mathbf{x}_T) \partial_T^2 \partial_+ \left(W[A_+^{[y, y+\eta]}] - W^\dagger[A_+^{[y, y+\eta]}] \right) \right\}.
$$



- **Induced interactions** of particles and Reggeons [Lipatov '95, '97; Bondarenko, Zubkov '18]:

$$L = \frac{i}{g_s}\,\mathrm{tr}\left[R_+\partial_\perp^2\partial_-\left(W\left[A_-\right]-W^\dagger\left[A_-\right]\right)+(+\leftrightarrow-)\right],$$

with $W_{x_\mp}[x_\pm, \mathbf{x}_T, A_\pm] = P\exp\left[\frac{-ig_s}{2}\int\limits_{-\infty}^{x_\mp} dx'_\mp A_\pm(x_\pm, x'_\mp, \mathbf{x}_T)\right] = \left(1+ig_s\partial_\pm^{-1}A_\pm\right)^{-1}.$

- Expansion of the Wilson line generates **induced vertices**:

$$\mathrm{tr}\left[R_+\partial_\perp^2 A_- + (-ig_s)(\partial_\perp^2 R_+)(A_-\partial_-^{-1}A_-)\right.$$
$$\left. +(-ig_s)^2(\partial_\perp^2 R_+)(A_-\partial_-^{-1}A_-\partial_-^{-1}A_-) + O(g_s^3) + (+\leftrightarrow-)\right].$$

- The *Eikonal propagators* $\partial_\pm^{-1} \to -i/(k^\pm)$ lead to **rapidity divergences**, which are regularized by tilting the Wilson lines from the light-cone [Hentschinski, Sabio Vera, Chachamis *et. al.*, '12-'13; M.N. '19]:

$$n_\pm^\mu \to \tilde{n}_\pm^\mu = n_\pm^\mu + r n_\mp^\mu,\ r \ll 1:\ \tilde{k}^\pm = \tilde{n}^\pm k.$$

- To keep the action Gauge-invariant at finite r one has to substitute $\delta(x_\pm) \to \delta(x_\pm - r x_\mp)$ [MN, 2019]

Feynman rules

Rg-**transition vertex** ("nonsense polarisation"):

$$L_{Rg} \supset \frac{i}{g_s}\mathrm{tr}\left[R_-\partial_\rho^2\partial_+(-2ig_s)\partial_+^{-1}A_+\right] \to \Delta^{ab}_{-\mu}(q) = (-iq^2)n^+_\mu\delta_{ab},$$

Rgg **induced vertex:**

$$\frac{i}{g_s}\mathrm{tr}\left[R_-\partial_\rho^2\partial_+(-g_s^2)\left(T^{b_1}T^{b_2}-T^{b_2}T^{b_1}\right)\partial_+^{-1}A_+^{b_1}\partial_+^{-1}A_+^{b_2}\right] = -ig_s\frac{if^{ab_1b_2}}{2}R^a_-\partial_\rho^2A_+^{b_1}\partial_+^{-1}A_+^{b_2}$$

$$\to \Delta^{ab_1b_2}_{-\mu_1\mu_2}(q,k_1) = g_s(n^+_{\mu_1}n^+_{\mu_2})\frac{q^2}{2}\left(\frac{f^{ab_1b_2}}{k_2^+ + i\varepsilon}+\frac{f^{ab_2b_1}}{k_1^+ + i\varepsilon}\right) = g_sq^2(n^+_{\mu_1}n^+_{\mu_2})\frac{f^{ab_1b_2}}{[k_1^+]},$$

Rggg **and** *Rgggg* **induced vertices:**

$$\Delta^{ab_1b_2b_3}_{-\mu_1\mu_2\mu_3} = -ig_s^2q^2(n^+_{\mu_1}n^+_{\mu_2}n^+_{\mu_3})\sum_{(i_1,i_2,i_3)\in S_3}\frac{\mathrm{tr}\left[T^a\left(T^{b_{i_1}}T^{b_{i_2}}T^{b_{i_3}}+T^{b_{i_3}}T^{b_{i_2}}T^{b_{i_1}}\right)\right]}{(k_{i_3}^+ + i\varepsilon)(k_{i_3}^+ + k_{i_2}^+ + i\varepsilon)},$$

$$\Delta^{ab_1b_2b_3b_4}_{-\mu_1\mu_2\mu_3\mu_4} = -ig_s^3q^2(n^+_{\mu_1}n^+_{\mu_2}n^+_{\mu_3}n^+_{\mu_4})$$

$$\times\sum_{(i_1,i_2,i_3,i_4)\in S_4}\frac{\mathrm{tr}\left[T^a\left(T^{b_{i_1}}T^{b_{i_2}}T^{b_{i_3}}T^{b_{i_4}}-T^{b_{i_4}}T^{b_{i_3}}T^{b_{i_2}}T^{b_{i_1}}\right)\right]}{(k_{i_4}^+ + i\varepsilon)(k_{i_4}^+ + k_{i_3}^+ + i\varepsilon)(k_{i_4}^+ + k_{i_3}^+ + k_{i_2}^+ + i\varepsilon)},$$

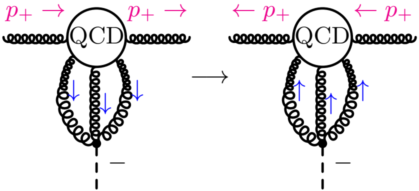
and so on...

Signature of induced vertices

In the LLA the Reggeized gluon has negative signature w.r.t. $s \to -s$:

$$\mathcal{M}_{1R}(gg \to gg) \propto f^{a_1 c a_2} f_{a_3 c a_4} \frac{s}{t} \left[\left(\frac{s}{-t} \right)^{\omega_g^{(1)}(-t)} + \left(\frac{-s}{-t} \right)^{\omega_g^{(1)}(-t)} \right] \delta_{\lambda_1 \lambda_2} \delta_{\lambda_3 \lambda_4},$$

we want to keep this property to all orders in the EFT.
Signature $p_+ \to -p_+$:



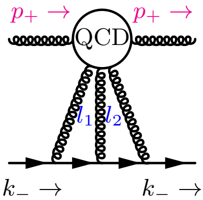
Simple graph-theoretic arguments show that the signature of $Rg \dots g$ vertex with n-gluons ($O(g_s^{n-1})$) is $(-1)^{n-1}$.
This property should be respected by $i\varepsilon$-prescriptions for Eikonal poles.

Signature of QCD part is $(-1)^{(\#\text{ 3g vert.})}$

The vertices from Hermitian version of the EFT satisfy the signature property.

The sgn(ε) independence

The induced vertices come from QCD diagrams like:



$$\propto \sum_{\text{perms.}} \frac{T^{a_1} \dots T^{a_n}}{(l_1^+ + i\varepsilon/k_-)(l_1^+ + l_2^+ + i\varepsilon/k_-)\dots},$$

factorisation requires **independence** on the sign of $k_- \leftrightarrow$ sign of ε.
This property is automatically satisfied by the EFT vertices:

$$S_{Rg} \supset \frac{i}{g_s} \int d^2\mathbf{x}_T \int dx_- \mathrm{tr}\left\{R_-(\mathbf{x}_T)\partial_T^2 \frac{\partial}{\partial x_-}\left[W_{(-\infty_-,x_-)}[A_+] - W^\dagger_{(-\infty_-,x_-)}[A_+]\right]\right\}$$
$$= \frac{i}{g_s} \int d^2\mathbf{x}_T \mathrm{tr}\left\{R_-(\mathbf{x}_T)\partial_T^2 \left[W_{(-\infty_-,+\infty_-)}[A_+] - W_{(+\infty_-,-\infty_-)}[A_+]\right]\right\}.$$

Additionally in [Hentschinski, '11] the *maximal anti-symmetry* of the colour factor in the induced vertices had been imposed. The physical motivation for this choice is less clear for me.

Relation with $\ln W$-definition

In [Caron-Huot, '12] an alternative definition of the Reggeized gluon operator had been proposed:

$$S_{Rg} \supset \int d^2\mathbf{x}_T \frac{f^{abc}}{C_A g_s} R_-^a(\mathbf{x}_T) \left\{ \ln \left[W^{\text{adj.}}_{(-\infty_-,+\infty_+,\mathbf{x}_T)}[A_+] \right] \right\}_{bc},$$

where the infinite lightlike adjoint Wilson line is:

$$W^{\text{adj.}}_{(-\infty_-,+\infty_+,\mathbf{x}_T)}[A_+] = 1 +$$

$$\sum_{n=1}^{\infty} (-g_s)^n f^{ba_1c_1} f^{c_1a_2c_2} \dots f^{c_{n-1}a_nc} \int\limits_{-\infty}^{+\infty} dx_- \partial_+ (\partial_+^{-1} A_+^{a_1} \dots \partial_+^{-1} A_+^{a_n}).$$

For tree-level $Rg \dots g$ vertices (i.e. without $i\varepsilon$) all three definitions agree

(checked up to $n = 4$, MH has the all-order proof)

Three definitions differ if one takes into account $i\varepsilon$ prescriptions. For $Rggg$ vertex the difference between all three approaches is proportional to:

$$\delta(k_1^+)\delta(k_2^+) \sum_{(i_1,i_2,i_3)\in S_3} \text{tr}\left[T^a T_{i_1} T_{i_2} T_{i_3}\right],$$

which **does not contribute to 2-loop Regge trajectory** but starts to contribute at 3 loops.

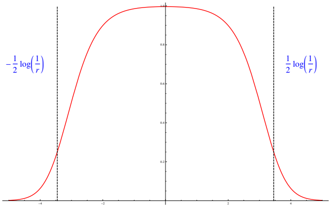
Regularisation by tilted Wilson lines

The *Eikonal propagators* $\partial_\pm^{-1} \to -i/(k^\pm)$ lead to **rapidity divergences**, which are regularized by tilting the Wilson lines from the light-cone [Hentschinski, Sabio Vera, Chachamis *et. al.*, '12-'13; M.N. '19]:

$$n_\pm^\mu \to \tilde{n}_\pm^\mu = n_\pm^\mu + r n_\mp^\mu,\ r \ll 1:\ \tilde{k}^\pm = \tilde{n}^\pm k.$$

To keep the action Gauge-invariant at finite r one has to substitute $\delta(x_\pm) \to \delta(x_\pm - r x_\mp)$ [MN, 2019]

For real emissions this is equivalent to a smooth cutoff in rapidity ($\eta = \ln r$):



The square of regularized Lipatov's $(R_+ R_- g)$ vertex:

$$\Gamma_{+\mu-}\Gamma_{+\nu-}P^{\mu\nu} = \frac{16\mathbf{q}_{T1}^2\mathbf{q}_{T2}^2}{\mathbf{k}_T^2} f(y),$$

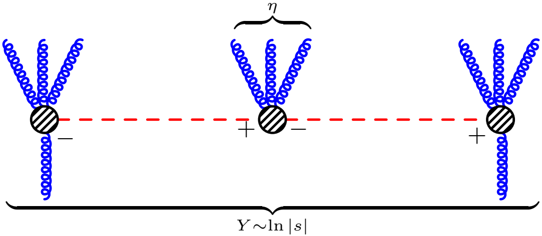
$$\longleftarrow f(y) = \frac{1}{(re^{-y}+e^{y})^2(re^{y}+e^{-y})^2},$$

$$\int\limits_{-\infty}^{+\infty} dy\ f(y) = -1 - \ln r + O(r)$$

The pre-RFT action

$$S = \int dy \int d^2\mathbf{x}_T 2\,\mathrm{tr}\left\{R_+(y,\mathbf{x}_T)\partial_T^2 \frac{\partial}{\partial y} R_-(y+\eta,\mathbf{x}_T)\right\}$$
$$+\frac{i}{g_s}\int dy \int d^4x\ \mathrm{tr}\left\{\delta(x_+)R_-(y,\mathbf{x}_T)\partial_T^2\partial_+\left(W[A_+^{[y,y+\eta]}] - W^\dagger[A_+^{[y,y+\eta]}]\right) + (+\leftrightarrow -)\right\}$$
$$+\int dy\left(S_{\mathrm{QCD}}\left[A_\mu^{[y,y+\eta]}\right] + S_{\mathrm{RFT}}^{(\mathrm{int.})}\left[R_+(y,\mathbf{x}_T), R_-(y,\mathbf{x}_T)\right]\right),$$

Integrating-out usual gluons (A_μ) and quarks we will obtain the RFT in QCD.



Bare Reggeon propagator:

$$\langle R_-^a R_+^b \rangle = \frac{i\delta^{ab}}{2\mathbf{q}_T^2}\theta(y_1 - y_2 - \eta)$$

Regulator:

$$1 \ll \eta \ll Y.$$

The dependence on the regulator η should cancel between integrations in y and the dependence of vertices on $\eta \Rightarrow$ *Rapidity renormalization group.*

Building the RFT

We construct the RFT interactions:

$$S_{\mathrm{RFT}}^{(\mathrm{int.})} = \int d^2\mathbf{x}_T d^2\mathbf{x}'_T \; R_+(y, \mathbf{x}_T) \left(K_{-+}(\mathbf{x}_T, \mathbf{x}'_T) \frac{\partial}{\partial y} - \omega_g(\mathbf{x}_T, \mathbf{x}'_T) \right) R_+(y+\eta, \mathbf{x}'_T)$$

$$+ \int d^2\mathbf{x}_T d^2\mathbf{x}'_{T1} d^2\mathbf{x}'_{T2} \; R_-(y, \mathbf{x}_T) K_{-++}(\mathbf{x}_T, \mathbf{x}'_{T1}, \mathbf{x}'_{T2}) R_+(y+\eta, \mathbf{x}'_{T1}) R_+(y+\eta, \mathbf{x}'_{T2})$$

$$+ \int d^2\mathbf{x}_{T1} d^2\mathbf{x}_{T2} d^2\mathbf{x}'_T \; R_-(y, \mathbf{x}_{T1}) R_-(y, \mathbf{x}_{T2}) K_{--+}(\mathbf{x}_{T1}, \mathbf{x}_{T2}, \mathbf{x}'_T) R_+(y+\eta, \mathbf{x}'_T)$$

$$+ \int d^2\mathbf{x}_{T1} d^2\mathbf{x}_{T2} d^2\mathbf{x}'_{T1} d^2\mathbf{x}'_{T2} \; R_-(y, \mathbf{x}_{T1}) R_-(y, \mathbf{x}_{T2}) K_{--++} R_+(y+\eta, \mathbf{x}'_{T1}) R_+(y+\eta, \mathbf{x}'_{T2})$$

$$+ \dots$$

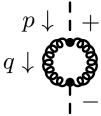
in such a way that the η-dependence cancels.

2-point function

The quadratic part of the RFT action leads to the "Reggeized" propagator:

$$\langle R_-(y_1,\mathbf{q}_T)R_+(y_2,\mathbf{q}_T)\rangle = \frac{i}{2\mathbf{q}_T^2}\theta(y_1-y_2-\eta)e^{\omega_g(\mathbf{q}_T^2)(y_1-y_2-\eta)},$$

while the Reggeon self-energy contains a rapidity-divergent contribution:



$$= \quad g_s^2 C_A \delta_{ab}\int \frac{d^d q}{(2\pi)^D}\frac{\left(\mathbf{p}_T^2(n_+n_-)\right)^2}{q^2(p-q)^2[q^+][q^-]}\theta\left(\eta-\frac{1}{2}\mathrm{Re}\ln\frac{q_+}{q_-}\right)$$

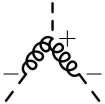
$$= \quad \eta\omega_g^{(1)}(\mathbf{p}_T^2) \text{ (or } \omega_g^{(1)}(\mathbf{p}_T^2)\ln r \text{ in TWL regularization).}$$

where $\frac{1}{[q_\pm]}=\frac{1}{2}\left(\frac{1}{q_\pm+i\varepsilon}+\frac{1}{q_\pm-i\varepsilon}\right)$ and *one-loop Regge trajectory of a gluon* is:

$$\omega_g^{(1)}(\mathbf{p}_T^2) = C_A g_s^2\int\frac{\mathbf{p}_T^2 d^{D-2}\mathbf{q}_T}{\mathbf{q}_T^2(\mathbf{p}_T-\mathbf{q}_T)^2}.$$

The cancellation of η-dependence requires $\omega_g(\mathbf{p}_T^2)=\omega_g^{(1)}(\mathbf{p}_T^2)+O(\alpha_s^2)$

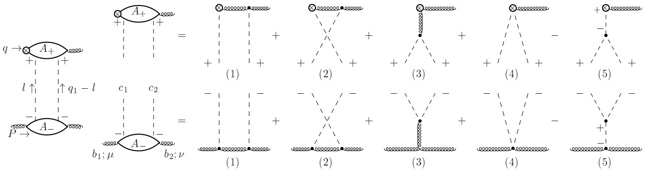
3-point function



$$= 4g_s f^{abc}\mathbf{p}_T^2 \int \frac{dk^-}{[k^-]}.$$

Two interpretations:

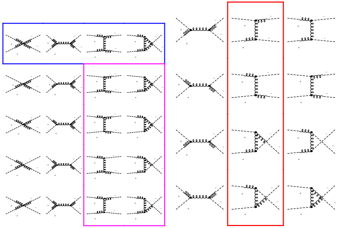
- Put integral $\int \frac{dk^-}{[k^-]} = 0 \Rightarrow$ all **even-odd** transitions are forbidden (*Gribov's signature conservation rule*).
- Put this vertex into RFT with the opposite sign, as the subtraction term for ill-defined light-cone momentum integrals [Hentchinski PhD thesis; M.N. 2019] .



$$f^{abc}\int_{-\infty}^{+\infty} dl_- \left\{ \frac{1}{l_- + \mathbf{l}_T^2/P_+ - i\varepsilon} + \frac{1}{l_- + (2\mathbf{l}_T\mathbf{q}_{T1} - \mathbf{l}_T^2)/P_+ + i\varepsilon} - \frac{1}{l_- - i\varepsilon} - \frac{1}{l_- + i\varepsilon} \right\} = 0.$$

4-point function, BFKL equation

Connected diagrams:



Lead to the rapidity-divergent contribution [Bartels, Lipatov, Vacca, 2012] :

$$\langle R_+^{a_1}(\mathbf{p}_{T1})R_+^{a_2}(\mathbf{p}_{T2})R_-^{b_1}(\mathbf{k}_{T1})R_-^{b_2}(\mathbf{k}_{T2})\rangle$$

$$= -i\alpha_s\eta[f^{a_1cb_1}f^{ca_2b_2}K_0 + (b_1 \leftrightarrow b_2, \mathbf{k}_{T1} \leftrightarrow \mathbf{k}_{T2})]$$

$$K_0 = \frac{\mathbf{k}_{T2}^2\mathbf{p}_{T1}^2 + \mathbf{k}_{T1}^2\mathbf{p}_{T2}^2}{\mathbf{k}_T^2} - \mathbf{q}_T^2,$$

where $\mathbf{k}_T = \mathbf{k}_{T1} - \mathbf{p}_{T1}$,
$\mathbf{q}_T = \mathbf{k}_{T2} - \mathbf{k}_{T1}$.

Together with the disconnected part form Regge trajectory, we get *the BFKL equation* for $2R$ Green's function, e.g. for **1** pair [BFKL, '76] :

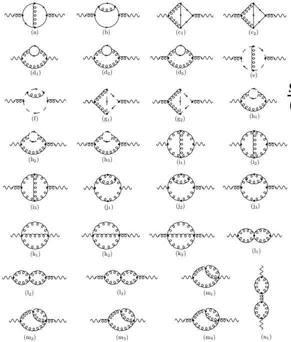
$$\frac{\partial}{\partial Y}G_Y(\mathbf{p}_{T1},\mathbf{p}_{T2}) = \frac{\alpha_s C_A}{\pi}\int d^{2-2\epsilon}\mathbf{k}_{T1,2}\Big[K_0(\mathbf{p}_{T1},\mathbf{p}_{T2},\mathbf{k}_{T1},\mathbf{k}_{T2})G_Y(\mathbf{k}_{T1},\mathbf{k}_{T2})$$

$$+(\omega_g^{(1)}(\mathbf{p}_{T1}) + \omega_g^{(1)}(\mathbf{p}_{T2}))G_Y(\mathbf{p}_{T1},\mathbf{p}_{T2})\Big],$$

For the **1** RR-pair the IR divergence at $\mathbf{k}_T \to 0$ cancels within the kernel.

2-loop Regge trajectory from the EFT

The EFT formalism had been tested at 2 loops in [Chachamis, *et al.*, 2013] .



$$\frac{g^4 N_c^2}{(4\pi)^4}\Bigg(\left\{\frac{2}{\epsilon^2}+\frac{4(1-\Xi)}{\epsilon}+4(1-\Xi)^2-\frac{\pi^2}{3}\right\}\ln^2 r+\Bigg\{\frac{7}{\epsilon^2}-\frac{14\Xi}{\epsilon} -\frac{1-\pi^2}{3\epsilon}-2\frac{\Xi(\pi^2-1)}{3}+14(1+\Xi^2)+\frac{2}{9}-\frac{\pi^2}{2}-2\zeta(3) -i\pi\left[\frac{2}{\epsilon^2}+4\frac{1-\Xi}{\epsilon}+\frac{1}{3}(12(1-\Xi)^2-\pi^2)\right]\Bigg\}\ln r\Bigg),$$

$$\text{where } \Xi = 1-\gamma_E-\ln\frac{\mathbf{q}_T^2}{4\pi\mu^2}.$$

The coefficient in front of $\ln^2 r$ coincides with $[\omega_g^{(1)}(\mathbf{q}_T^2)]^2/2$ (exponentiation!). After subtracting it, the coefficient in front of $\ln r$ reproduces the QCD result for 2-loop Regge trajectory [Fadin, Fiore, Kotsky '96] .

II. One-loop corrections to massless impact-factors

Rapidity divergences at one loop

Only log-divergence $\sim \log r$ (Blue cells in the table) is related with Reggeization of particles in t-channel.
Integrals which **do not** have log-divergence may still contain the power-dependence on r:

- $r^{-\epsilon} \to 0$ for $r \to 0$ and $\epsilon < 0$.
- $r^{+\epsilon} \to \infty$ for $r \to 0$ and $\epsilon < 0$ – **weak-power divergence** (Pink cells in the table)
- $r^{-1+\epsilon} \to \infty$ – **power divergence.** (Red)

(# LC prop.) \ (# quadr. prop.)	1	2	3	4
1	$A_{[-]}$	$B_{[-]}$	$C_{[-]}$	...
2	$A_{[+-]}$	$B_{[+-]}$	$C_{[+-]}$	...
3	...	...	...	...

The **weak-power** and **power-divergences** cancel between Feynman diagrams describing one region in rapidity, so only log-divergences are left.

Scalar integrals with power RDs.

Notation: $\left\{\frac{\mu}{k}\right\}^{2\epsilon} = \frac{1}{2}\left[\left(\frac{\mu}{k-i\varepsilon}\right)^{2\epsilon} + \left(\frac{\mu}{-k-i\varepsilon}\right)^{2\epsilon}\right].$

Tadpoles:

$$A_{[-]}(p) = -\frac{\tilde{p}^- \; r^{-1+\epsilon}}{\cos(\pi\epsilon)} \frac{1}{2\epsilon(1-2\epsilon)} \left\{\frac{\mu}{\tilde{p}^-}\right\}^{2\epsilon},$$

$$A_{[--]}(p) = \frac{1}{\tilde{p}_-} A_{[-]}(p).$$

Bubbles:

$$B_{[-]}(p) = \frac{1}{p^-\epsilon^2}\left(\frac{\mu^2}{-p^2}\right)^{\epsilon} + \frac{1-2\epsilon}{\epsilon} \frac{r \cdot A_{[-]}(p)}{\tilde{p}_-^2} + \Delta B_{[-]}(-p^2, p_-) + O(r),$$

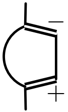
$$B_{[--]}(p) = \frac{2}{\tilde{p}_-} B_{[-]}(p),$$

where:

$$\Delta B_{[-]}(-p^2, p_-) = -\frac{1}{p_-}\left(\frac{p_-^2\mu^2}{(-p^2)^2}\right)^{\epsilon} \frac{\Gamma^2(1-2\epsilon)\Gamma(1+2\epsilon) \cdot r^{-\epsilon}}{2\epsilon^2\Gamma^2(1-\epsilon)}.$$

Logarithmic RDs

- $[+-]$-bubble in transverse kinematics $p^- = p^+ = 0$:



$$B_{[+-]}(\mathbf{p}_T) = \frac{1}{\mathbf{p}_T^2}\left(\frac{\mu^2}{\mathbf{p}_T^2}\right)^{\epsilon} \frac{i\pi + 2\log r}{\epsilon},$$

- $[+-]$-bubble in $p^- = 0$ kinematics:

$$\begin{aligned} B_{[+-]}(\mathbf{p}_T, p^+) &= \frac{1}{\mathbf{p}_T^2}\left(\frac{\mu^2}{\mathbf{p}_T^2}\right)^{\epsilon} \frac{\Gamma^2(1+\epsilon)\Gamma(2+\epsilon)\sin(\pi\epsilon)}{\pi\epsilon^2} \\ &\times \left[i\pi + \log r - \log\frac{p_+^2}{\mathbf{p}_T^2} - \psi(1+\epsilon) + \psi(1)\right] + O(r^{1/2}) \end{aligned}$$

- $[+-]$-bubble in light-like kinematics $p^2 = 0$:

$$B_{[+-]}(\mathbf{p}_T^2, p^2 = 0) = \int \frac{[d^d l]}{l^2(l+p)^2[l^+][l^- + p^-]} = \frac{-2\Gamma(1-\epsilon)\Gamma(1+\epsilon)}{\mathbf{p}_T^2\epsilon^2}\left(\frac{\mu^2}{\mathbf{p}_T^2}\right)^{\epsilon}.$$

Triangle integrals, logarithmic RD

Result for $Q^2 = 0$:

$$C_{[-]}(t_1, 0, q^-) = \frac{1}{q^- t_1}\left(\frac{\mu^2}{t_1}\right)^{\epsilon}\frac{1}{\epsilon}\left[\log r + i\pi - \log\frac{|q_-|^2}{t_1}\right.$$
$$\left. -\psi(1+\epsilon) - \psi(1) + 2\psi(-\epsilon)\right] + O(r^{1/2}),$$

coincides with the result of [G. Chachamis, *et. al.*, '12] .

Result for $Q^2 \neq 0$ [M.N., '19] :

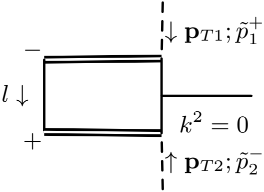
$$C_{[-]}(t_1, Q^2, q_-) = C_{[-]}(t_1, 0, q_-) + \left(\frac{\mu^2}{t_1}\right)^{\epsilon}\frac{I(Q^2/t_1)}{q_- t_1} - \frac{1}{t_1}\Delta B_{[-]}(Q^2, q_-),$$

where

$$\begin{aligned} I(X) &= -\frac{2X^{-\epsilon}}{\epsilon^2} - \frac{2}{\epsilon}\int\limits_0^X \frac{(1-x^{-\epsilon})dx}{1-x} \\ &= -\frac{2X^{-\epsilon}}{\epsilon^2} + 2\left[-\mathrm{Li}_2(1-X) + \frac{\pi^2}{6}\right] + O(\epsilon). \end{aligned}$$

Triangle with two light-cone propagators

Usual one-loop Feynman integrals with more than 4 propagators are reducible to more simple integrals up to terms $O(\epsilon)$.



We apply method of [Bern, Dixon, Kosower, '92]. The $O(\epsilon)$ remnant is proportional to $(d-4)I^{(d+2)}$ and integral $I^{(6)}$ is finite.
The resiut in Euclidean region ($p_1^+ > 0$, $-p_2^- > 0$, $\mathbf{p}_{T1,2}^2 > 0$):

$$C_{[+-]}(\mathbf{p}_{T1}^2, \mathbf{p}_{T2}^2, p_1^+, -p_2^-) = \frac{(-1)}{2\mathbf{p}_{T1}^2 \mathbf{p}_{T2}^2 \mathbf{k}_T^2} \times$$
$$\{\mathbf{p}_{T1}^2(\mathbf{p}_{T2}^2 - \mathbf{p}_{T1}^2 + \mathbf{k}_T^2)\left[B_{[+-]}(\mathbf{p}_{T1}^2, p_1^+) + (-p_2^-)C_{[-]}(\mathbf{p}_{T1}^2, \mathbf{p}_{T2}^2, -p_2^-)\right]$$
$$+\mathbf{p}_{T2}^2(\mathbf{p}_{T1}^2 - \mathbf{p}_{T2}^2 + \mathbf{k}_T^2)\left[B_{[+-]}(\mathbf{p}_{T2}^2, -p_2^-) + p_1^+ C_{[+]}(\mathbf{p}_{T2}^2, \mathbf{p}_{T1}^2, p_1^+)\right]$$
$$-\mathbf{k}_T^2(\mathbf{p}_{T1}^2 + \mathbf{p}_{T2}^2 - \mathbf{k}_T^2)B_{[+-]}(\mathbf{k}_T^2, k^2 = 0)\},$$

where $\mathbf{k}_T^2 = p_1^+(-p_2^-)$.
The $\log r$-divergence cancels within square brackets, as expected.

Forward scattering vertices (impact factors) at one loop

The one-loop results for quark and gluon impact-factors had been reproduced in [Chachamis, *et al.*, '12] .
The Particle-Particle-Reggeon wis two scales of virtuality [MN, '19] :

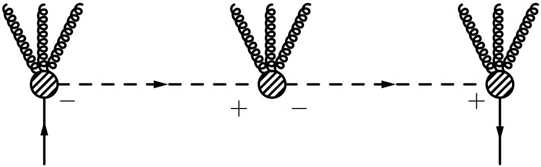
$$\gamma^{\star}(q) + Q_{+}(q_1) \to q(q + q_1), \qquad (1)$$
$$\mathcal{O}(q) + R_{+}(q_1) \to g(q + q_1), \qquad (2)$$

where $q^2 = Q^2 \neq 0$ and $\tilde{q}_1^- = 0$ to ensure GI at $r \neq 0$ and

$$\mathcal{O}(x) = \mathrm{tr}[G_{\mu\nu}G^{\mu\nu}],$$

with $G_{\mu\nu}$ being the QCD field-strength tensor.
The result (2) had later been used by [Hetnschinski *et al.*; Fucilla *et al.*] for the computation of the Higgs production impact factor at NLO.

EFT for QMRK-processes with quark exchange



EFT for Reggeized quarks [Lipatov, Vyazovsky, '01] :

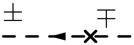
$$L_Q = \bar{Q}_- i\hat{\partial}\left(Q_+ - W^\dagger\left[A_+\right]\psi\right) + \bar{Q}_+ i\hat{\partial}\left(Q_- - W^\dagger\left[A_-\right]\psi\right) + \text{h.c.},$$

where $\hat{p} = p_\mu \gamma^\mu$, QMRK kinematic constraints:

$$\partial_\pm Q_\mp = \partial_\pm \bar{Q}_\mp = 0,$$
$$\hat{n}^\pm Q_\mp = 0,\ \bar{Q}_\mp \hat{n}^\pm = 0. \Rightarrow$$

Reggeized quark propagator ($\hat{P}_\pm = \hat{n}_\mp \hat{n}_\pm / 4$):

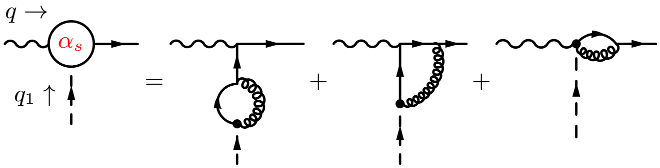
$= \hat{P}_\pm \frac{i\hat{q}}{q^2},$



$= \frac{i\hat{q}}{q^2} \hat{P}_\pm.$

$Q\gamma^\star q$-vertex

$\Gamma^{(1)}_{+\mu} =$



$$= C[\Gamma]\cdot\Gamma^{(0)}_{+\mu}(q_1,q) + C[\Delta^{(1)}]\cdot\Delta^{(1)}_{+\mu}(q_1,q) + C[\Delta^{(2)}]\cdot\Delta^{(2)}_{+\mu}(q_1,q)$$

Lorentz structures:

$$\Gamma^{(0)}_{+\mu}(q_1,k,q_2) = \gamma_\mu + \frac{\hat{q}_1 n^-_\mu}{[\tilde{k}^-]}, \quad \longleftarrow$$ [Fadin, Sherman, '76]

$$\Delta^{(1)}_{+\mu}(q_1,q) = \frac{\hat{q}}{q_-}\left(n^-_\mu - \frac{2(q_1)_\mu}{q_1^+}\right), \; \Delta^{(2)}_{+\mu}(q_1,q) = \frac{\hat{q}}{q_-}\left(n^-_\mu - \frac{q_\mu}{q^+}\right)$$

Cancellation of RDs:

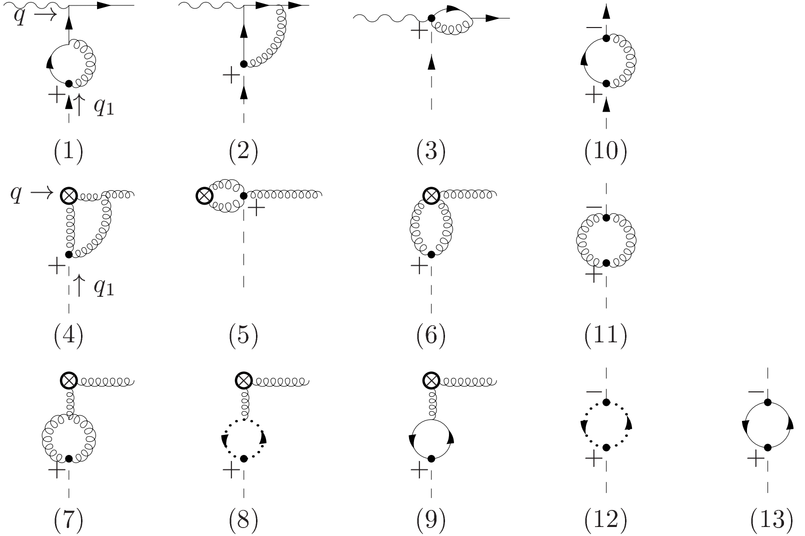
- $A_{[-]} \sim r^{-1+\epsilon}$ – cancels between diagrams
- $O(r^\epsilon)$-terms cancel between $B_{[-]}(q)$ and $B_{[-]}(q+q_1)$
- $O(r^{-\epsilon})$-terms cancel between $B_{[-]}(q)$ and $C_{[-]}$.
- **only $O(\log r)$-divergence from $C_{[-]}$ is left**

Expressions for the coefficients

$$C[\Gamma] = -\frac{\bar{\alpha}_s C_F}{4\pi}\frac{1}{2}\left\{\frac{[(d-8)Q^2+(d-6)t_1]B(t_1)-2(d-7)Q^2B(Q^2)}{Q^2-t_1}\right.$$
$$\left. -2\left[(Q^2-t_1)C(t_1,Q^2)-q_-\left(t_1C_{[-]}(t_1,Q^2,q_-)+(B_{[-]}(q)-B_{[-]}(q+q_1))\right)\right]\right\},$$
$$C[\Delta^{(1)}] = -\frac{\bar{\alpha}_s C_F}{4\pi}\frac{(Q^2+t_1)}{2(Q^2-t_1)^2}\left[\left((d-2)Q^2-(d-4)t_1\right)B(t_1)-2Q^2B(Q^2)\right],$$
$$C[\Delta^{(2)}] = -\frac{\bar{\alpha}_s C_F}{4\pi}\frac{Q^2}{(Q^2-t_1)^2}\left[\left((d-6)t_1-(d-8)Q^2\right)B(Q^2)+2(t_1-2Q^2)B(t_1)\right],$$

were $\bar{\alpha}_s = \frac{\mu^{-2\epsilon}g_s^2}{(4\pi)^{1-\epsilon}}r_\Gamma$, $t_1 = -q_1^2$.

$RO g$-vertex



$R\mathcal{O}g$-vertex

The one-loop correction is proportional to the Born vertex:

$$G^{(0)}_{+\mu} = \frac{i}{2}\left((Q^2 - t_1)n^-_\mu - 2q_-(q_1)_\mu\right),$$

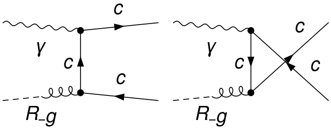
with the coefficient

$$C\left[G^{(0)}_+\right] = \frac{\bar{\alpha}_s}{4\pi}\frac{1}{2}\left\{\frac{B(t_1)}{(d-2)(d-1)(Q^2-t_1)^2}\right.$$
$$\times\left[C_A\left((d-2)(5d-4)Q^4 - 2(d(7d-24)+16)Q^2 t_1\right.\right.$$
$$\left.\left. +(d-2)(5d-4){t_1}^2\right) - 2(d-2)^2 n_F (Q^2-t_1)^2\right]$$
$$-\frac{2C_A(d-4)Q^2 B(Q^2)}{(d-2)(Q^2-t_1)^2}\left[(d-4)Q^2 - (d-2)t_1\right]$$
$$\left.-2C_A\left[q_-\left(t_1 C_{[-]}(t_1,Q^2,q_-) + B_{[-]}(q) - B_{[-]}(q+q_1)\right) + (t_1 - Q^2)C(t_1,Q^2)\right]\right\}.$$

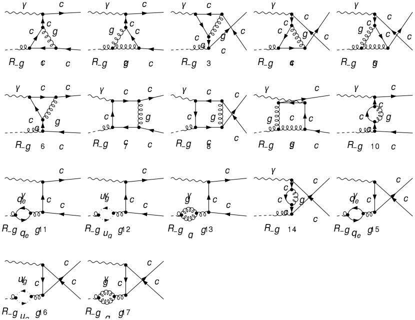
III. One-loop corrections to quarkonium impact-factors

$R\gamma \to c\bar{c}\left[{}^1S_0^{[8]}\right]$ @ 1 loop

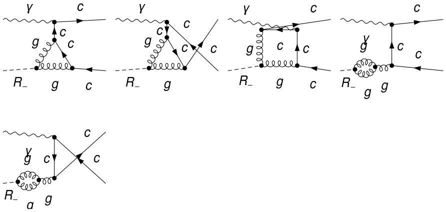
Interference with LO:



Rg-coupling diagrams:

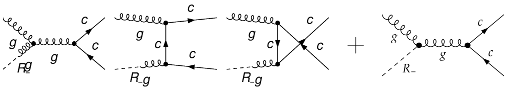


Induced Rgg coupling diagrams:

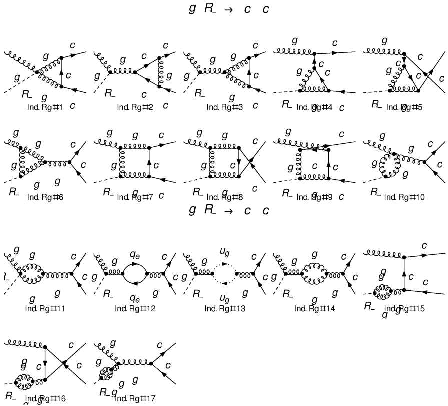


- Diagrams had been generated using custom `FeynArts` model-file, projector on the $c\bar{c}\left[{}^1S_0^{(8)}\right]$-state is inserted
- heavy-quark momenta $= p_{\mathcal{Q}}/2 \Rightarrow$ need to resolve linear dependence of quadratic denominators in some diagrams before IBP
- IBP reduction to master integrals has been performed using `FIRE`

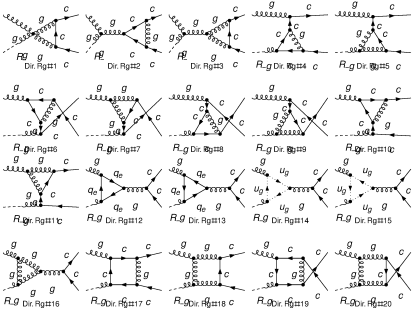
$Rg \to c\bar{c}\left[{}^1S_0^{[1]}\right]$ and $c\bar{c}\left[{}^3S_1^{[8]}\right]$ @ 1 loop

Interference with LO: 

Induced Rgg coupling diagrams:



Some Rg-coupling diagrams:



and so on...

Integrals with massive internal lines

In presence of the linear denominator the massive propagator can be converted to the massless one:

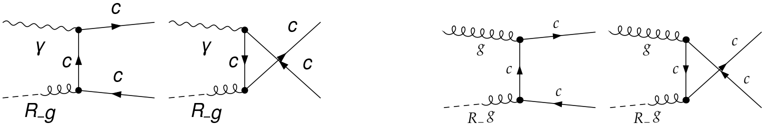
$$\frac{1}{((\tilde{n}_+ l) + k_+)(l^2 - m^2)} = \frac{1}{((\tilde{n}_+ l) + k_+)(l + \kappa\tilde{n}_+)^2} + \frac{\cancel{2\kappa\left[(\tilde{n}_+ l) + \frac{m^2 + \tilde{n}_+^2 \kappa^2}{2\kappa}\right]}}{\cancel{((\tilde{n}_+ l) + k_+)}(l + \kappa\tilde{n}_+)^2 (l^2 - m^2)}$$

$\Rightarrow$ all the masses can be moved to integrals with **only quadratic propagators**. New **massless** scalar integrals with RDs arise($k^2 = 0$, $p^2 = 4m^2$, $p = q + q_1$, $q^2 = 0$):

$$\begin{aligned}
B_{[+]}(-k, k-q) &= \int \frac{d^D l}{[\tilde{l}^+](l-k)^2 (l+k-q)^2}, \\
C_{[+]}(0, -k, k-q) &= \int \frac{d^D l}{[\tilde{l}^+] l^2 (l-k)^2 (l+k-q)^2}, \\
B_{[+]}(p, k) &= \int \frac{d^D l}{[\tilde{l}^+](l+p)^2 (l+k)^2}, \\
C_{[+]}(p, k, q_1) &= \int \frac{d^D l}{[\tilde{l}^+](l+p)^2 (l+k)^2 (l+q_1)^2},
\end{aligned}$$

but they have the same complexity as already encountered ones.

Results: $R\gamma \to c\bar{c}\left[{}^1S_0^{[8]}\right]$ vs. $Rg \to c\bar{c}\left[{}^1S_0^{[1]}\right]$ @ 1 loop



Results for $2\Re\left[\frac{H_{\mathbf{1L\ x\ LO}}(\mathbf{q}_T)-\mathbf{(On\text{-}shell\ mass\ CT)}}{(\alpha_s/(2\pi))H_{\rm LO}(\mathbf{q}_T)}\right]$:

$$ {}^1S_0^{[8]}:\left(\frac{\mu^2}{\mathbf{q}_T^2}\right)^{\epsilon}\frac{1}{\epsilon}\left[N_c\left(\ln\frac{\mathbf{q}_T^2}{M^2}+\ln\frac{q_-^2}{\mathbf{q}_T^2 r}+\frac{19}{6}\right)-\frac{2n_F}{3}-\frac{3}{2N_c}\right]+F_{{}^1S_0^{[8]}}(\mathbf{q}_T^2/M^2) $$

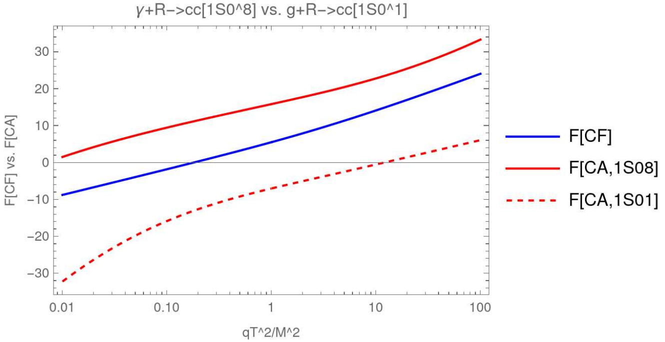
$$ {}^1S_0^{[1]}:\left(\frac{\mu^2}{\mathbf{q}_T^2}\right)^{\epsilon}\left\{-\frac{N_c}{\epsilon^2}+\frac{1}{\epsilon}\left[N_c\left(\ln\frac{q_-^2}{\mathbf{q}_T^2 r}+\frac{25}{6}\right)-\frac{2n_F}{3}-\frac{3}{2N_c}\right]\right\}+F_{{}^1S_0^{[1]}}(\mathbf{q}_T^2/M^2) $$

$$ \begin{aligned} F_{{}^1S_0^{[1]}}(\tau) &= -\frac{10}{9}n_F+\Re[C_F F^{(C_F)}_{{}^1S_0^{[1]}}(\tau)+C_A F^{(C_A)}_{{}^1S_0^{[1]}}(\tau)], \\ F^{(C_F)}_{{}^1S_0^{[1]}}(\tau) &= F^{(C_F)}_{{}^1S_0^{[8]}}(\tau), \end{aligned} $$

while $F^{(C_A)}_{{}^1S_0^{[1]}}(\tau) \neq F^{(C_A)}_{{}^1S_0^{[8]}}(\tau)$.

The C_F coefficient

$$F^{(C_F)}_{{}^1S_0^{[8]}}(\tau) = \frac{\mathcal{L}_2 + \mathcal{L}_7(1-2\tau)}{\tau+1}$$
$$+\frac{1}{6(\tau+1)(2\tau+1)^2}\{144L_1\tau^2 + 144L_1\tau + 36L_1 - 16\pi^2\tau^3 - 72\tau^3 + 72\tau^3\log(2)$$
$$-156\tau^2 + 12\tau^2\log^2(2\tau+1) + 168\tau^2\log(2) - 24\left(3\tau^2+5\tau+2\right)\tau\log(\tau+1)$$
$$+12\pi^2\tau - 108\tau + 12\tau\log^2(2\tau+1) + 3\log^2(2\tau+1) + 132\tau\log(2)$$
$$+18(\tau+1)(2\tau+1)^2\log(\tau) + 4\pi^2 - 24 + 36\log(2)\}$$



The C_A coefficient for $R\gamma \to c\bar{c}\left[{}^1S_0^{[8]}\right]$

$$
\begin{aligned}
F^{(C_A)}_{{}^1S_0^{[8]}}(\tau) \quad = \quad & \frac{1}{2(\tau-1)(\tau+1)^3}\{(\tau+1)^2\left(-4\mathcal{L}_4\left(\tau^2-1\right)+\mathcal{L}_2(\tau+1)(2\tau+1)+\mathcal{L}_7\tau(2\tau-3)+\mathcal{L}_7\right) \\
& +2\mathcal{L}_6(\tau(\tau((\tau-4)\tau-6)-4)+1)\} \\
+ \quad & \frac{1}{36(\tau-1)(\tau+1)^3(2\tau+1)}\{-216L_1\tau^4-324L_1\tau^3+108L_1\tau^2+324L_1\tau+108L_1 \\
& +120\pi^2\tau^5+608\tau^5-36\tau^5\log^2(\tau+1)+36\tau^5\log^2(2\tau+1)-36\tau^5\log^2(2) \\
& -72\tau^5\log(2)\log(\tau+1)+216\tau^5\log(\tau+1)+72\tau^5\log(2)+228\pi^2\tau^4+1520\tau^4 \\
& -306\tau^4\log^2(\tau+1)+144\tau^4\log^2(2\tau+1)-306\tau^4\log^2(2) \\
& +252\tau^4\log(2)\log(\tau+1)+432\tau^4\log(\tau+1)+360\tau^4\log(2)+84\pi^2\tau^3+608\tau^3 \\
& -360\tau^3\log^2(\tau+1)+225\tau^3\log^2(2\tau+1)-360\tau^3\log^2(2)+576\tau^3\log(2)\log(\tau+1) \\
& +72\tau^3\log(\tau+1)+72\tau^3\log(2)-120\pi^2\tau^2-1216\tau^2-108\tau^2\log^2(\tau+1) \\
& +171\tau^2\log^2(2\tau+1)-108\tau^2\log^2(2)+504\tau^2\log(2)\log(\tau+1)-360\tau^2\log(\tau+1) \\
& -360\tau^2\log(2)-72(\tau+1)^3\left(2\tau^2-\tau-1\right)\log(\tau-1)(\log(2)-\log(\tau+1)) \\
& +36(2\tau+1)\log(\tau)\left[-\tau^4+\tau^4\log(8)-6\tau^2\log(2)+\left(-\tau^3+4\tau^2+6\tau+4\right)\tau\log(\tau+1)\right. \\
& \left.-8\tau\log(2)-\log(2\tau+2)+1\right]-18\left(2\tau^5+17\tau^4+20\tau^3+6\tau^2-6\tau-3\right)\log^2(\tau) \\
& -84\pi^2\tau-1216\tau+108\tau\log^2(\tau+1)+63\tau\log^2(2\tau+1)+108\tau\log^2(2) \\
& +54\log^2(\tau+1)+9\log^2(2\tau+1)+72\tau\log(2)\log(\tau+1)-288\tau\log(\tau+1) \\
& -144\tau\log(2)-36\log(2)\log(\tau+1)-72\log(\tau+1)-12\pi^2-304+54\log^2(2)\}
\end{aligned}
$$

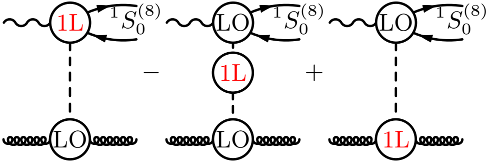
The C_A coefficient for $Rg \to c\bar{c}\left[{}^1S_0^{[1]}\right]$

$$
\begin{aligned}
F^{(C_A)}_{{}^1S_0^{[1]}}(\tau) \quad = \quad & \frac{1}{(\tau-1)(\tau+1)^3}\{2\mathcal{L}_1\left(\tau^2+\tau-2\right)(\tau+1)^3+\tau\left[2\mathcal{L}_5\left(\tau(\tau+1)\left(\tau^2-2\right)+1\right)\right. \\
& -\mathcal{L}_7\left(\tau^2+\tau-1\right)-\left(\mathcal{L}_2(\tau+2)(\tau+1)^2\right)+\mathcal{L}_6(\tau(\tau(6-(\tau-4)\tau)+4)-1)\Big] \\
& +2\mathcal{L}_3(\tau-1)(\tau+1)^3+2\mathcal{L}_5+\mathcal{L}_7\} \\
- \quad & \frac{1}{18(\tau-1)(\tau+1)^3}\{6\pi^2\tau^5-36\tau^5\log(2)\log(\tau+1)+36\tau^5\log(\tau+1)\log(\tau+2)+63\pi^2\tau^4 \\
& -98\tau^4-63\tau^4\log^2(\tau+1)+9\tau^4\log^2(2\tau+1)-63\tau^4\log^2(2)+54\tau^4\log(2)\log(\tau+1) \\
& -36\tau^4\log(\tau+1)+36\tau^4\log(\tau+1)\log(\tau+2)+36\tau^4\log(2)+138\pi^2\tau^3-196\tau^3 \\
& -72\tau^3\log^2(\tau+1)+36\tau^3\log^2(2\tau+1)-72\tau^3\log^2(2)+144\tau^3\log(2)\log(\tau+1) \\
& -36\tau^3\log(\tau+1)-72\tau^3\log(\tau+1)\log(\tau+2)-36\tau^3\log(2)+18\pi^2\tau^2 \\
& -18\tau^2\log^2(\tau+1)+45\tau^2\log^2(2\tau+1)-18\tau^2\log^2(2)+108\tau^2\log(2)\log(\tau+1) \\
& +36\tau^2\log(\tau+1)-72\tau^2\log(\tau+1)\log(\tau+2)-36\tau^2\log(2) \\
& -18\left(4\tau^4+5\tau^3+\tau^2-3\tau-1\right)\log^2(\tau)+18\log(\tau)\left[\tau^5\log(2)-\tau^4(\log(4)-2)\right. \\
& \left.-\tau^3\log(4)-2\tau^2(1+\log(4))-\left(\tau^4-4\tau^3-6\tau^2-4\tau+1\right)\tau\log(\tau+1)-\tau\log(8)-\log(4)\right] \\
& -120\pi^2\tau+196\tau+36\tau\log^2(\tau+1)+18\tau\log^2(2\tau+1)+36\tau\log^2(2)+9\log^2(\tau+1) \\
& -36\tau\log(2)\log(\tau+1)+36\tau\log(\tau+1)+36\tau\log(\tau+1)\log(\tau+2)+36\tau\log(2) \\
& -36(\tau-1)(\tau+1)^3\log(\tau-1)(\log(2)-\log(\tau+1))-18\log(2)\log(\tau+1) \\
& +36\log(\tau+1)\log(\tau+2)-69\pi^2+98+9\log^2(2)\}
\end{aligned}
$$

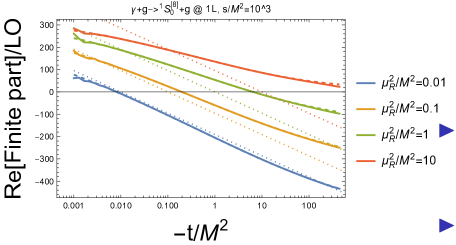
$$
\begin{aligned}
L_1 &= \sqrt{\tau(1+\tau)}\ln\left[1+2\tau+2\sqrt{\tau(1+\tau)}\right], \\
\mathcal{L}_1 &= \mathrm{Li}_2\left(\frac{1}{\tau}+1\right) \\
\mathcal{L}_2 &= \mathrm{Li}_2\left(\frac{1}{-2\tau-1}\right) \\
\mathcal{L}_3 &= \mathrm{Li}_2\left(\frac{1}{\tau}\right)+\mathrm{Li}_2\left(\frac{\tau-1}{\tau+1}\right)-\mathrm{Li}_2\left(\frac{\tau+1}{2\tau}\right)+\frac{\mathrm{Li}_2\left(\frac{1}{4}\right)}{2}+\mathrm{Li}_2(-2) \\
\mathcal{L}_4 &= \mathrm{Li}_2\left(1+\frac{1}{\tau}\right)+\mathrm{Li}_2\left(\frac{1}{\tau}\right)+\mathrm{Li}_2\left(\frac{\tau-1}{\tau+1}\right)-\mathrm{Li}_2\left(\frac{\tau+1}{2\tau}\right)+\frac{\mathrm{Li}_2\left(\frac{1}{4}\right)}{2}+\mathrm{Li}_2(-2) \\
\mathcal{L}_5 &= \mathrm{Li}_2\left(-\frac{1}{\tau+1}\right)-\mathrm{Li}_2(\tau+2)+\frac{1}{2}\mathrm{Li}_2\left(\frac{2\tau+1}{2\tau+2}\right) \\
\mathcal{L}_6 &= -\mathrm{Li}_2\left(-\frac{2\tau+1}{\tau^2}\right)+\mathrm{Li}_2\left(-\frac{-2\tau^2+\tau+1}{2\tau^2}\right)+\mathrm{Li}_2\left(\frac{1}{2}-\frac{\tau}{2}\right)+\mathrm{Li}_2\left(-\frac{1}{\tau}\right) \\
&\quad -\mathrm{Li}_2\left(\frac{\tau-1}{2\tau}\right)-\mathrm{Li}_2(-\tau)+\mathrm{Li}_2\left(\frac{1-\tau}{\tau+1}\right) \\
\mathcal{L}_7 &= \mathrm{Li}_2(-2\tau-1)-\mathrm{Li}_2\left(\frac{2\sqrt{\tau}}{\sqrt{\tau}-\sqrt{\tau+1}}\right)-\mathrm{Li}_2\left(\frac{2\sqrt{\tau}}{\sqrt{\tau}+\sqrt{\tau+1}}\right)
\end{aligned}
$$

$R\gamma \to c\bar{c}\left[{}^1S_0^{[8]}\right]$ @ 1 loop, cross-check

In the combination of 1-loop results in the EFT:



the $\ln r$ cancels and it should reproduce the the Regge limit($s \gg -t$) of the *real part* of the $2 \to 2$ 1-loop QCD amplitude:



Solid lines – QCD, dashed lines – EFT, dotted lines – $-2C_A \ln(-t/\mu_R^2)\ln(s/M^2)$

$$\gamma + g \to c\bar{c}\left[{}^1S_0^{(8)}\right] + g.$$

- The $2 \to 2$ QCD 1-loop amplitude can be computed numerically using `FormCalc` (with some tricks, due to Coulomb divergence)
- The Regge limit of $1/\epsilon$ divergent part agrees with the EFT result
- For the finite part agreement within few % is reached, need to push to higher s

Conclusions and outlook

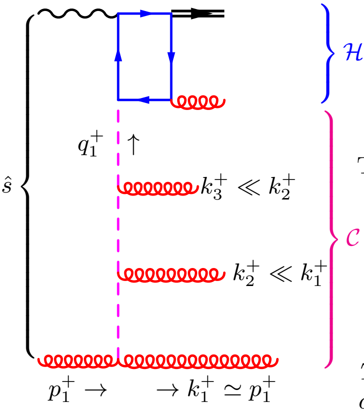
- High-Energy EFT is a well-tested computational tool in High-Energy QCD at one loop with some applications at 2 loops
- The computations of one-loop corrections to impact-factors of heavy quarkonium production are progressing
- EFT+tilted WL regularization is a convenient framework to compute NNLO correction to the (Regge-pole part of the) BFKL kernel

Thank you for your attention!

High-Energy Factorization (J/ψ photoproduction)

The **LLA** ($\sum_n \alpha_s^n \ln^{n-1}$) formalism [Collins, Ellis, '91; Catani, Ciafaloni, Hautmann, '91,'94]

Physical picture in the **LLA** for photoproduction:



The LLA in $\ln \frac{1}{\xi} = \ln \frac{p_1^+}{q_1^+} \sim \ln(1+\eta)$:

$$\hat{\sigma}_{\mathrm{HEF}}^{\ln(1/\xi)}(\eta) \propto \int\limits_{1/z}^{1+\eta} \frac{dy}{y} \int\limits_0^{\infty} d\mathbf{q}_{T1}^2 \mathcal{C}\left(\frac{y}{1+\eta}, \mathbf{q}_{T1}^2, \mu_F, \mu_R\right) \mathcal{H}(y, \mathbf{q}_{T1}^2),$$

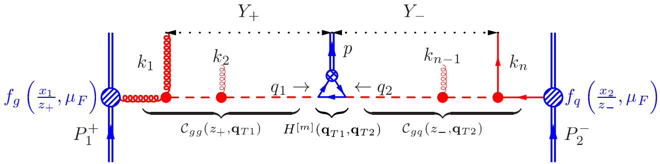
The **strict LLA** in $\ln(1+\eta) = \ln \frac{\hat{s}}{M^2}$:

$$\hat{\sigma}_{\mathrm{HEF}}^{\ln(1+\eta)}(\eta) \propto \int\limits_0^{\infty} d\mathbf{q}_{T1}^2 \mathcal{C}\left(\frac{1}{1+\eta}, \mathbf{q}_{T1}^2, \mu_F, \mu_R\right) \int\limits_{1/z}^{\infty} \frac{dy}{y} \mathcal{H}(y, \mathbf{q}_{T1}^2).$$

The LLA($\ln(1/\xi)$) contains some (N..)NLLA contributions relative to the LLA($\ln(1+\eta)$).

The coefficient function $\mathcal{H}$ has been calculated at LO [Kniehl, Vasin, Saleev, '06] and decreases as $1/y^2$ for $y \gg 1$.

High-Energy Factorization (η_c hadroproduction)



Small parameter: $z = \frac{M^2}{\hat{s}}$, LLA in $\alpha_s^n \ln^{n-1} \frac{1}{z}$:

$$\hat{\sigma}_{ij}^{[m],\text{ HEF}}(z,\mu_F,\mu_R) = \int\limits_{-\infty}^{\infty} d\eta \int\limits_{0}^{\infty} d\mathbf{q}_{T1}^2 d\mathbf{q}_{T2}^2\, \mathcal{C}_{gi}\left(\frac{M_T}{M}\sqrt{z}e^{\eta}, \mathbf{q}_{T1}^2, \mu_F, \mu_R\right)$$

$$\times \mathcal{C}_{gj}\left(\frac{M_T}{M}\sqrt{z}e^{-\eta}, \mathbf{q}_{T2}^2, \mu_F, \mu_R\right) \int\limits_{0}^{2\pi} \frac{d\phi}{2} \frac{H^{[m]}(\mathbf{q}_{T1}^2, \mathbf{q}_{T2}^2, \phi)}{M_T^4}$$

The coefficient functions $H^{[m]}$ are known at LO in α_s [Hagler *et.al*, 2000; Kniehl, Vasin, Saleev 2006] for $m = {}^1S_0^{(1,8)}, {}^3P_J^{(1,8)}, {}^3S_1^{(8)}$.

The $H^{[m]}$ is a tree-level "squared matrix element" of the $2 \rightarrow 1$-type process:

$$R_+(\mathbf{q}_{T1}, q_1^+) + R_-(\mathbf{q}_{T2}, q_2^-) \rightarrow c\bar{c}[m].$$

LLA evolution w.r.t. $\ln 1/\xi$

In the LL($\ln 1/\xi$)-approximation, the $Y = \ln 1/\xi$-evolution equation for *collinearly un-subtracted* $\tilde{\mathcal{C}}$-factor has the form:

$$\tilde{\mathcal{C}}(\xi, \mathbf{q}_T) = \delta(1-\xi)\delta(\mathbf{q}_T^2) + \hat{\alpha}_s \int\limits_{\xi}^{1} \frac{dz}{z} \int d^{2-2\epsilon}\mathbf{k}_T K(\mathbf{k}_T^2, \mathbf{q}_T^2) \tilde{\mathcal{C}}\left(\frac{\xi}{z}, \mathbf{q}_T - \mathbf{k}_T\right)$$

with $\hat{\alpha}_s = \alpha_s C_A/\pi$ and

$$K(\mathbf{k}_T^2, \mathbf{q}_T^2) = \frac{1}{\pi(2\pi)^{-2\epsilon}\mathbf{k}_T^2} + \delta^{(2-2\epsilon)}(\mathbf{k}_T)\ 2\omega_g(\mathbf{q}_T^2),$$

where $\omega_g(\mathbf{q}_T^2)$ – 1-loop Regge trajectory of a gluon. It is convenient to go from $(z, \mathbf{q}_T)$-space to $(N, \mathbf{x}_T)$-space:

$$\tilde{\mathcal{C}}(N, \mathbf{x}_T) = \int d^{2-2\epsilon}\mathbf{q}_T\ e^{i\mathbf{x}_T\mathbf{q}_T} \int\limits_0^1 dx\ x^{N-1}\ \tilde{\mathcal{C}}(x, \mathbf{q}_T),$$

because:

- Mellin convolutions over z turn into products: $\int \frac{dz}{z} \to \frac{1}{N}$
- Large logs map to poles at $N=0$: $\boxed{\alpha_s^{k+1} \ln^{k} \frac{1}{\xi} \to \frac{\alpha_s^{k+1}}{N^{k+1}}}$
- All *collinear divergences* are contained inside $\mathcal{C}$ in $\mathbf{x}_T$-space.

Exact LL solution and the DLA

In $(N, \mathbf{q}_T)$-space, subtracted $\mathcal{C}$, which resums all terms $\propto (\hat{\alpha}_s/N)^n$ (complete LLA) has the form [Collins, Ellis, '91; Catani, Ciafaloni, Hautmann, '91,'94]:

$$\mathcal{C}(N, \mathbf{q}_T, \mu_F) = R(\gamma_{gg}(N, \alpha_s)) \frac{\gamma_{gg}(N, \alpha_s)}{\mathbf{q}_T^2} \left(\frac{\mathbf{q}_T^2}{\mu_F^2}\right)^{\gamma_{gg}(N,\alpha_s)},$$

where $\gamma_{gg}(N, \alpha_s)$ is the solution of [Jaroszewicz, '82]:

$$\frac{\hat{\alpha}_s}{N} \chi(\gamma_{gg}(N, \alpha_s)) = 1, \text{ with } \chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma),$$

where $\psi(\gamma) = d \ln \Gamma(\gamma)/d\gamma$ – Euler's ψ-function. The first few terms:

$$\gamma_{gg}(N, \alpha_s) = \underbrace{\underbrace{\frac{\hat{\alpha}_s}{N}}_{\text{DLA [Blümlein, '95]}} + 2\zeta(3)\frac{\hat{\alpha}_s^4}{N^4} + 2\zeta(5)\frac{\hat{\alpha}_s^6}{N^6} + \dots}_{\text{LLA}}$$

$\frac{\hat{\alpha}_s}{N} \leftrightarrow P_{gg}(z \to 0) = \frac{2C_A}{z} + \dots$
The function $R(\gamma)$ is

$$R(\gamma_{gg}(N, \alpha_s)) = 1 + O(\alpha_s^3).$$

Fixed-order asymptotics from HEF

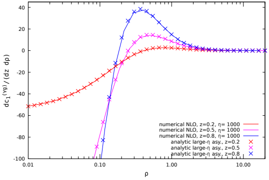
When expanded up to $O(\alpha_s)$ the HEF resummation should predict the $\hat{s} \gg M^2$ asymptotics of the CF coefficient function $\hat{\sigma}$

For the $g + g \to c\bar{c}\left[{}^1S_0^{(1)}, {}^3P_0^{(1)}, {}^3P_2^{(1)}\right]$ the NLO and NNLO($\alpha_s^2 \ln(1/z)$) terms in $\hat{\sigma}$ are predicted [M.N., Lansberg, Ozcelik '22]:

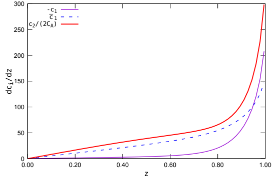
State	$A_0^{[m]}$	$A_1^{[m]}$	$A_2^{[m]}$	$B_2^{[m]}$
${}^1S_0$	1	$\mathbf{-1}$	$\frac{\pi^2}{6}$	$\frac{\pi^2}{6}$
${}^3S_1$	0	1	0	$\frac{\pi^2}{6}$
${}^3P_0$	1	$-\mathbf{\frac{43}{27}}$	$\frac{\pi^2}{6}+\frac{2}{3}$	$\frac{\pi^2}{6}+\frac{40}{27}$
${}^3P_1$	0	$\frac{5}{54}$	$-\frac{1}{9}$	$-\frac{2}{9}$
${}^3P_2$	1	$-\mathbf{\frac{53}{36}}$	$\frac{\pi^2}{6}+\frac{1}{2}$	$\frac{\pi^2}{6}+\frac{11}{9}$

$$\hat{\sigma}_{gg}^{[m]}(z \to 0) = \sigma_{\rm LO}^{[m]} \left\{ A_0^{[m]} \delta(1-z) \right.$$
$$+ \frac{\alpha_s}{\pi} 2C_A \left[A_1^{[m]} + A_0^{[m]} \ln \frac{M^2}{\mu_F^2} \right]$$
$$+ \left(\frac{\alpha_s}{\pi}\right)^2 \ln\frac{1}{z} \cdot C_A^2 \left[2A_2^{[m]} + B_2^{[m]} \right.$$
$$\left. \left. + 4A_1^{[m]} \ln\frac{M^2}{\mu_F^2} + 2A_0^{[m]} \ln^2 \frac{M^2}{\mu_F^2} \right] + O(\alpha_s^3) \right\},$$

For the $\gamma + g \to c\bar{c}\left[{}^3S_1^{(1)}\right] + g$ we have computed $\eta \to \infty$ limit of the z and $\rho = \mathbf{p}_T^2/M^2$-differential NLO "scaling functions" in closed analytic form,



and obtained numerical results for NNLO "scaling function" c_2 in front of $\alpha_s \ln(1+\eta)$.



Quarkonium in the potential model

Cornell potential:

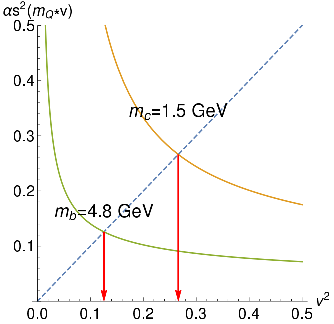
$$V(r) = -C_F \frac{\alpha_s(1/r)}{r} + \sigma r,$$

neglect linear part, because quarkonium is "small" (~ 0.3 fm) $\rightarrow$ Coulomb wavefunction (for effective mass $\frac{m_1 m_2}{m_1+m_2} = \frac{m_Q}{2}$):

$$R(r) = \frac{\sqrt{m_Q^3 \alpha_s^3 C_F^3}}{2} e^{-\frac{\alpha_s C_F}{2} m_Q r}$$

$$\langle v^2 \rangle = \frac{C_F^2 \alpha_s^2}{2}, \langle r \rangle = \frac{3}{2C_F} \frac{1}{m_Q v}$$

$$\Rightarrow \boxed{\alpha_s^2(m_Q v) \simeq v^2}$$



Non-relativistic QCD

The velocity-expansion for quarkonium eigenstate is carbon-copy of corresponding arguments from atomic physics (hierarchy of E-dipole/M-dipole with ΔS/M-dipole transitions):

$$\begin{aligned} |J/\psi\rangle &= O(1)\left|c\bar{c}\left[{}^3S_1^{(1)}\right]\right\rangle + O(v)\left|c\bar{c}\left[{}^3P_J^{(8)}\right]+g\right\rangle \\ &+ O(v^{3/2})\left|c\bar{c}\left[{}^1S_0^{(8)}\right]+g\right\rangle + O(v^2)\left|c\bar{c}\left[{}^3S_1^{(8)}\right]+gg\right\rangle + \ldots, \end{aligned}$$

for validity of this arguments, we should work in *non-relativistic EFT*, dynamics of which conserves number of heavy quarks. In such EFT, $Q\bar{Q}$-pair is produced in a point, by local operator:

$$\mathcal{A}_{\mathrm{NRQCD}} = \langle J/\psi + X|\, \chi^\dagger(0)\kappa_n\psi(0)\,|0\rangle\,,$$

Different operators “couple” to different Fock states:

$$\chi^\dagger(0)\psi(0) \leftrightarrow \left|c\bar{c}\left[{}^1S_0^{(1)}\right]\right\rangle,\ \chi^\dagger(0)\sigma_i\psi(0) \leftrightarrow \left|c\bar{c}\left[{}^3S_1^{(1)}\right]\right\rangle,$$

$$\chi^\dagger(0)\sigma_i T^a\psi(0) \leftrightarrow \left|c\bar{c}\left[{}^3S_1^{(8)}\right]\right\rangle,\ \chi^\dagger(0)D_i\psi(0) \leftrightarrow \left|c\bar{c}\left[{}^1P_1^{(8)}\right]\right\rangle, \ldots$$

squared NRQCD amplitude (=LDME):

$$\sum_X |\mathcal{A}|^2 = \langle 0|\, \underbrace{\psi^\dagger\kappa_n^\dagger\chi a_{J/\psi}^\dagger a_{J/\psi}\chi^\dagger\kappa_n\psi}_{\mathcal{O}_n^{J/\psi}}\,|0\rangle = \left\langle \mathcal{O}_n^{J/\psi}\right\rangle,$$

Non-relativistic QCD

Velocity-scaling of LDMEs follows from velocity-scaling of corresponding Fock states and of operators $\chi^\dagger \kappa_n \psi$:

	${}^1S_0^{(1)}$	${}^3S_1^{(1)}$	${}^1S_0^{(8)}$	${}^3S_1^{(8)}$	${}^1P_1^{(1)}$	${}^3P_0^{(1)}$	${}^3P_1^{(1)}$	${}^3P_2^{(1)}$	${}^1P_1^{(8)}$	${}^3P_0^{(8)}$	${}^3P_1^{(8)}$	${}^3P_2^{(8)}$
η_c	1		v^4	v^3					v^4			
J/ψ		1	v^3	v^4						v^4	v^4	v^4
h_c			v^2		v^2							
χ_{c0}				v^2		v^2						
χ_{c1}				v^2			v^2					
χ_{c2}				v^2				v^2				

Matching procedure between QCD and NRQCD:

$$v \to 0 : \mathcal{A}_{\mathrm{QCD}}(gg \to Y_{Q\bar{Q}(v)}) = \sum_n f_n \left\langle Y_{Q\bar{Q}(v)} \right| \chi^\dagger(0)\kappa_n\psi(0) \left|0\right\rangle + O(v^{\#}),$$

$\Rightarrow$ NRQCD factorization formula ("theorem") [Bodwin, Braaten, Lepage 95'] :

$$\sigma(gg \to \mathcal{H} + X) = \sum_n \sigma(gg \to Q\bar{Q}[n] + X) \left\langle \mathcal{O}_n^{\mathcal{H}} \right\rangle .$$