# Transverse momentum dependent (TMD) factorization theorem

**Alexey Vladimirov**
Universidad Complutense de Madrid

**Overlap between QCD resummations**


**Aussois, Centre Paul Langevin**
**January 14, 2024**

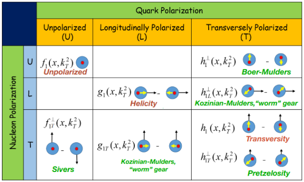
## Outline

**PART I (15.01)**

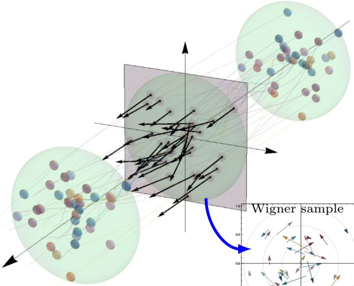
- TMD factorization theorem: conditions, and approaches
- TMD factorization theorem: structure
- Wilson lines and process (in)dependence
- Divergences
- TMD soft factor

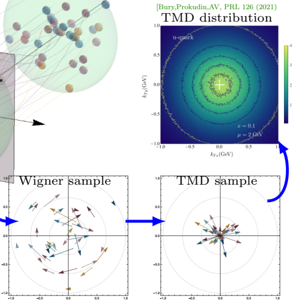
**PART II (16.01)**

- Renormalization of rapidity divergences
- Soft-rapidity correspondence
- Small-b OPE
- Phenomenology



| | | Quark Polarization | | |
|---|---|---|---|---|
| | | Unpolarized (U) | Longitudinally Polarized (L) | Transversely Polarized (T) |
| Nucleon Polarization | U | $f_1(x,k_T^2)$ Unpolarized | | $h_1^\perp(x,k_T^2)$ Boer-Mulders |
| | L | | $g_1(x,k_T^2)$ Helicity | $h_{1L}^\perp(x,k_T^2)$ Kozinian-Mulders, "worm" gear |
| | T | $f_{1T}^\perp(x,k_T^2)$ Sivers | $g_{1T}(x,k_T^2)$ Kozinian-Mulders, "worm" gear | $h_1(x,k_T^2)$ Transversity; $h_{1T}^\perp(x,k_T^2)$ Pretzelosity |






- CSS=[Collins, Soper, Sterman] (1981-1984)
- Sivers effect (1990) → non-perturbative TMDs
- Classification [Mulders, et al, 90's]
- Gauge structure, rapidity divergences, ... 00's
- Proof of TMD factorization [Collins,2011] [SCET,2009 - 2012]
- 2-loop [2015 – 2018], 3-loop [2019-2022], 4-loop [2022-..]
- Power corrections [2021-...]

Preprints: JLAB-THY-23-3780, LA-UR-21-20798, MIT-CTP/5386

# TMD Handbook

Renaud Boussarie[1], Matthias Burkardt[2], Martha Constantinou[3], William Detmold[4], Markus Ebert[4,5], Michael Engelhardt[2], Sean Fleming[6], Leonard Gamberg[7], Xiangdong Ji[8], Zhong-Bo Kang[9], Christopher Lee[10], Keh-Fei Liu[11], Simonetta Liuti[12], Thomas Mehen[13], Andreas Metz[3], John Negele[4], Daniel Pitonyak[14], Alexei Prokudin[7,16], Jian-Wei Qiu[16,17], Abha Rajan[12,18], Marc Schlegel[2,19], Phiala Shanahan[4], Peter Schweitzer[20], Iain W. Stewart[4], Andrey Tarasov[21,22], Raju Venugopalan[18], Ivan Vitev[10], Feng Yuan[23], Yong Zhao[24,4,18]

ArXiV: 2304.03302

- 350+ pages on TMD factorization and related topic
- Good introduction to the topic [with a strong SCET flavor]
- Various topics, from definitions to models, small-x, sub-leading power, etc.

# TMD factorization

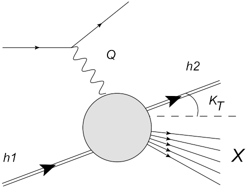
**CSS**-approach
- Method of regions
- Diagram-based
- Particular regularization
- $\sqrt{\text{soft-factors}}$
- difficult to generalize
- [Collins:2011zzd]

**SCET**-approach
- Modes defined by method of regions
- SCET operators
- $\sqrt{\text{soft-factor}}$
- Mostly unpolarized case
- [Becher:2010tm]
- [Echevarria:2011epo]

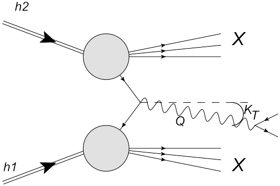
**Background field**-approach
- Modes defined by physics
- QCD with 2 background fields
- Renormalization of rap.div
- [Vladimirov:2021hdn]

- All approaches agrees with each other (in physical terms)
- The main difference is the treatment of the soft-factor/contribution/rapidity divergences
- I will discuss using **background field**-approach (because it is clearer and simpler)

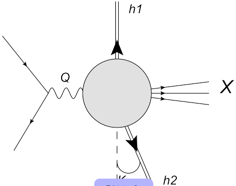
## Standard TMD processes




**SIDIS**




**Drell-Yan** This talk




**SIA**

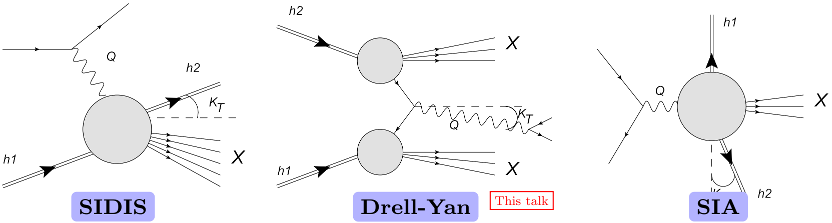
$q^2 = \pm Q^2$ momentum of hard probe

$q_T^\mu$ transverse component

**Hadron tensor:**

$$W^{\mu\nu}(q) = \int d^4y e^{-i(qy)} \langle p_1, p_2 | J^\mu(y) | X \rangle \langle X | J^\nu(0) | p_1, p_2 \rangle, \qquad J^\mu = \bar{q}\gamma^\mu q$$

## Standard TMD processes




$q^2 = \pm Q^2$ momentum of hard probe
$q_T^\mu$ transverse component

**Limit of TMD factorization**

$$Q \to \infty, \qquad s \to \infty, \qquad \text{such that} \qquad \frac{Q^2}{s} = \text{const}$$

$$q_T = \text{finite}$$

---

$$\frac{q_T}{Q} \to 0, \qquad \frac{\Lambda}{Q} \to 0, \qquad \text{but} \qquad q_T \not\to 0$$

## Background field method for parton physics (in a nutshell)

$$\langle h|T\, J^\mu(z)J^\nu(0)|h\rangle = \int [D\bar{q}DqDA]e^{iS_{\text{QCD}}}\Psi^*[\bar{q},q,A]J^\mu(z)J^\nu(0)\Psi[\bar{q},q,A]$$

Cannot be integrated since $\Psi$ is unknown

## Background field method for parton physics
(in a nutshell)

$$\langle h|T\, J^\mu(z)J^\nu(0)|h\rangle = \int [D\bar{q}DqDA]e^{iS_{\rm QCD}}\Psi^*[\bar{q},q,A]J^\mu(z)J^\nu(0)\Psi[\bar{q},q,A]$$

**Parton model**

$\Psi$ contains only collinear particles

$$\Psi[\bar{q},q,A]\to\Psi[\bar{q}_{\bar{n}},q_{\bar{n}},A_{\bar{n}}]$$

$$\{\partial_+,\partial_-,\partial_T\}q_{\bar{n}}\lesssim\{1,\lambda^2,\lambda\}q_{\bar{n}}$$

Integral can be partially computed

## Background field method for parton physics
## (in a nutshell)

$$\langle h|T\, J^\mu(z)J^\nu(0)|h\rangle = \int [D\bar q D q D A] e^{iS_{\rm QCD}}\Psi^*[\bar q, q, A] J^\mu(z) J^\nu(0)\Psi[\bar q, q, A]$$

**Parton model**

$\Psi$ contains only collinear particles

$\Psi[\bar q, q, A] \to \Psi[\bar q_{\bar n}, q_{\bar n}, A_{\bar n}]$

$\{\partial_+, \partial_-, \partial_T\} q_{\bar n} \lesssim \{1, \lambda^2, \lambda\} q_{\bar n}$

**Background technique**

$$\begin{aligned} q &= q_{\bar n} + \psi \\ A &= A_{\bar n} + B \end{aligned}$$

- $q_{\bar n}, A_{\bar n}$: background (external field)
- $\psi, B$: dynamical (to be integrated)

Integral can be partially computed

$$\langle h|T\, J^\mu(z)J^\nu(0)|h\rangle = \int [D\bar q_{\bar n} D q_{\bar n} D A_{\bar n}] e^{iS_{\rm QCD}}\Psi^*[\bar q, q, A] \mathcal{J}^{\mu\nu}_{\rm eff}[\bar q_{\bar n}, q_{\bar n}, A_{\bar n}](z)\Psi[\bar q, q, A]$$

$$\mathcal{J}^{\mu\nu}_{\rm eff} = \int [D\bar\psi D\psi D B] e^{iS_{\rm QCD} + iS_{\rm back}[\bar q, q, A]} J^\mu[q+\psi](z) J^\nu[q+\psi](0)$$

Generating function for operator product expansion

## Background QCD with 2-component background

$$q \to q_n + q_{\bar{n}} + \psi \qquad A^\mu \to A^\mu_n + A^\mu_{\bar{n}} + B^\mu$$

collinear-fields
(associated with hadron 1)

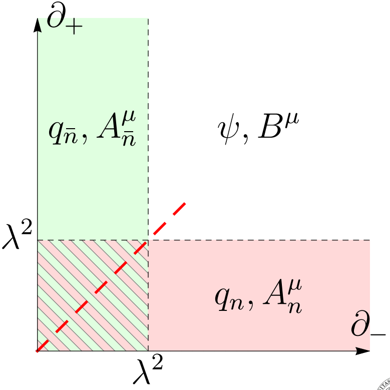
$$\{\partial_+, \partial_-, \partial_T\}\, q_{\bar{n}} \lesssim Q\{1, \lambda^2, \lambda\}\, q_{\bar{n}},$$

$$\{\partial_+, \partial_-, \partial_T\}\, A^\mu_{\bar{n}} \lesssim Q\{1, \lambda^2, \lambda\}\, A^\mu_{\bar{n}},$$

anti-collinear-fields
(associated with hadron 2)

$$\{\partial_+, \partial_-, \partial_T\}\, q_n \lesssim Q\{\lambda^2, 1, \lambda\}\, q_n,$$

$$\{\partial_+, \partial_-, \partial_T\}\, A^\mu_n \lesssim Q\{\lambda^2, 1, \lambda\}\, A^\mu_n.$$



**TMD operator expansion**
is conceptually similar to ordinary OPE
**The only difference** is counting rule for $y$

$$W^{\mu\nu}(q)=\int d^4y e^{-i(qy)}\langle p_1,p_2|J^\mu(y)|X\rangle\langle X|J^\nu(0)|p_1,p_2\rangle, \qquad J^\mu=\bar q\gamma^\mu q$$

$$(q\cdot y)\sim 1 \qquad \Rightarrow \qquad \{y^+,y^-,y_T\}\sim\{\frac{1}{q^-},\frac{1}{q^+},\frac{1}{q_T}\}\sim\frac{1}{Q}\{1,1,\lambda^{-1}\}$$

To be accounted in operator expansion

$$z_T^\mu\partial_\mu q\sim \text{NLP}, \qquad y_T^\mu\partial_\mu q\sim \text{LP}$$

$$\int [dq...]e^{iS[...]} \underbrace{\Psi_1^\dagger \Psi_2^\dagger}_{\langle p_1,p_2|} \underbrace{(\bar q_{\bar n}+\bar q_n)\gamma^\nu(q_{\bar n}+q_n)(y)}_{J^\mu(y)} \underbrace{(\bar q_{\bar n}+\bar q_n)\gamma^\mu(q_{\bar n}+q_n)(0)}_{J^\nu(0)} \underbrace{\Psi_1\Psi_2}_{|p_1,p_2\rangle}$$

$$= \langle p_1|\langle p_2|\bar q_{\bar n}\gamma^\mu q_n(y)|X\rangle\langle X|\bar q_n\gamma^\nu q_{\bar n}(0)|p_1\rangle|p_2\rangle + ...$$

$$= -\frac{1}{N_c}\sum_{n,m}\frac{\mathrm{tr}(\gamma^\mu\overline{\Gamma}_n\gamma^\nu\overline{\Gamma}_m)}{4}\Phi^{[\Gamma_n]}(-y)\Phi^{[\Gamma_m]}(y) + ...$$

**TMD distribution**

$$\Phi^{[\Gamma]}(y) = \langle p|\bar q(y)...\Gamma...q(0)|p\rangle.$$

$$\int [dq...]e^{iS[...]} \underbrace{\Psi_1^\dagger \Psi_2^\dagger}_{\langle p_1, p_2|} \underbrace{(\bar q_{\bar n} + \bar q_n)\gamma^\nu (q_{\bar n} + q_n)(y)}_{J^\mu(y)} \underbrace{(\bar q_{\bar n} + \bar q_n)\gamma^\mu (q_{\bar n} + q_n)(0)}_{J^\nu(0)} \underbrace{\Psi_1 \Psi_2}_{|p_1, p_2\rangle}$$

$$= \langle p_1|\langle p_2|\bar q_{\bar n}\gamma^\mu q_n(y)|X\rangle\langle X|\bar q_n \gamma^\nu q_{\bar n}(0)|p_1\rangle|p_2\rangle + ...$$

$$= -\frac{1}{N_c}\sum_{n,m}\frac{\mathrm{tr}(\gamma^\mu \overline{\Gamma}_n \gamma^\nu \overline{\Gamma}_m)}{4}\Phi^{[\Gamma_n]}(-y)\Phi^{[\Gamma_m]}(y) + ...$$

**TMD distribution**

$$\Phi^{[\Gamma]}(y) = \langle p|\bar q(y)...\Gamma...q(0)|p\rangle.$$

However, $\partial_- q \sim \lambda^2$ (but $y_\mu \partial_T^\mu q \sim 1$)

$$\Phi^{[\Gamma]}(y) \to \langle p|\bar q(y^- n + y_T)...\Gamma...q(0)|p\rangle + \mathcal{O}(\lambda^2)$$

$$\int [dq...]e^{iS[...]} \underbrace{\Psi_1^\dagger \Psi_2^\dagger}_{\langle p_1, p_2|} \underbrace{(\bar q_{\bar n} + \bar q_n)\gamma^\nu (q_{\bar n} + q_n)(y)}_{J^\mu(y)} \underbrace{(\bar q_{\bar n} + \bar q_n)\gamma^\mu (q_{\bar n} + q_n)(0)}_{J^\nu(0)} \underbrace{\Psi_1 \Psi_2}_{|p_1, p_2\rangle}$$

$$= \langle p_1|\langle p_2|\bar q_{\bar n}\gamma^\mu q_n(y)|X\rangle\langle X|\bar q_n \gamma^\nu q_{\bar n}(0)|p_1\rangle|p_2\rangle + ...$$

$$= -\frac{1}{N_c}\sum_{n,m}\frac{\mathrm{tr}(\gamma^\mu \overline{\Gamma}_n \gamma^\nu \overline{\Gamma}_m)}{4}\Phi^{[\Gamma_n]}(-y)\Phi^{[\Gamma_m]}(y) + ...$$

**TMD distribution**

$$\Phi^{[\Gamma]}(y) = \langle p|\bar q(y)...\Gamma...q(0)|p\rangle.$$

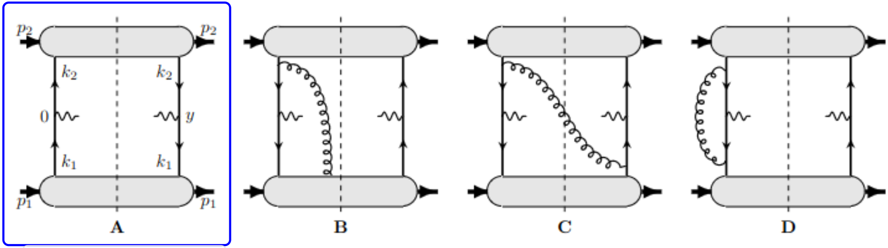
However, $\partial_- q \sim \lambda^2$ (but $y_\mu \partial_T^\mu q \sim 1$)

$$\Phi^{[\Gamma]}(y) \to \langle p|\bar q(y^- n + y_T)...\Gamma...q(0)|p\rangle + \mathcal{O}(\lambda^2)$$

Also, taking into account EOM $\gamma^-\gamma^+ q \sim \lambda^2$

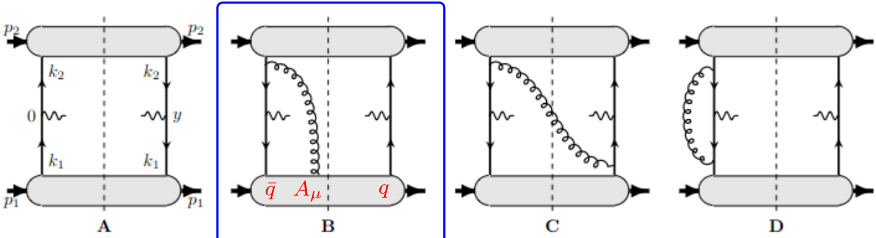
$$\Phi^{[\Gamma]}(y) \to \langle p|\bar q(y^- n + y_T)...\Gamma^+...q(0)|p\rangle + \mathcal{O}(\lambda^2)$$

$\Gamma^+$ can be only $\gamma^+$, $\gamma^+\gamma^5$ or $\sigma^{\alpha+}$.




**LO**

$\Phi^{[\Gamma_n]}(-y)\Phi^{[\Gamma_m]}(y)$



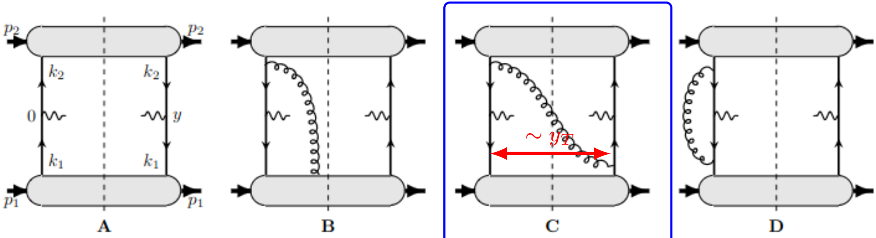

**NLO**

$(\bar{q}_{\bar{n}} A_+ q_n)(\bar{q}q) \sim$ **LP**

$(\bar{q}_{\bar{n}} A_T q_n)(\bar{q}q) \sim$ **NLP**

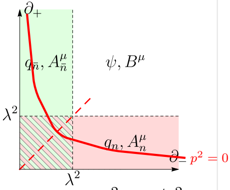
$(\bar{q}_{\bar{n}} A_- q_n)(\bar{q}q) \sim$ **N$^2$LP**

Wilson lines

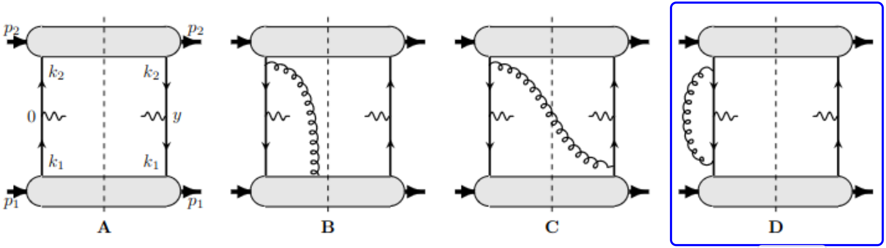



**NLO**

$$\frac{1}{y_T^2}(\bar{q}_{\bar{n}} q_n)(\bar{q}q) \sim \mathbf{N^2LP}$$

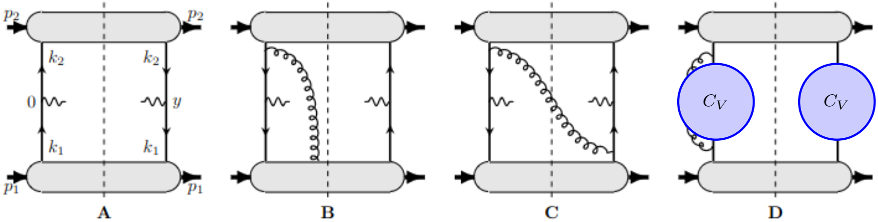



$p_+ p_- \sim p_T^2 \sim 1/y_T^2$




**NLO**

Coefficient function



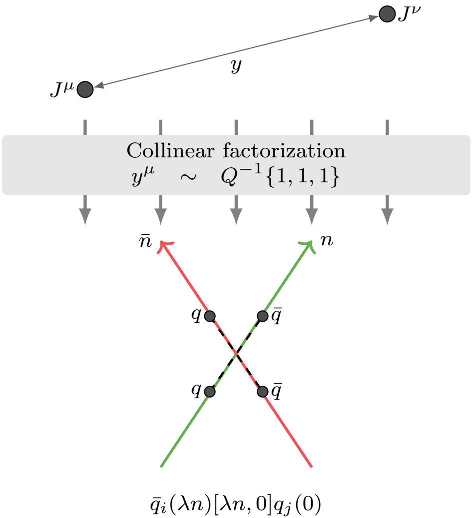

$$W^{\mu\nu} = -\frac{1}{N_c}\sum_{n,m}\frac{\text{tr}(\gamma^\mu\overline{\Gamma}_n\gamma^\nu\overline{\Gamma}_m)}{4}\int d^4y e^{-iqy}\Big|C_V\left(\frac{Q}{\mu}\right)\Big|^2\Phi^{[\Gamma_n]}(y^+n+y_T;\mu)\Phi^{[\Gamma_n]}(y^-\bar{n}+y_T;\mu)$$

$$C_V = 1 + a_sC_F\left(2\ln^2\left(\frac{-Q^2}{\mu^2}\right) + 2\ln\left(\frac{-Q^2}{\mu^2}\right) + \frac{\pi^2}{3}\right) + a_s^2....$$

$$\Phi^{[\Gamma]}(y,\mu) \to Z_V^2(\mu)\langle p|\bar{q}(y)[\text{Wilson line}]\Gamma^+q(0)|p\rangle$$

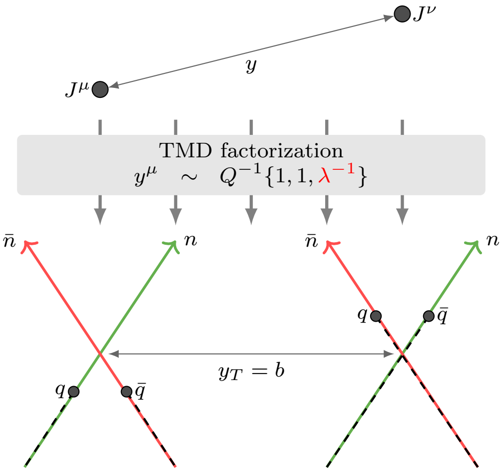
Dependence on $\mu$ cancels
(not that simple...)

# Process dependence



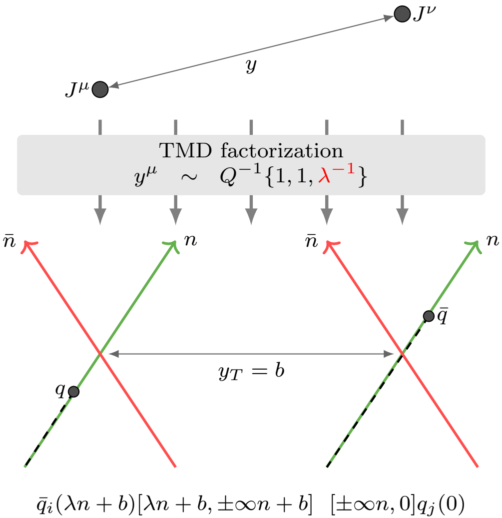

TMD operator expansion has different geometry

**Two**
light-cone operators
⇓
**Two**
parton distribution function
PDFs & FFs




TMD operator expansion has different geometry

**Four**
light-cone operators
⇓
**Two**
TMD distributions
TMDPDFs & TMDFFs




TMD operator expansion has different geometry

**Four**
light-cone operators
$\Downarrow$
**Two**
TMD distributions
TMDPDFs & TMDFFs

## Process dependence

The background can be taken in any gauge (since it is gauge invariant)

- Light-cone gauge kills operators with $A_{+,\bar{n}}$ and $A_{-,n}$ ($\sim 1$ in power counting).
- Convenient choice of gauges
  - Collinear field $A_+ = 0$
  - Anti-Collinear field $A_- = 0$
  - Dynamical field: **Feynman gauge**
- **However** one needs to specify boundary condition. The result depends on it.

$$A^{\mu}_{\bar{n}}(z) = -g\int_{-\infty}^{0} d\sigma F^{\mu+}_{\bar{n}}(z+n\sigma) \quad \textbf{vs.} \quad A^{\mu}_{\bar{n}}(z) = -g\int_{+\infty}^{0} d\sigma F^{\mu+}_{\bar{n}}(z+n\sigma)$$

$$\bar{q}[z, z-\infty n] \quad \textbf{vs.} \quad \bar{q}[z, z+\infty n]$$

**etc.**

To specify boundary and WL direction, we should go to NLO

## **NLO expression** in position space

$$I = \int_{-\infty}^{\infty} dz^+ dz^- \frac{f_{\bar{n}}(z^-) f_n(z^+)}{[-2z^+z^- + i0]^\alpha}$$

$f$'s are TMDPDFs or TMDFFs →

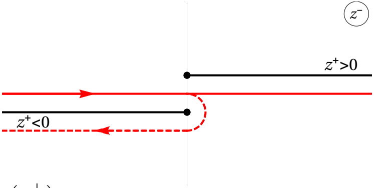
| | for DY | for SIDIS | for SIA | |
|---|---|---|---|---|
| $f_{\bar{n}}(z^-)$ is analytical in | lower | lower | upper | half-plane. |
| $f_n(z^+)$ is analytical in | lower | upper | upper | half-plane. |

## **NLO expression** in position space

$$I = \int_{-\infty}^{\infty} dz^+ dz^- \frac{f_{\bar{n}}(z^-) f_n(z^+)}{[-2z^+z^- + i0]^\alpha}$$

$f$'s are TMDPDFs or TMDFFs →

| | for DY | for SIDIS | for SIA | |
|---|---|---|---|---|
| $f_{\bar{n}}(z^-)$ is analytical in | lower | lower | upper | half-plane. |
| $f_n(z^+)$ is analytical in | lower | upper | upper | half-plane. |

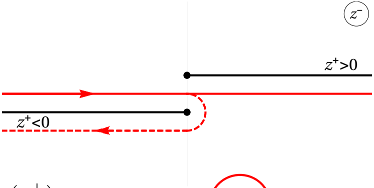



$$I = \int_{-\infty}^{0} dz^+ \frac{f_n(z^+)}{(-2z^+)^\alpha} \left(I_0 + I_1 + I_2 + I_\infty\right), \qquad I_C = \int_C \frac{f_{\bar{n}}(z^-)}{(z^-)^\alpha}$$

$$I=\int_{-\infty}^{\infty}dz^+dz^-\frac{f_{\bar n}(z^-)f_n(z^+)}{[-2z^+z^-+i0]^\alpha}$$

$f$'s are TMDPDFs or TMDFFs

| | for DY | for SIDIS | for SIA | |
|---|---|---|---|---|
| $f_{\bar n}(z^-)$ is analytical in | lower | lower | upper | half-plane. |
| $f_n(z^+)$ is analytical in | lower | upper | upper | half-plane. |




$$I=\int_{-\infty}^{0}dz^+\frac{f_n(z^+)}{(-2z^+)^\alpha}\left(I_0+I_1+I_2+I_\infty\right),\qquad I_C=\int_C\frac{f_{\bar n}(z^-)}{(z^-)^\alpha}$$

$I_\infty \to 0$

| | | |
|---|---|---|
| for DY: | $\lim_{z^-\to-\infty}A^\mu_{\bar n}(z)=0,$ | $\lim_{z^+\to-\infty}A^\mu_n(z)=0,$ |
| for SIDIS: | $\lim_{z^-\to+\infty}A^\mu_{\bar n}(z)=0,$ | $\lim_{z^+\to-\infty}A^\mu_n(z)=0,$ |
| for SIA: | $\lim_{z^-\to+\infty}A^\mu_{\bar n}(z)=0,$ | $\lim_{z^+\to+\infty}A^\mu_n(z)=0.$ |

Fields at $\infty$
(= interaction with transverse link)

Reproduce ordinary rules!

# Divergences of TMDs

**TMD-twist-(1,1)**
*(Usual TMDs)*

$U_1 = [..]\xi =$ good-component of quark field (twist-1)

$$\widetilde{\Phi}_{11}^{[\Gamma]}(\{z_1, z_2\}, b) = \langle p, s|\bar{\xi}(z_1 n + b)..\frac{\Gamma}{2}..\xi(z_2 n)|p, s\rangle$$

## TMD-twist-(1,1)

*(Usual TMDs)*

$U_1 = [..]\xi$ = good-component of quark field (twist-1)

$$\widetilde{\Phi}_{11}^{[\Gamma]}(\{z_1, z_2\}, b) = \langle p, s|\bar{\xi}(z_1 n + b)..\frac{\Gamma}{2}..\xi(z_2 n)|p, s\rangle$$


UV divergence

- Local (number)
- Anomalous dimension of quark field in LC gauge

$$\text{1-loop} = a_s C_F \frac{1}{\epsilon}\left(\frac{1}{\epsilon} + \frac{3}{2} + \ln\left(\frac{\mu^2}{2p^+\nu^-}\right)\right)$$

vs.

$$\text{coef.func} = 1 - a_s C_F\left[\frac{1}{\epsilon}\left(\frac{1}{\epsilon} + \frac{3}{2} + \ln\left(\frac{\mu^2}{Q^2}\right)\right) + ...\right]$$

requires (see later)

$$\zeta\bar{\zeta} = Q^4$$

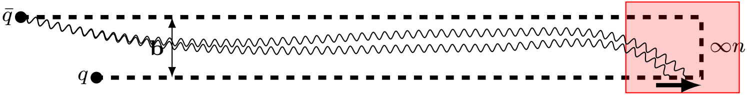
$$\mu^2 \frac{d}{d\mu^2} F_{f\leftarrow h}(x, b; \mu, \zeta) = \frac{\gamma_F^f(\mu, \zeta)}{2} F_{f\leftarrow h}(x, b; \mu, \zeta)$$

## TMD-twist-(1,1)

*(Usual TMDs)*

$U_1 = [..]\xi$ = good-component of quark field (twist-1)

$$\widetilde{\Phi}^{[\Gamma]}_{11}(\{z_1, z_2\}, b) = \langle p, s|\bar{\xi}(z_1 n + b)..\frac{\Gamma}{2}..\xi(z_2 n)|p, s\rangle$$




Rapidity divergence

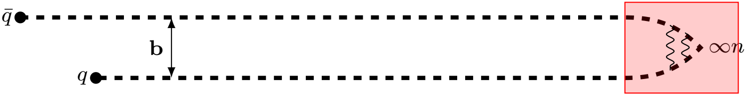
- Non-Local (depends on $b$)
- Not regularized by dim.reg.
- Rapidity anomalous dimension

$$\zeta \frac{d}{d\zeta} F_{f\leftarrow h}(x, b; \mu, \zeta) = -\mathcal{D}^f(b, \mu) F_{f\leftarrow h}(x, b; \mu, \zeta)$$

## TMD-twist-(1,1)
*(Usual TMDs)*

$U_1 = [..]\xi =$ good-component of quark field (twist-1)

$$\widetilde{\Phi}^{[\Gamma]}_{11}(\{z_1, z_2\}, b) = \langle p, s|\bar{\xi}(z_1 n + b)..\frac{\Gamma}{2}..\xi(z_2 n)|p, s\rangle$$

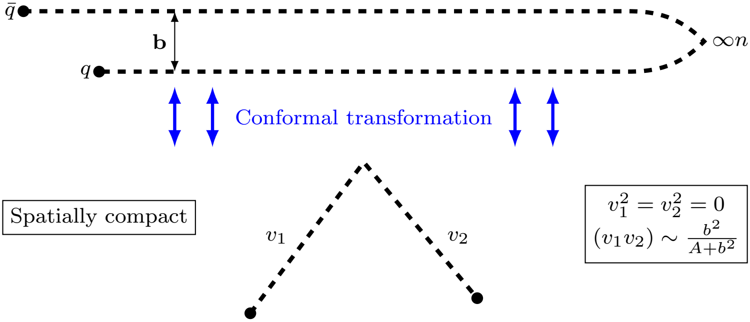



In non-singular gauges infinity is a single point

Rapidity divergence

- = Anomalous dimension of a distant cusp
- Distance $b \leftrightarrow$ angle

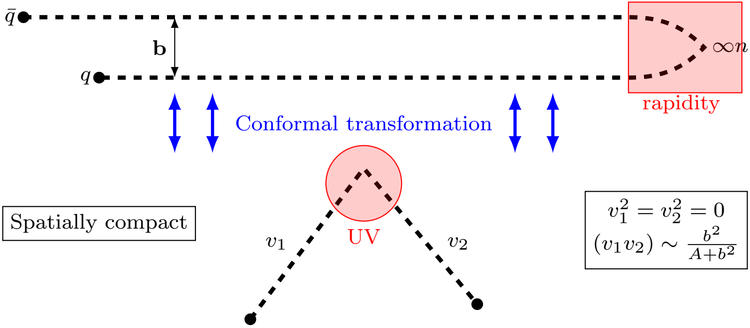
$$\widetilde{\Phi}_{11}^{[\Gamma]}(\{z_1, z_2\}, b) = \langle p, s|\bar{\xi}(z_1 n + b)..\frac{\Gamma}{2}..\xi(z_2 n)|p, s\rangle$$




Spatially compact

$$v_1^2 = v_2^2 = 0$$
$$(v_1 v_2) \sim \frac{b^2}{A+b^2}$$

$$\widetilde{\Phi}_{11}^{[\Gamma]}(\{z_1, z_2\}, b) = \langle p, s|\bar{\xi}(z_1 n + b)..\frac{\Gamma}{2}..\xi(z_2 n)|p, s\rangle$$




$$2\mathcal{D}(\mu, b) = \gamma_S((v_1 v_2), \mu)$$

RAD $\leftrightarrow$ SAD correspondance

Exact in conformal field theories

In QCD:

$$2\mathcal{D}\Big(\mu, b; \epsilon^*\Big) = \gamma_S((v_1 v_2), \mu)$$

$$\epsilon^* = \frac{\beta(\alpha_s)}{\alpha_s} \text{ (Wilson-Fisher critical dimension)}$$

**Consequences**

- Rapidity divergence is **multiplicatively renormalizable**, by factor $R$

$$\mathcal{D}(b, \mu) = \frac{1}{2} R^{-1}(b, \mu; \nu) \frac{d}{d \ln \nu} R(b, \mu; \nu)$$

- Same RAD for all TMDs of twist-2 and twist-3 (same soft-factor at sub-leading power)
- ...
- N-loop RAD + (N+1)-loop SAD ⇒ (N+1)-loop RAD — checked at N$^3$LO
- Absence of odd-color structures in SAD

$$W^{\mu\nu} \simeq \int d^2 b e^{-i(q_T b)} \Big| C_{\text{bare}}\left(\frac{Q}{\mu}; \epsilon\right) \Big|^2 \Phi^{[\Gamma]}_{\text{bare}}(x_1, b; \epsilon, \delta^+) \Phi^{[\Gamma]}_{\text{bare}}(x_2, b; \epsilon, \delta^-)$$

**Here:**

$$\Phi^{[\Gamma]}(x, b) = \int dy^- e^{-ixp^+ y^-} \Phi^{[\Gamma]}(y^- n + b)$$

$\epsilon$ for IR/UV divergences

$\delta$ for rapidity divergences

$$W^{\mu\nu} \simeq \int d^2 b e^{-i(q_T b)} \left| C_V\left(\frac{Q}{\mu};\mu\right) Z_V(\epsilon,\mu)\right|^2 |Z_1|^{-2}(\epsilon,\mu) R(\delta^+,\zeta)\Phi^{[\Gamma]}(x_1,b;\mu,\zeta)|Z_1|^{-2}(\epsilon,\mu)R(\delta^-,\bar\zeta)\Phi^{[\Gamma]}(x_2,b;\mu,\bar\zeta)$$

Should be:

$$Z_V(\epsilon,\mu)Z_1^{-2}(\epsilon,\mu)\sqrt{R(\delta^+,\zeta)R(\delta^-,\bar\zeta)} \sim 1$$

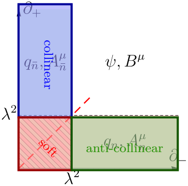
But there is $\delta^{\pm}$

Here we come to the issue of the soft factor

# Soft-factor

**Standard approach (CSS, SCET)**

introduce extra modes (collinear, anti-collinear, (ultra)SOFT)




It results into "soft factor" in $W$

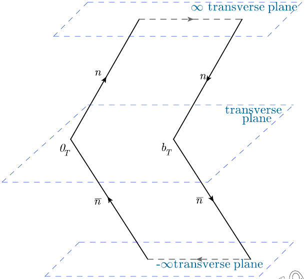
$$W \sim \Phi(x,b;\delta^+)S(b,\delta^+\delta^-)\Phi(x,b;\delta^-)$$

**TMD soft factor**

$S(b) = \langle 0|[\text{Wilson loop}]|0\rangle$

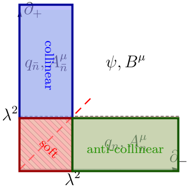
contains rapidity divergences

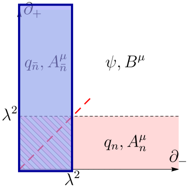
in "+" and "−" directions

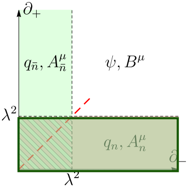


## Standard approach (CSS, SCET)

introduce extra modes (collinear, anti-collinear, (ultra)SOFT)




It results into "soft factor" in $W$

$$W \sim \Phi(x,b;\delta^+) S(b,\delta^+\delta^-) \Phi(x,b;\delta^-)$$

Define each TMD in "full" domain

$$\Phi(x,b;\delta^+) \sim \frac{\Phi_{\text{QCD}}(x,b;\zeta)}{Z.b.(b,\zeta,\delta^+\delta^-)}$$

**Zero-bin subtraction (SCET)**

not very well defined

$$Z.b.(b) \simeq \int_{\text{soft}} [d\bar{q}dqdA] e^{iS} W$$

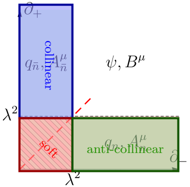
see [Manohar:2006nz]

**Collins**

$$Z.b.(b) \simeq S(b; \text{different regulator})$$

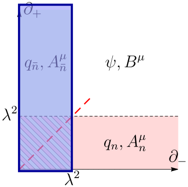
## Standard approach (CSS, SCET)

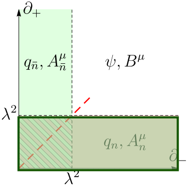
introduce extra modes (collinear, anti-collinear, (ultra)SOFT)




It results into "soft factor" in $W$

$$W \sim \Phi(x,b;\delta^+)S(b,\delta^+\delta^-)\Phi(x,b;\delta^-)$$




Define each TMD in "full" domain

$$\Phi(x,b;\delta^+) \sim \frac{\Phi_{\rm QCD}(x,b;\zeta)}{Z.b.(b,\zeta,\delta^+\delta^-)}$$

**Altogether**

$$W \sim \Phi_{\rm QCD}(x,b;\zeta)\frac{S(b,\delta^+\delta^-)}{Z.b.(b,\zeta,\delta^+\delta^-)Z.b.(b,\bar\zeta,\delta^+\delta^-)}\Phi_{\rm QCD}(x,b;\bar\zeta)$$

this extra factor compensates $R(\delta^+)R(\delta^-)$

## Origin of scales $\zeta$

- Renormalization of rapidity divergences introduces boost-dependent scales $\nu^{\pm}$

$$\Phi(x,b;\nu^{\pm})R(\delta^{\pm},\nu^{\pm})$$

- Soft factor is boost-invariant $S(b;\delta^{+}\delta^{-})$
- Thus, the combination of soft factor and R's is boost-invariant (because $\delta^{\pm}$ cancel)

$$R(\delta^{+},\nu^{+})\frac{S(b,\delta^{+}\delta^{-})}{Z.b.(b,\zeta,\delta^{+}\delta^{-})Z.b.(b,\bar{\zeta},\delta^{+}\delta^{-})}R(\delta^{-},\nu^{-})=\Sigma_0(b,\nu^{+}\nu^{-})$$

- Factor $\Sigma_0$ can be distributed between TMDs

$$\Phi(x,b,\nu^{+})\Sigma_0(b,\nu^{+}\nu^{-})\Phi(x,b,\nu^{-})=\underbrace{\Phi(x,b,\nu^{+})\sqrt{\Sigma_0(b,\nu^{+}\nu^{-})}}_{\text{"physical" TMD}}\underbrace{\sqrt{\Sigma_0(b,\nu^{+}\nu^{-})}\Phi(x,b,\nu^{-})}_{\text{"physical" TMD}}$$

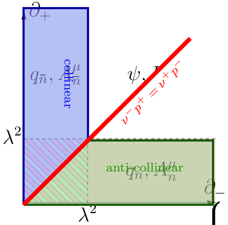
- "Physical" TMD depends on boost invariant combinations

$$\zeta\sim 2(p^{+})^2\frac{\nu^{+}}{\nu^{-}},\qquad \bar{\zeta}\sim 2(p^{-})^2\frac{\nu^{-}}{\nu^{+}}$$

- To cancel IR divergences one must set $\zeta\bar{\zeta}=Q^4$

## Non-Standard approach

avoid overlap + renormalize



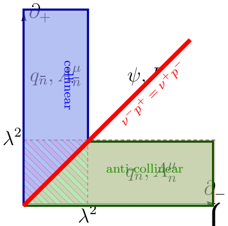

The rapidity divergences are (naturally) regularized by $\left\{ \begin{array}{c} p_1^+ > \dfrac{\nu^+}{\nu^-} p_1^- \\ p_2^- > \dfrac{\nu^-}{\nu^+} p_2^+ \end{array} \right\}$, and renormalized

"physical" TMDs are defined on this "reduced" set of fields.

The renormalization condition is that $W \simeq \Phi_1 \Phi_2$ (without any extra factor $\Sigma_0$)

**Non-Standard approach**

avoid overlap + renormalize




The rapidity divergences are (naturally) regularized by $\begin{Bmatrix} p_1^+ > \dfrac{\nu^+}{\nu^-} p_1^- \\ p_2^- > \dfrac{\nu^-}{\nu^+} p_2^+ \end{Bmatrix}$, and renormalized

"physical" TMDs are defined on this "reduced" set of fields.

The renormalization condition is that $W \simeq \Phi_1 \Phi_2$ (without any extra factor $\Sigma_0$)

Final expressions from both approaches are **identically the same**.
However, "renormalization" approach is easier to generalize (e.g. beyond LP)
(Also, there are indications that soft factor formula does not work at $b^2$ order...)

# TMD factorization theorem for DY

$$W^{\mu\nu} = \frac{-1}{4N_c} \sum_{n,m} \mathrm{tr}(\Gamma_n \gamma^\mu \Gamma_m \gamma^\nu) \int \frac{d^2 b}{(2\pi)^2} e^{-i(qb)} \Big| C_V\left(\frac{Q}{\mu}\right)\Big|^2 \Phi^{[\Gamma_n]}_{q\leftarrow h_1}(x_1, b; \mu, \zeta) \Phi^{[\Gamma_n]}_{\bar q\leftarrow h_2}(x_2, b; \mu, \bar\zeta)$$

$$+\mathcal{O}\left(\frac{q_T}{Q}, \frac{1}{bQ}, \frac{\Lambda}{Q}\right)$$

$$x_1 = \frac{q^+}{p^+} = \sqrt{\frac{Q^2 + \mathbf{q}_T^2}{s}} e^y, \qquad x_2 = \frac{q^-}{p^-} = \sqrt{\frac{Q^2 + \mathbf{q}_T^2}{s}} e^{-y}$$

(Is $q_T^2$ a power correction (??) $\Rightarrow$ **frame-dependence**)

# TMD factorization theorem for DY

$$W^{\mu\nu} = \frac{-1}{4N_c}\sum_{n,m}\mathrm{tr}(\Gamma_n\gamma^\mu\Gamma_m\gamma^\nu)\int\frac{d^2b}{(2\pi)^2}e^{-i(qb)}\Big|C_V\left(\frac{Q}{\mu}\right)\Big|^2\Phi^{[\Gamma_n]}_{q\leftarrow h_1}(x_1,b;\mu,\zeta)\Phi^{[\Gamma_n]}_{\bar{q}\leftarrow h_2}(x_2,b;\mu,\bar{\zeta})$$
$$+\mathcal{O}\left(\frac{q_T}{Q},\frac{1}{bQ},\frac{\Lambda}{Q}\right)$$

**Evolution equations**

$$\mu^2\frac{d}{d\mu^2}\Phi^{[\Gamma]}(\mu,\zeta) = \gamma_F(\mu,\zeta)\Phi^{[\Gamma]}(\mu,\zeta) \qquad (1)$$
$$\zeta\frac{d}{d\zeta}\Phi^{[\Gamma]}(\mu,\zeta) = -\mathcal{D}(b,\mu)\Phi^{[\Gamma]}(\mu,\zeta) \qquad (2)$$

- Evolution is universal for **all polarizations**, depends only on color-representation (quark/gluon)
- Hard coefficient function is also universal for all polarizations
- Hard coefficient function and ADs are known at 4-loops
- Collins-Soper kernel is **independent** non-perturbative function

# TMD factorization theorem for DY

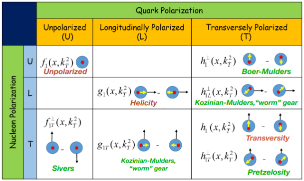
$$W^{\mu\nu} = \frac{-g_T^{\mu\nu}}{4N_c} \sum_q \int \frac{d^2 b}{(2\pi)^2} e^{-i(qb)} \left| C_V \left( \frac{Q}{\mu} \right) \right|^2 f_1(x_1, b; \mu, \zeta) f_2(x_2, b; \mu, \bar{\zeta})$$

$$+\mathcal{O}\left( \frac{q_T}{Q}, \frac{1}{bQ}, \frac{\Lambda}{Q} \right)$$

## Parametrization of TMDs

$$\Phi^{[\gamma^+]}(x, b) \quad = \quad f_1(x, b) - i(b \times s) M f_{1T}^{\perp}(x, b),$$

$$etc.$$



# TMD factorization theorem for DY

$$W^{\mu\nu} = \frac{-g_T^{\mu\nu}}{4N_c} \sum_q \int \frac{d^2b}{(2\pi)^2} e^{-i(qb)} \Big| C_V \left(\frac{Q}{\mu}\right) \Big|^2 f_1(x_1, b; \mu, \zeta) f_2(x_2, b; \mu, \bar{\zeta})$$
$$+\mathcal{O}\left(\frac{q_T}{Q}, \frac{1}{bQ}, \frac{\Lambda}{Q}\right)$$

LP expression is incomplete:
1. Violation of gauge invariance

$$q_\mu W^{\mu\nu} \sim q_T^\nu$$

2. Frame-dependence

$$n^\mu \to n^\mu + \frac{\Delta^\mu}{q^-} - \bar{n}^\nu \frac{\Delta^2}{2(q^-)^2}$$

Factorization theorem holds, but variables flow....

**Solution:** kinematic power corrections

# TMD evolution

$$\mu^2 \frac{d}{d\mu^2} \Phi^{[\Gamma]}(\mu,\zeta) = \frac{\gamma_F(\mu,\zeta)}{2} \Phi^{[\Gamma]}(\mu,\zeta)$$

$$\zeta \frac{d}{d\zeta} \Phi^{[\Gamma]}(\mu,\zeta) = -\mathcal{D}(b,\mu) \Phi^{[\Gamma]}(\mu,\zeta)$$

Integrability condition (CS-equation)

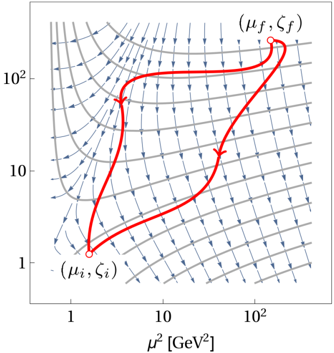
$$\zeta \frac{d}{d\zeta} \frac{\gamma_F(\mu,\zeta)}{2} = -\mu^2 \frac{d}{d\mu^2} \mathcal{D}(b,\mu) = -\frac{\Gamma_{\text{cusp}}}{2}$$

- It provides evolution for Collins-Soper kernel

$$\mathcal{D}(b,\mu) = \mathcal{D}(b,\mu_0) + \int_{\mu_0}^{\mu} \frac{d\mu}{\mu} \Gamma_{\text{cusp}}(\mu)$$

- It specifies the structure of $\gamma_F$

$$\gamma_F(\mu,\zeta) = \Gamma_{\text{cusp}}(\mu) \ln\left(\frac{\mu^2}{\zeta}\right) - \gamma_v(\mu)$$

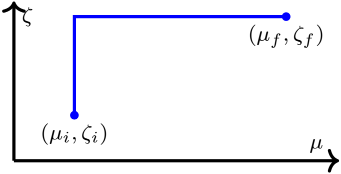



**TMD evolution is 2D evolution**

| | |
|---|---|
| **Evolution field** | $\mathbf{E} = \left(\frac{\gamma_F}{2}, -\mathcal{D}\right)$ |
| **is conservative** | $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = 0$ |
| **Evol.potential:** | $\mathbf{E} = \boldsymbol{\nabla} U$ |
| **Evolution eqn:** | $\boldsymbol{\nabla} \ln \Phi^{[\Gamma]} = \mathbf{E} = \boldsymbol{\nabla} U$ |

$$R[\mathbf{b}; i \to f] = \exp \int_P d\boldsymbol{\nu} \cdot \mathbf{E} = \exp(U_f - U_i) =$$

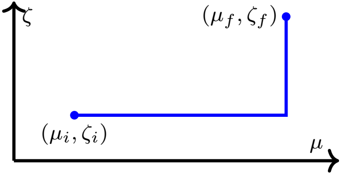
$$\exp\left[\int_P \left(\gamma_F(\mu,\zeta)\frac{d\mu}{\mu} - \mathcal{D}(\mu,\mathbf{b})\frac{d\zeta}{\zeta}\right)\right]$$




Example 1

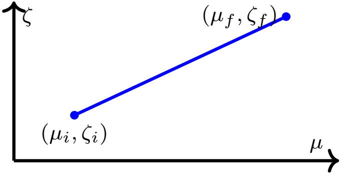
$$\ln R = \int_{\mu_i}^{\mu_f} \frac{d\mu}{\mu} \gamma_F(\mu, \zeta_f) - \mathcal{D}(\mu_i, b) \ln\left(\frac{\zeta_f}{\zeta_i}\right)$$

given in [Collins' textbook]




Example 2

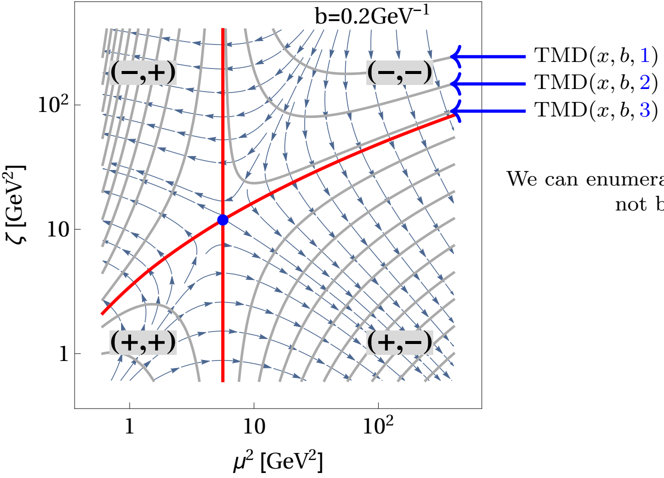
$$\ln R = \int_{\mu_i}^{\mu_f} \frac{d\mu}{\mu} \gamma_F(\mu, \zeta_i) - \mathcal{D}(\mu_f, b) \ln\left(\frac{\zeta_f}{\zeta_i}\right)$$




Solution 3

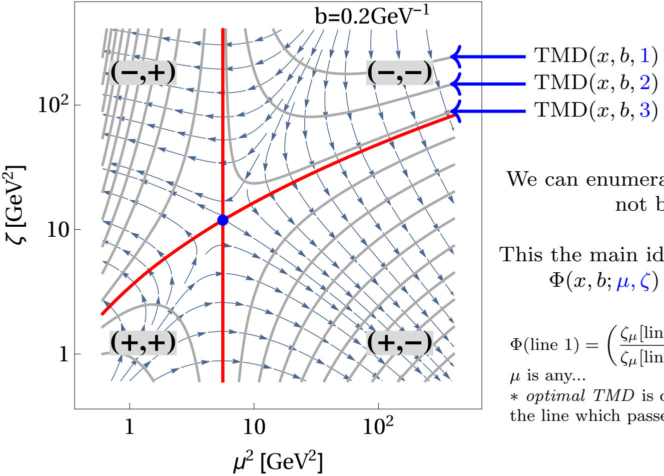
$$\ln R = \int_0^1 \Big(\gamma_F(\mu(t), \zeta(t)) \frac{\mu_f - \mu_i}{(\mu_f - \mu_i)t + \mu_i} - \mathcal{D}(\mu(t), b) \frac{\zeta_f - \zeta_i}{(\zeta_f - \zeta_i)t + \zeta_i}\Big) dt$$

TMD distributions on the same equipotential line are equivalent.




We can enumerate them by a lines not by $(\mu, \zeta)$

TMD distributions on the same equipotential line are equivalent.



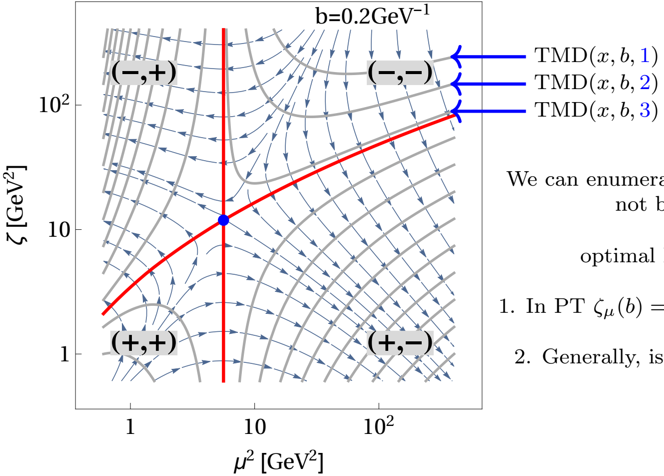

We can enumerate them by a lines not by $(\mu, \zeta)$

This the main idea of $\zeta$-prescription
$\Phi(x, b; \mu, \zeta) \to \Phi(z, b; \text{line})$

$$\Phi(\text{line } 1) = \left(\frac{\zeta_\mu[\text{line } 1]}{\zeta_\mu[\text{line } 2]}\right)^{-\mathcal{D}(b,\mu)} \Phi(\text{line } 2)$$

$\mu$ is any...
$*$ *optimal TMD* is defined on the line which passes though saddle point.

TMD distributions on the same equipotential line are equivalent.




We can enumerate them by a lines not by $(\mu, \zeta)$

optimal line $=\zeta_\mu(b)$

1. In PT $\zeta_\mu(b) = \frac{\mu}{b} 2e^{-\gamma_E}(1 + a_s ...)$
2. Generally, is a functional of $\mathcal{D}$

# Small-b regime

(From TMD factorization to “resummation” approach)

### “Resummation” regime

$$\Lambda \ll q_T \ll Q$$

$$W \sim \int d^2b e^{-ibq_T} \widetilde{W}(b) \xrightarrow{q_T \gg \Lambda} \int d^2b e^{-ibq_T} \lim_{b\to 0} \widetilde{W}(b) + ...$$

$$W(b) = \underbrace{|C_V(Q)|^2}_{\text{hard.c.}} \underbrace{R^2[\mathcal{D}(b)](Q \to \mu_0)}_{\text{evol.}} \underbrace{\Phi(x_1, b; \mu_0)\Phi(x_2, b; \mu_0)}_{\text{TMDs}}$$

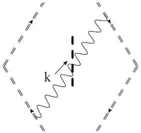
One needs:

- Collins-Soper kernel at small-b
- TMDs at small-b (”matching”)

## Collins-Soper kernel at small-b

There are several ways:

1. Compute soft-factor and extract rapidity divergence
2. Compute TMD and extract rapidity divergence
3. Use RAD/SAD correspondence (actual 4-loop computation)




$$\sim \int_0^\infty d\sigma_1 \int_0^\infty d\sigma_2 \frac{g^2 C_F}{[-2\sigma_1\sigma_2 + \mathbf{b}^2 + i0]^{1-\epsilon}} e^{-\sigma_1\delta^+} e^{-\sigma_2\delta^-}$$

$$= a_s C_F (\mu^2 \mathbf{b}^2)^\epsilon \Gamma(-\epsilon) \ln(\delta^+\delta^- \mathbf{b}^2) + ...$$

$$\mathcal{D} = 2a_s C_F \mathbf{L} + a_s^2 \left[\frac{\Gamma_0\beta_0}{4}\mathbf{L}^2 + \frac{\Gamma_1}{2}\mathbf{L} + C_F C_A \left(\frac{404}{27} - 14\zeta_3\right) - \frac{56}{27} C_F N_F\right] + ...$$

$$\mathbf{L} = \ln\left(\frac{\mu^2 \mathbf{b}^2}{4e^{-2\gamma_E}}\right)$$

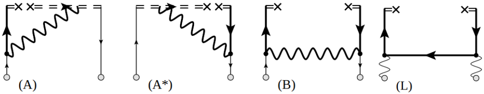
## TMDs at small-b

(small-b OPE)

$$\bar{q}(\lambda n+b)[\lambda n+b,\infty n+b]\Gamma[\infty n,0]q(0)=\int dz C(z,\mathbf{L})\bar{q}(zn)[zn,0]\Gamma q(0)+\mathcal{O}(b^2)$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$\underbrace{f_1(x,b;\mu,\zeta)}_{\text{TMD}}=\int\frac{dy}{y}\underbrace{C\left(\frac{x}{y},\mathbf{L};\mu,\zeta,\mu_{\text{OPE}}\right)}_{\text{coef.func.}}\underbrace{f_1(y,\mu_{\text{OPE}})}_{\text{PDF}}+\mathcal{O}(b^2)$$




$$\widetilde{\mathcal{U}}_{\mathbf{A}}=2a_sC_F\Gamma(-\epsilon)\boldsymbol{b}^{2\epsilon}\int_{-\infty}^{z_1}d\sigma\int_0^1 d\alpha\,\bar{\alpha}\;\bar{q}(z_1n+\boldsymbol{b})\gamma^+\overrightarrow{\partial_+}q(z_{2\sigma}^{\alpha}n-(1-2\alpha)\boldsymbol{b}),$$

$$\widetilde{\mathcal{U}}_{\mathbf{A}^*}=2a_sC_F\Gamma(-\epsilon)\boldsymbol{b}^{2\epsilon}\int_{-\infty}^{z_2}d\sigma\int_0^1 d\alpha\,\bar{\alpha}\;\bar{q}(z_{1\sigma}^{\alpha}n+(1-2\alpha)\boldsymbol{b})\;\overleftarrow{\partial_+}\gamma^+q(z_2n-\boldsymbol{b}),$$

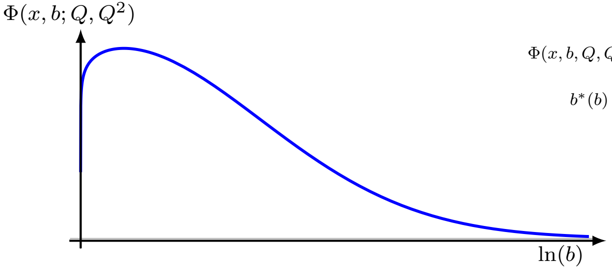
$$\widetilde{\mathcal{U}}_{\mathbf{B}}=2a_sC_F\Gamma(-\epsilon)\boldsymbol{b}^{2\epsilon}\int[d\alpha d\beta d\gamma]\Big\{(1-\epsilon)\;\bar{q}(z_{12}^{\alpha}n+\boldsymbol{b}\,(1-2\alpha))\gamma^+\,q(z_{21}^{\beta}n-\boldsymbol{b}\,(1-2\beta))$$

$$f_1(x,\boldsymbol{b};\mu,\zeta)=f_1(x)+a_s(\mu)\Big\{-2\mathbf{L}_\mu P\otimes f_1+C_F\left(-\mathbf{L}_\mu^2+2\mathbf{l}_\zeta\mathbf{L}_\mu+3\mathbf{L}_\mu-\frac{\pi^2}{6}\right)f_1(x)$$
$$+\int d\xi\int_0^1 dy\delta(x-y\xi)\Big[C_F2\bar{y}f_1(\xi)+2y\bar{y}g(\xi)\Big]\Big\}+O(a_s^2)+O(\boldsymbol{b}^2),$$

Table from [Moos, AV,2008.01744]

| Name | Function | Twist of leading matching | Twist-2 distributions in matching | Twist-3 distributions in matching | Order of leading power coef.function | Ref. |
|---|---|---|---|---|---|---|
| unpolarized | $f_1(x,b)$ | tw-2 | $f_1(x)$ | – | $\text{N}^3\text{LO}$ ($\alpha_s^3$) | [21, 22] |
| Sivers | $f_{1T}^{\perp}(x,b)$ | tw-3 | – | $T(-x,0,x)$ | NLO ($\alpha_s^1$) | [23] |
| helicity | $g_{1L}(x,b)$ | tw-2 | $g_1(x)$ | $\mathcal{T}_g(x)$ | NLO ($\alpha_s^1$) | [16, 17] |
| worm-gear T | $g_{1T}(x,b)$ | tw-2/3 | $g_1(x)$ | $\mathcal{T}_g(x)$ | NLO ) | [13, 14] |
| transversity | $h_1(x,b)$ | tw-2 | $h_1(x)$ | $\mathcal{T}_h(x)$ | NNLO ($\alpha_s^2$) | [19] |
| Boer-Mulders | $h_1^{\perp}(x,b)$ | tw-3 | – | $\delta T_{\epsilon}(-x,0,x)$ | NLO ) | [14] |
| worm-gear L | $h_{1L}^{\perp}(x,b)$ | tw-2/3 | $h_1(x)$ | $\mathcal{T}_h(x)$ | NLO ) | [13, 14] |
| pretzelosity | $h_{1T}^{\perp}$ | tw-3/4 | – | $\mathcal{T}_h(x)$ | LO ($\alpha_s^0$) | eq.(4.8) |

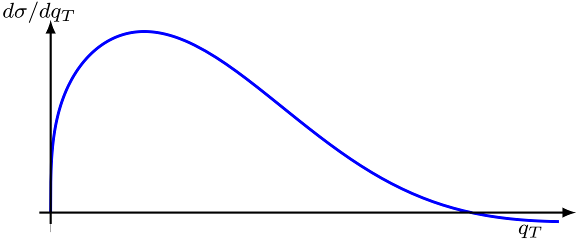
[Rein, Rodini, et al,2209.00962]

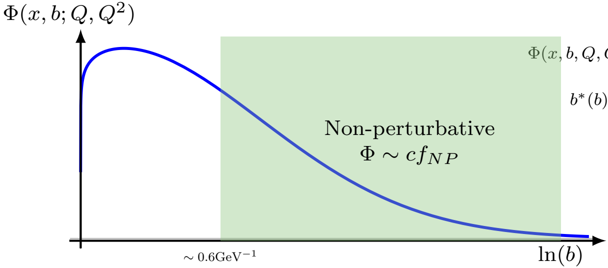



$$\Phi(x,b,Q,Q^2)=R[b,Q]\Phi(x,b;\frac{1}{b^*},\frac{1}{b^*})f_{\rm NP}(x,b)$$

$$b^*(b)\sim\begin{cases} b, & b\to 0\\ \text{const.}<\Lambda^{-1}, & b\to\infty\end{cases}$$

$$\lim_{b\to 0} f_{\rm NP}(x,b)=1$$

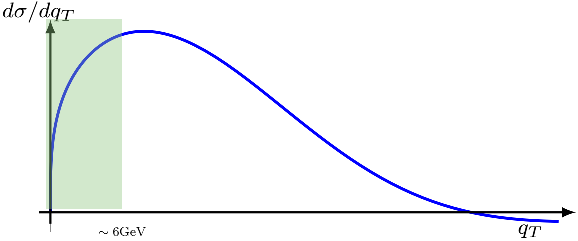


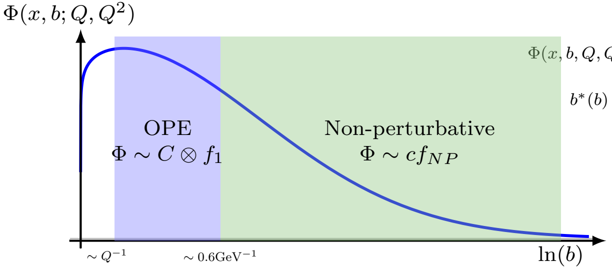



$$\Phi(x, b, Q, Q^2) = R[b, Q]\Phi(x, b; \frac{1}{b^*}, \frac{1}{b^*}) f_{\text{NP}}(x, b)$$

$$b^*(b) \sim \begin{cases} b, & b \to 0 \\ \text{const.} < \Lambda^{-1}, & b \to \infty \end{cases}$$

$$\lim_{b \to 0} f_{\text{NP}}(x, b) = 1$$

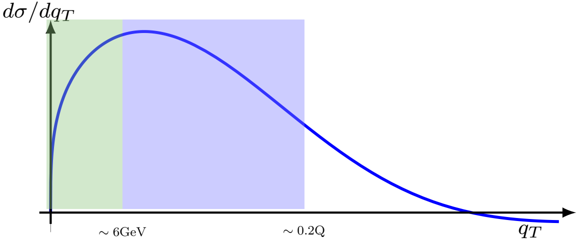


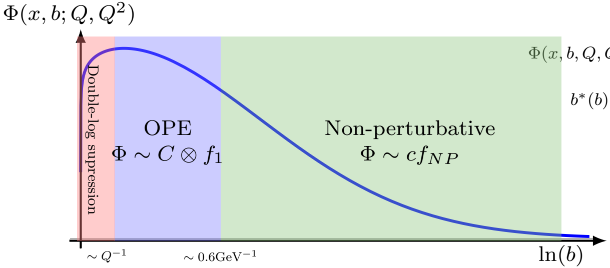



$$\Phi(x,b,Q,Q^2)=R[b,Q]\Phi(x,b;\frac{1}{b^*},\frac{1}{b^*})f_{\rm NP}(x,b)$$

$$b^*(b)\sim\begin{cases} b, & b\to 0\\ \text{const.}<\Lambda^{-1}, & b\to\infty\end{cases}$$

$$\lim_{b\to 0} f_{\rm NP}(x,b)=1$$

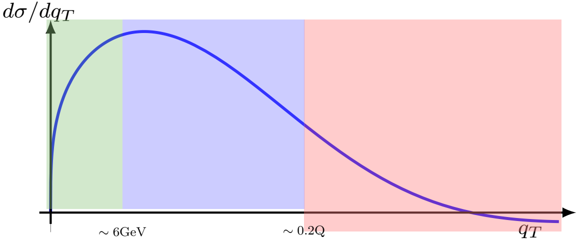


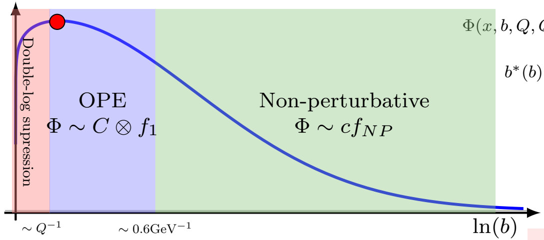



$$\Phi(x, b, Q, Q^2) = R[b, Q]\Phi(x, b; \frac{1}{b^*}, \frac{1}{b^*}) f_{\text{NP}}(x, b)$$

$$b^*(b) \sim \begin{cases} b, & b \to 0 \\ \text{const.} < \Lambda^{-1}, & b \to \infty \end{cases}$$

$$\lim_{b\to 0} f_{\text{NP}}(x, b) = 1$$

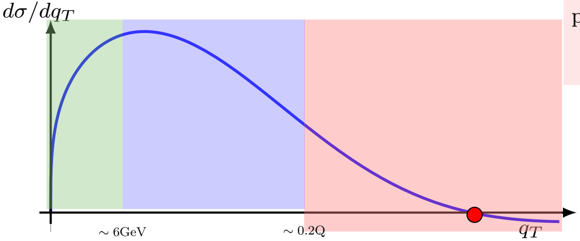




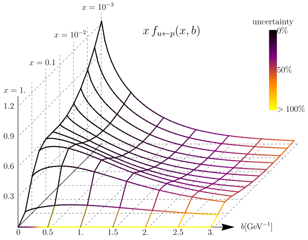

$$\Phi(x,b,Q,Q^2) = R[b,Q]\Phi(x,b;\frac{1}{b^*},\frac{1}{b^*})f_{\rm NP}(x,b)$$

$$b^*(b) \sim \begin{cases} b, & b \to 0 \\ \text{const.} < \Lambda^{-1}, & b \to \infty \end{cases}$$

$$\lim_{b\to 0} f_{\rm NP}(x,b) = 1$$




Node $\leftrightarrow$ zero
position depends on $Q$
$q_T \sim 0.5 - 0.7Q$
**power corrections**



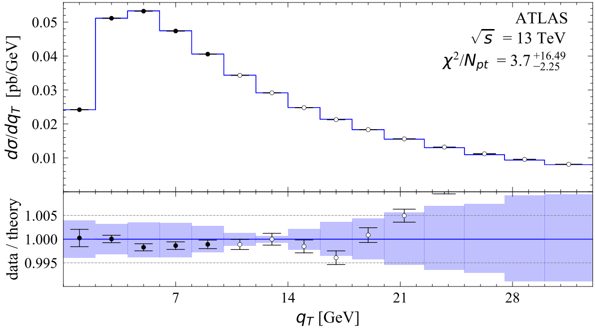

# ART23

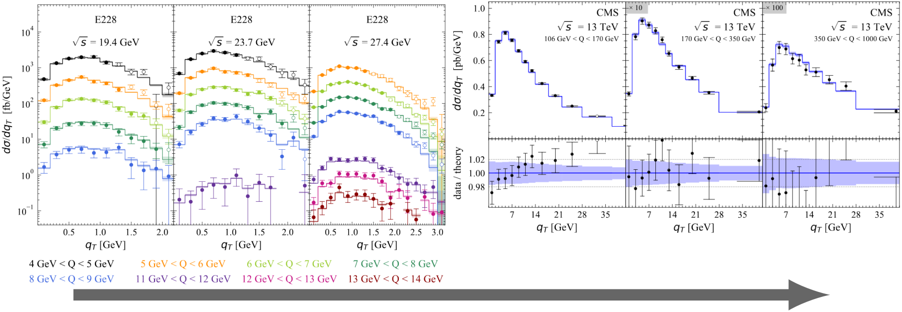
N$^4$LL

[V.Moos, I.Scimemi, AV, P.Zurita, 2305.07473]

627 data points

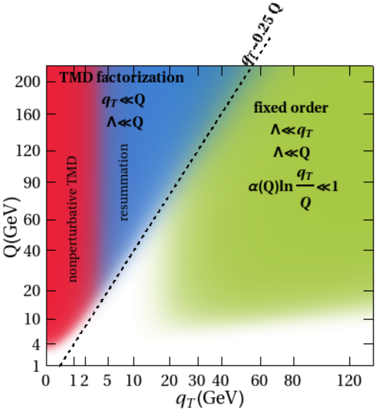
4GeV< $Q$ <1000GeV






4GeV 1000GeV

Very presice test of TMD evolution



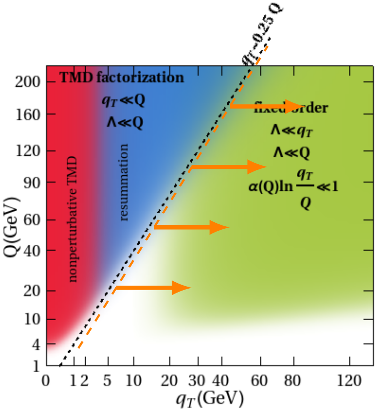

**Power corrections:**
(many works during last year)

- I.Stewart, A.Gao, et al,
- S.Rodini, AV, et al,
- I.Balitsky, et al,
- ...

**NLP TMD factorization is done!**
e.g. [2306.09495] for SIDIS
(it is much more complicated than one expected)

## TMD factorization at NLP

- 4 TMDFFs, 16 TMDPDFs of twist-3
- NLP restoration of frame-invariance, gauge invariance, boost invariance
- NLO expression for coefficient functions
- LO evolution for twist-3 TMDs
- Qiu-Sterman-like terms in TMD factorization



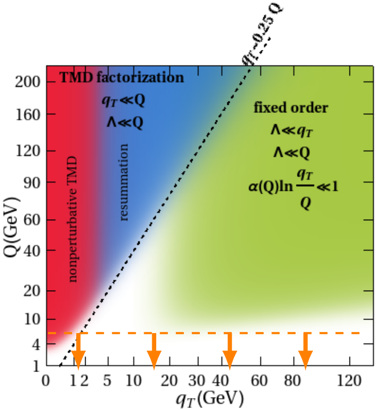

**Power corrections:**

**1.** $q_T/Q$-corrections
$Y$-term

**2.** $\Lambda/Q$ & $M/Q$-corrections
higher-twist
target-mass

**3.** $k_T/Q$-corrections
kinematic

[AV,2307.13054]



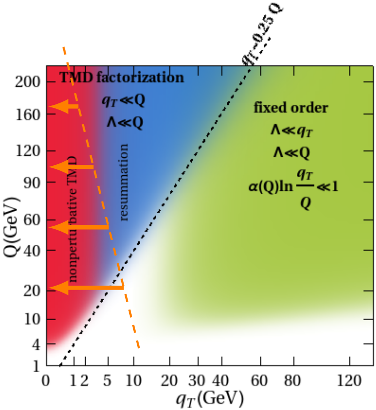

**Power corrections:**

1. $q_T/Q$-corrections
   $Y$-term

2. $\Lambda/Q$ & $M/Q$-corrections
   higher-twist
   target-mass

3. $k_T/Q$-corrections
   kinematic

[AV,2307.13054]



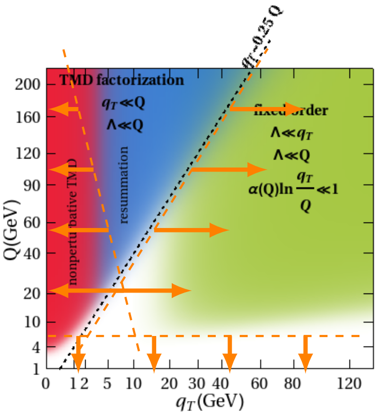

**Power corrections:**

**1.** $q_T/Q$-corrections
$Y$-term

**2.** $\Lambda/Q$ & $M/Q$-corrections
higher-twist
target-mass

**3.** $k_T/Q$-corrections
kinematic

[AV,2307.13054]



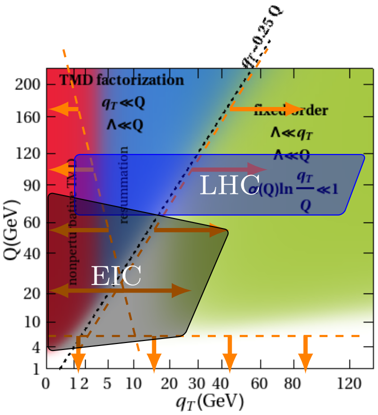

**Power corrections:**

**1.** $q_T/Q$-corrections
$Y$-term

**2.** $\Lambda/Q$ & $M/Q$-corrections
higher-twist
target-mass

**3.** $k_T/Q$-corrections
kinematic

[AV,2307.13054]




**Power corrections:**

1. $q_T/Q$-corrections
   $Y$-term
2. $\Lambda/Q$ & $M/Q$-corrections
   higher-twist
   target-mass
3. $k_T/Q$-corrections
   kinematic

[AV,2307.13054]

This explains why there are problems with low-$k_T$ at $Q \sim 10$GeV
LHC is "pure" perturbation theory
EIC will be "more interesting"

# (some) References


**TMD factorization**

- [1] J.Collins "*Foundations of Perturbative QCD*" [Collins:2011zzd]
- [2] Becher & Neubert, [1007.4005] (SCET, collinear anomaly)
- [3] Echevaria, Idilbi, Scimemi, [1111.4996](SCET)
- [4] Vladimirov, Moos, Scimemi [2109.09771] (background QCD)

**Divergences**

- [5] Chiu, Jain, Neill, Rothstein [1202.0814] (rap.div. vs. double logs)
- [6] Vladimirov [1707.07606] (proof of renormalization of rap.div.)
- [7] Vladimirov [1610.05791] (SAD/RAD correspondence)

**TMD evolution**

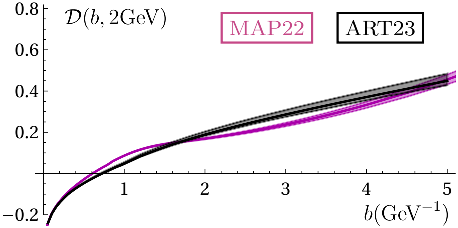
- [8] Aybat, Rogers [1101.5057] ("standard solution")
- [9] Scimemi, Vladimirov [1803.11089] (2D evolution)

**Small-b**

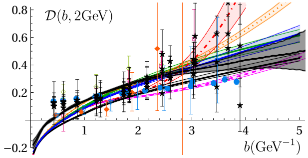
- [10] Moult, Zhu, Zhu, Jiao [2205.02249] (CS at small-b at 4-loops)
- [11] [Luo:2019szz, Ebert:2020yqt, Luo:2020epw] (3 loop for $f_1$)
- [12] Scimemi, Tarasov, Vladimirov [1901.04519] (Small-b in background QCD)

Backup

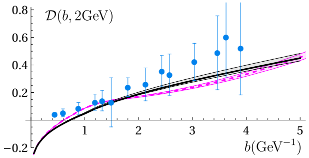
## Collins-Soper kernel




Very small uncertanties
(despite huge uncertanties in TMDPDFs)




all extractions 2019-2023
pheno + lattice




MAP22+ART23+mean lattice