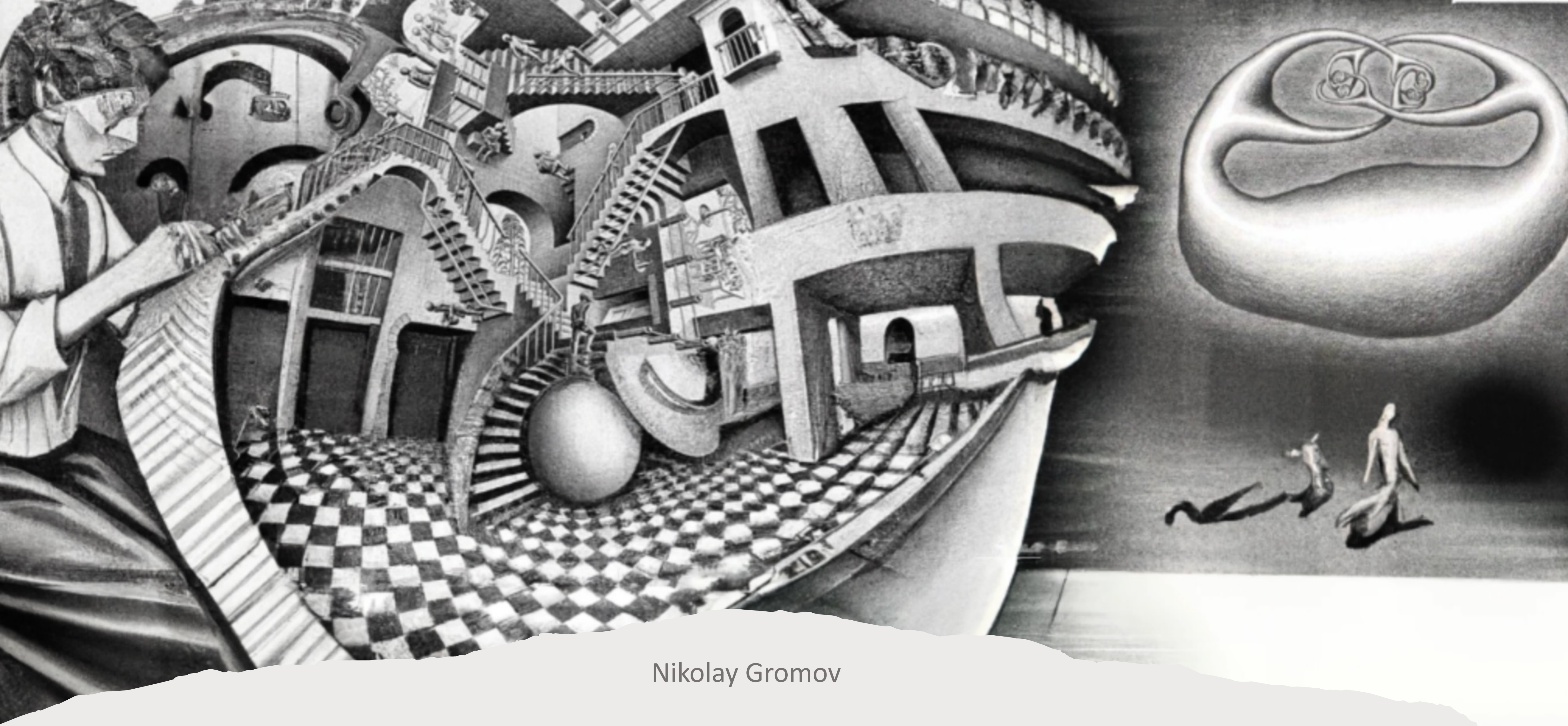


Nikolay Gromov

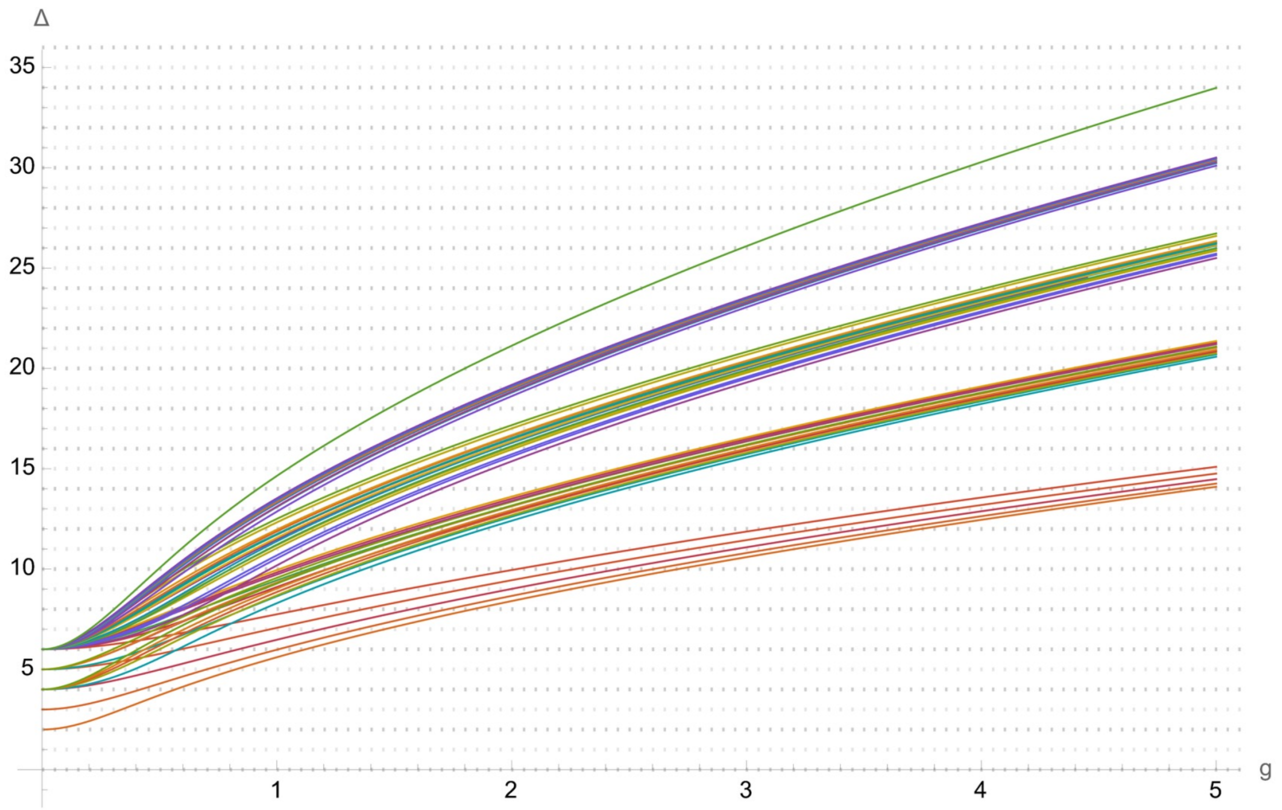
Quantum Spectral Curve and BFKL limit of N=4 SYM

European Research Council
Established by the European Commission

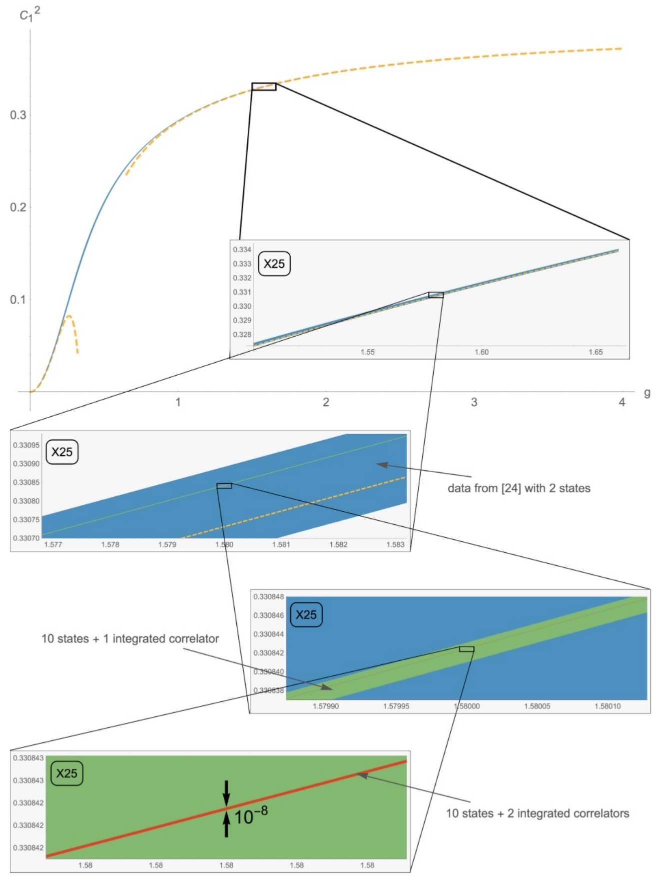
QSC in real life

What is QSC?

[NG, Kazakov, Leurent, Volin '13]



[NG, Julius, N.Sokolova '23]



[Cavaglia, NG, Julius, Preti '21]

Spectrum of integrable CFTs: N=4 SYM, 3D ABJM, 2D…, Non-susy (e.g. fishnets)

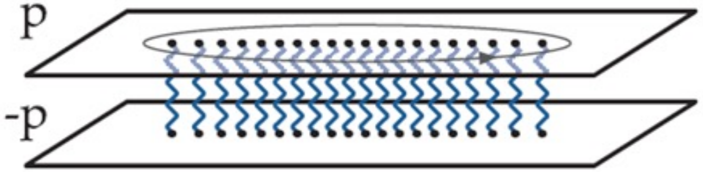
And beyond…

[Basso Georgoudis Sueiro '22]

Conceptual “derivation”

1. QQ-relations

Oscillator:



$$\Psi_1 = Q_1(x)e^{-x^2/2}$$

$$\Psi_2 = Q_2(x)e^{+x^2/2}$$

Satisfy Wronskian identity

$$\Psi_2(x)\Psi_1'(x) - \Psi_1(x)\Psi_2'(x) = 1$$

Derivative ⟶ Finite difference

Wronskian ⟶ Super-Wronskian

N=4 SYM:

$$Q_{a,i}(u+\tfrac{i}{2}) - Q_{a,i}(u-\tfrac{i}{2}) = \mathbf{P}_a(u)\mathbf{P}_b(u)Q_{b,i}(u+\tfrac{i}{2})$$

$$\mathbf{Q}_i(u) = \mathbf{P}_a(u)Q_{a,i}(u+\tfrac{i}{2})$$

N=4 SYM

The "simplest" generalization of QCD:

$$S = \frac{1}{4g_{YM}^2}\int d^4x \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu} + \ldots)$$ Plus extra scalar fields $\Phi_1, \ldots, \Phi_6$ and fermions

Parameters: $\lambda = g_{YM}^2 N_c$ and $N_c = \infty$

Symmetries:

Lorentz: $so(3,1)$

Conformal: $so(4,2)$ $\Leftrightarrow$ $su(2,2)$

$so(6)$ $\Leftrightarrow$ $su(4)$ - rotation of the scalars

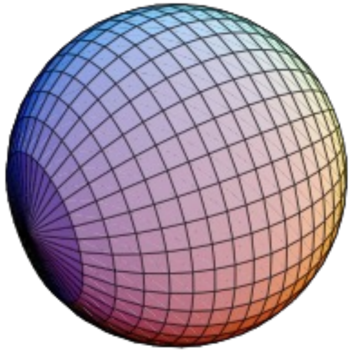
Super (graded) Lie algebra:

$$su(2,2|4)$$

Dual description

String in $AdS_5 x S^5$

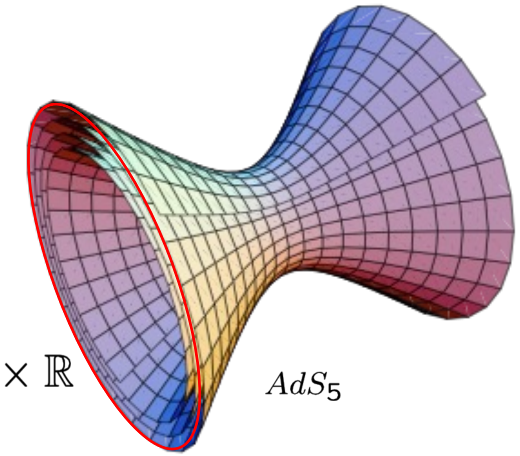
$$X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 = 1$$



S^5

x

$$Y_0^2 + Y_1^2 - Y_2^2 - Y_3^2 - Y_4^2 - Y_5^2 = 1$$



$\mathbb{S}^3 \times \mathbb{R}$

AdS_5

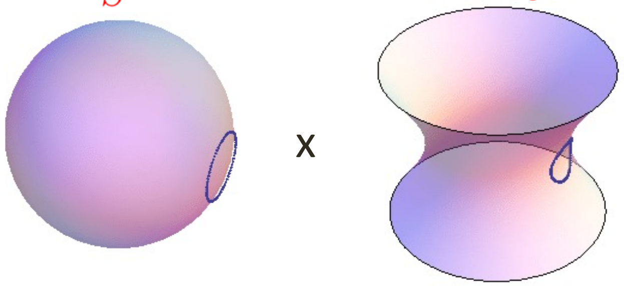
Symmetries:

$$so(6) \qquad so(4,2)$$

$$su(2,2|4)$$

Finding QSC

S^5 AdS_5



$\mathrm{SO}(6)$ $\mathrm{SO}(5,1)$

Super (graded) Lie algebra:

$$\mathrm{PSU}(2,2|4)$$

- From integrable spin-chains we know that for rank M symmetry we get M+1 Q-functions, for $\mathrm{PSU}(2,2|4)$ M=7
- Classical spectral curve has 8 sheets

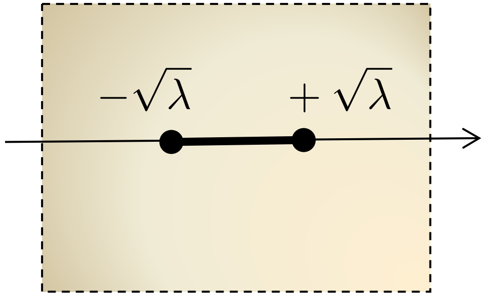
$$\Psi = Q(x)e^{-x^2/2} \longrightarrow (\underbrace{\mathbf{P}_1,\ \mathbf{P}_2,\ \mathbf{P}_3,\ \mathbf{P}_4}_{S^5}|\underbrace{\mathbf{Q}_1,\ \mathbf{Q}_2,\ \mathbf{Q}_3,\ \mathbf{Q}_4}_{AdS_5})$$

2. Analyticity

[N.G., Kazakov, Leuren, Volin]

[N.G., Levkovich-Maslyuk, Sizov]

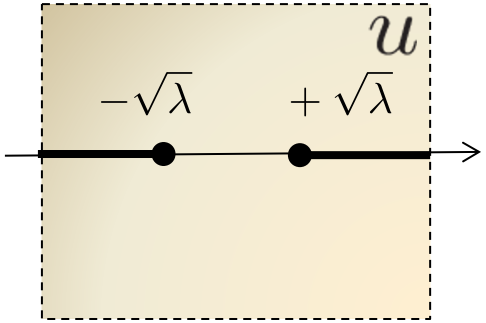
$\mathbf{P}_a$



$$\mathbf{P}_a \simeq u^{\text{R-charge}} \quad , \quad u \to \infty$$

$\mathrm{SO}(6)$

$\mathbf{Q}_a$



$$\mathbf{Q}_a \sim u^{\text{conformal charge}}$$

$\mathrm{SO}(5,1)$

BFKL POMERON

(Fadin, Lipatov'98; Kotikov, Lipatov'00)

The total crossection of the hight-energy scattering

$$\sigma(s) = \int \frac{d^2q\, d^2q'}{(2\pi)^2 q^2 q'^2} \Phi_A(q) \Phi_B(q') \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \left(\frac{s}{s_0}\right)^{\omega} G_\omega(q, q')$$

where $s_0 = |q||q'|$ and $s = 2p_A p_B$.

For the t-channel partial wave there holds the Bethe-Salpeter equation

$$\omega G_\omega(q, q_1) = \delta^{D-2}(q - q_1) + \int d^{D-2} q_2 K(q, q_2) G_\omega(q_2, q_1)$$

$$G = \frac{1}{\omega - K}$$

It appears to be possible to classify the Pomeron eigenvalues ω of the BFKL kernel K using two quantum numbers: integer n (conformal spin) and real ν

$$\omega = \omega(n, \nu)\ .$$

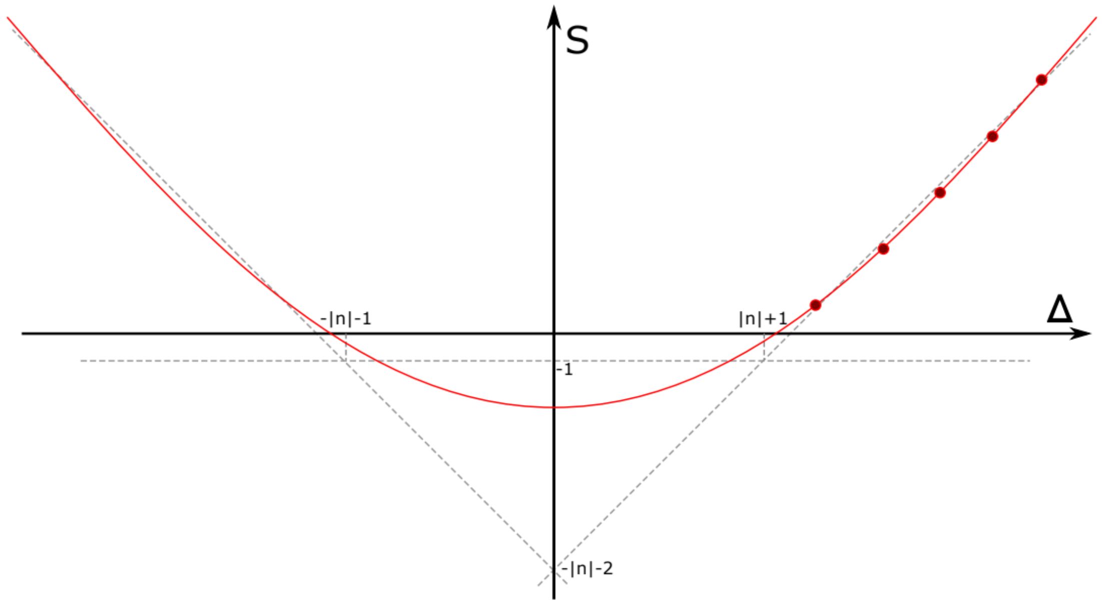
BFKL pomeron in N=4 SYM

Kotikov, Lipatov'00

- We consider important class of length-2 operators

$$\mathrm{tr} Z (D_+)^{S_1} (\partial_\perp)^{S_2} Z + \text{permutations} .$$

- Trajectory $S(\Delta, n)$, where $S = S_1$ and $n = S_2$, corresponding to the length-2 operator $\mathrm{tr}\, Z(D_+)^S(\partial_\perp)^n Z$ with the physical points depicted by the dots



The identification with the high-energy scattering regime is $\omega(n, \nu) = S + 1$, where $\nu = -i\Delta/2$.

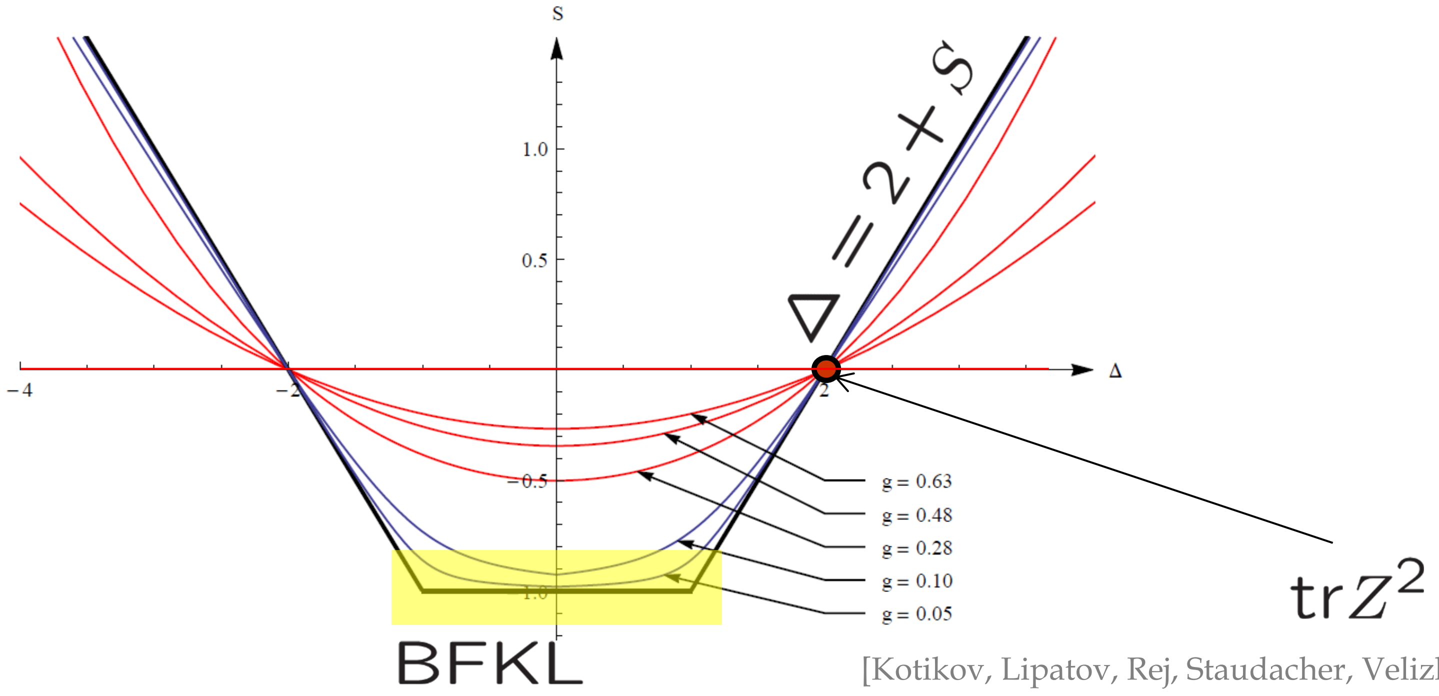
BFKL regime

Important class of single trace operators:

$$\mathrm{tr} D^S Z^2 + \text{permutations}$$

Spectrum for different spins:

[Brower, Polchinski, Strassler, -Itan `06]



[Kotikov, Lipatov, Rej, Staudacher, Velizhanin]

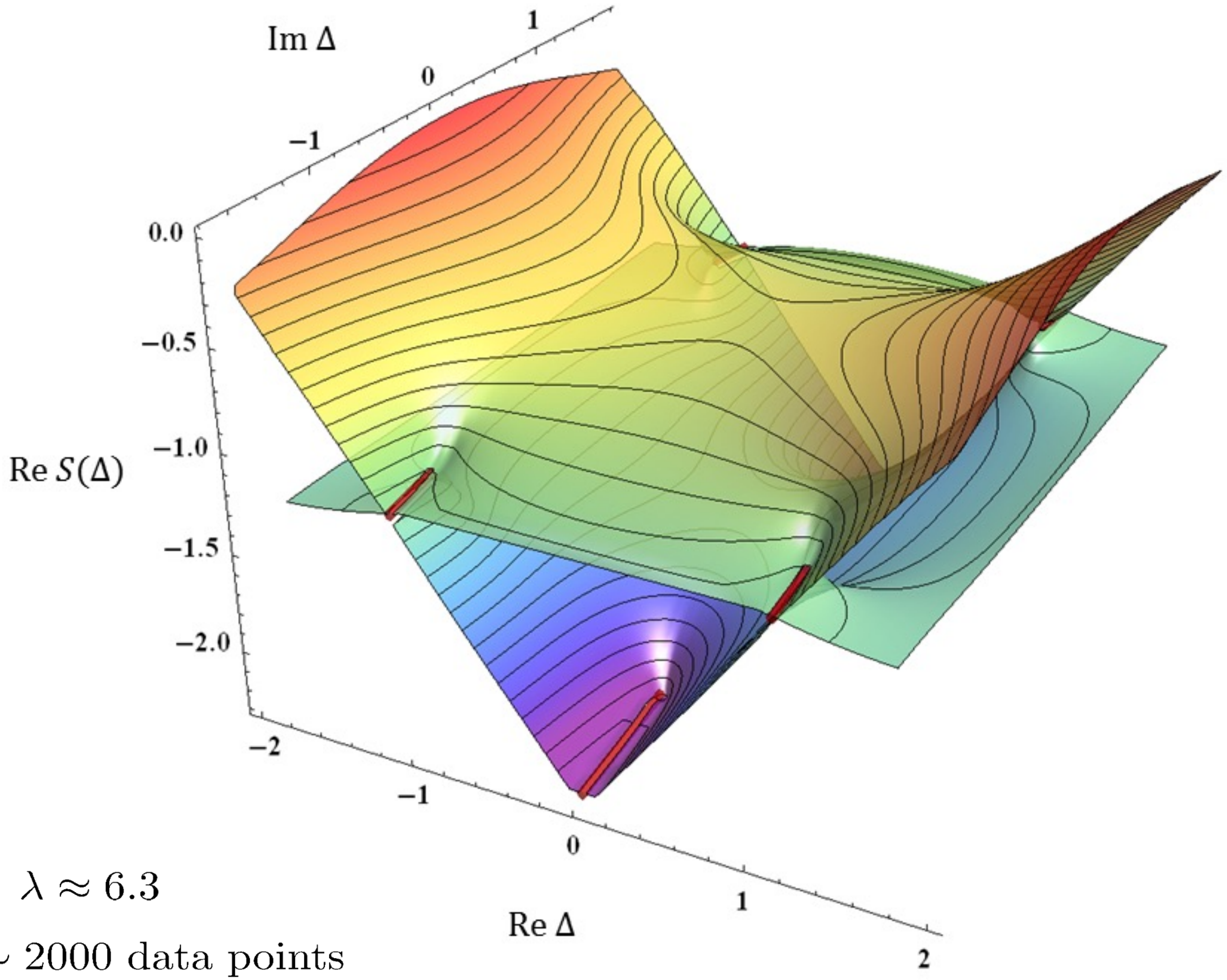
BFKL regime:

$S \to -1 \;,\; g \to 0$ So that: $\dfrac{g^2}{S+1} \simeq 1$ Resumming to all loops terms $\left(\dfrac{g^2}{S+1}\right)^n$

In this regime SYM is undistinguishable from the real QCD

Analytic structure at finite coupling

Gromov,FLM,Sizov 2015



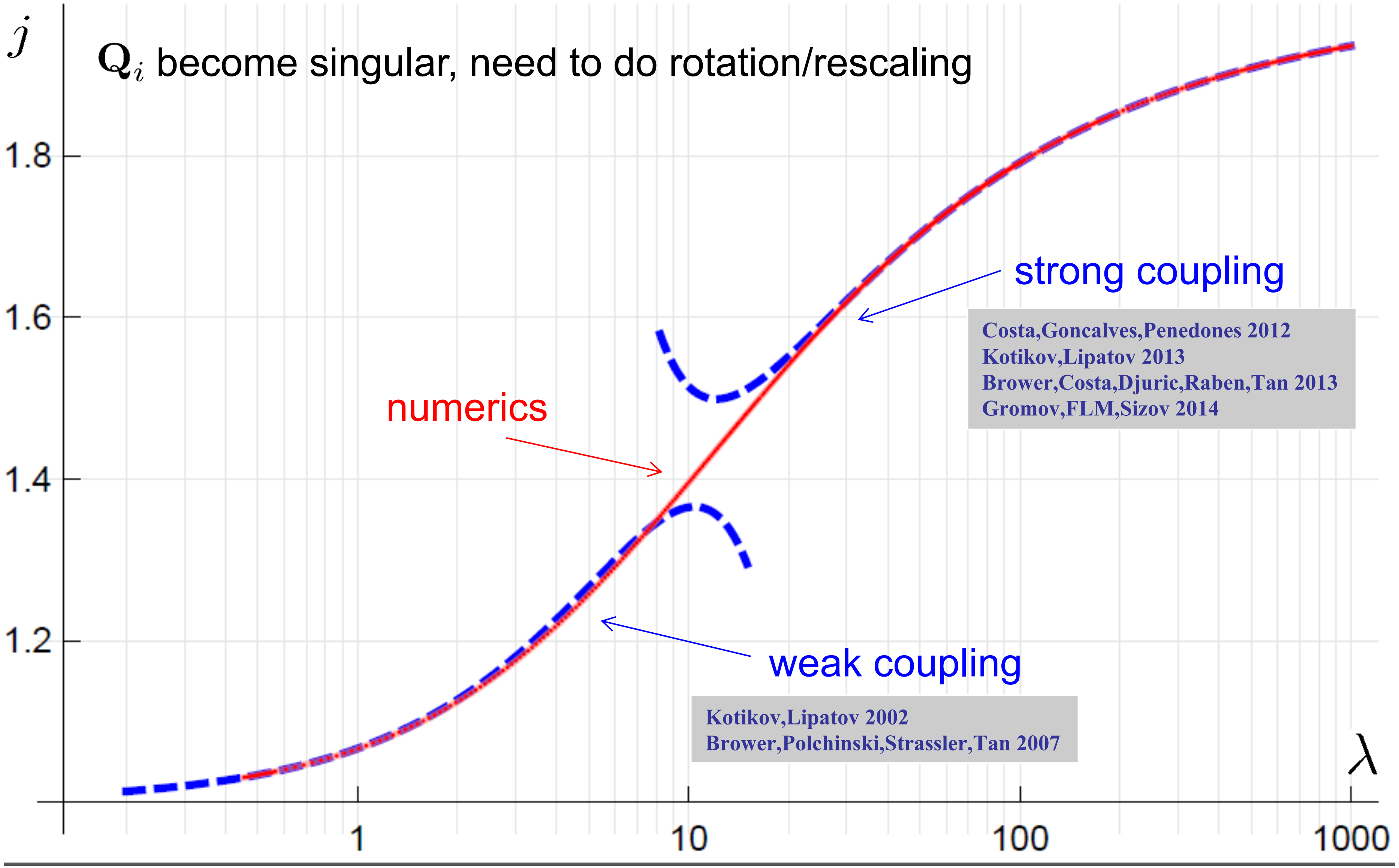
$\lambda \approx 6.3$

~ 2000 data points

BFKL pomeron intercept

$$j = 2 + S(\Delta)|_{\Delta=0}$$

Gromov,FLM,Sizov 2015



Known analytical results

BFKL in QCD

- At the LO:

Jaroszewicz, 1982
Lipatov 1986
Kotikov,Lipatov 2002

$$\chi(\gamma) = 2\Psi(1) - \Psi(\gamma) - \Psi(1-\gamma), \quad \Psi(\gamma) = \Gamma'(\gamma)/\Gamma(\gamma)$$

- At NLO:

Kotikov,Lipatov 2002
Kotikov,Lipatov 2000

$$\delta(\gamma) = -\Bigg[\left(\frac{11}{3} - \frac{2n_f}{3N_c}\right)\frac{1}{2}\left(\chi^2(\gamma) - \Psi'(\gamma) + \Psi'(1-\gamma)\right) - \left(\frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9}\frac{n_f}{N_c}\right)\chi(\gamma$$

$$-6\zeta(3) + \frac{\pi^2\cos(\pi\gamma)}{\sin^2(\pi\gamma)(1-2\gamma)}\left(3 + \left(1 + \frac{n_f}{N_c^3}\right)\frac{2+3\gamma(1-\gamma)}{(3-2\gamma)(1+2\gamma)}\right)$$

$$-\Psi''(\gamma) - \Psi''(1-\gamma) - \frac{\pi^3}{\sin(\pi\gamma)} + 4\phi(\gamma)\Bigg].$$

$$\phi(\gamma) = -\int_0^1 \frac{dx}{1+x}\left(x^{\gamma-1} + x^{-\gamma}\right)\int_x^1 \frac{dt}{t}\ln(1-t)$$

$$= \sum_{n=0}^{\infty}(-1)^n\left[\frac{\Psi(n+1+\gamma) - \Psi(1)}{(n+\gamma)^2} + \frac{\Psi(n+2-\gamma) - \Psi(1)}{(n+1-\gamma)^2}\right]$$

BFKL regime

$S \to -1, g \to 0, \frac{g^2}{S+1} = \text{fixed}$ Resums contributions from all loop orders

N=4 SYM should give the highest transcendentality part of the QCD result

$$S = -1 + \sum_{n=1}^{\infty} g^{2n} \left[F_n\left(\tfrac{\Delta-1}{2}\right) + F_n\left(\tfrac{-\Delta-1}{2}\right) \right]$$

Costa,Goncalves,Penedones 2012

$$S_a(x) = \sum_{k=1}^{x} \frac{(\text{sign}(a))^k}{k^{|a|}}$$

$$S_{a,b,c,\ldots}(x) = \sum_{k=1}^{x} (\text{sign}(a))^k S_{b,c,\ldots}(k)$$

Leading order $F_1(x) = -4S_1(x)$

Reproduced from QSC in [Alfimov,Gromov,Kazakov 2014]

NLO
$$F_2(x) = 4\left(-\frac{3}{2}\zeta(3) + \pi^2 \log 2 + \frac{\pi^2}{3} S_1(x) + 2S_3(x) + \pi^2 S_{-1}(x) - 4S_{-2,1}(x)\right)$$

NNLO $F_3(x) = ?????$

Basis for NNLO

Each term has transcendentality 5

$\Downarrow$

harmonic sums with transcendentality up to 5, and constants:

$$\pi, \log(2), \zeta(3), \zeta(5), \mathrm{Li}_4(1/2), \mathrm{Li}_5(1/2)$$

In total 288 elements

We derive analytic constraints from QSC
for expansion around poles at $\Delta_0 = 1, 3, 5, 7, \ldots$

E.g. around $\Delta = 5$ we get the expansion

$$S_{\text{NNLO}} = -\frac{1024}{\delta^5} + \frac{64\left(4\pi^2-33\right)}{3\delta^3} + \frac{16\left(-36\zeta_3+2\pi^2+31\right)}{\delta^2} + \frac{-288\zeta_3+\frac{232\pi^4}{45}-16\pi^2-296}{\delta}$$

$$-\frac{2}{15}\left[20\left(4\pi^2-75\right)\zeta_3+6300\zeta_5+\pi^4-215\pi^2+285\right]$$

$$+\delta\left[40\zeta_3^2+2\left(8\pi^2-123\right)\zeta_3-396\zeta_5+\frac{373\pi^6}{945}+\frac{11\pi^4}{9}-26\pi^2+\frac{1771}{4}\right]$$

$$+\delta^2\left[-48\zeta_3^2+\left(\frac{505}{2}-6\pi^2+\frac{43\pi^4}{45}\right)\zeta_3+\left(329+\frac{4\pi^2}{3}\right)\zeta_5-\frac{1001\zeta_7}{4}+\frac{31\pi^6}{252}+\frac{27\pi^4}{40}+\frac{147\pi^2}{8}-\frac{12387}{16}\right]$$

$$+\delta^3\left[-\frac{2}{3}\left(\pi^2-30\right)\zeta_3^2+\left(218\zeta_5+\frac{7\pi^4}{5}+\frac{4\pi^2}{3}-\frac{1779}{8}\right)\zeta_3+\left(8\pi^2-\frac{1161}{4}\right)\zeta_5\right.$$

$$\left.-\frac{2715\zeta_7}{8}+\frac{78S_{5,3}(\infty)}{5}-\frac{14233\pi^8}{3402000}-\frac{223\pi^4}{144}-\frac{557\pi^2}{48}+\frac{7625}{8}\right]+O\left(\delta^4\right)\,.$$

Get an overdetermined set of equations on coefficients of basis elements ⇒ the result

Only 37 out of 288 coefficients are nonzero!

Our result: BFKL at NNLO

$$S = -1 + \sum_{n=1}^{\infty} g^{2n} \left[F_n\left(\tfrac{\Delta-1}{2}\right) + F_n\left(\tfrac{-\Delta-1}{2}\right)\right]$$

NG,Levkovich-Maslyk,Sizov
Phys.Rev.Lett. 115 (2015)

$$\frac{1}{256}F_3 =$$
$$-\frac{5S_{-5}}{8} - \frac{S_{-4,1}}{2} + \frac{S_1 S_{-3,1}}{2} + \frac{S_{-3,2}}{2} - \frac{5S_2 S_{-2,1}}{4}$$
$$+\frac{S_{-4}S_1}{4} + \frac{S_{-3}S_2}{8} + \frac{3S_{3,-2}}{4} - \frac{3S_{-3,1,1}}{2} - S_1 S_{-2,1,1}$$
$$+S_{2,-2,1} + 3S_{-2,1,1,1} - \frac{3S_{-2}S_3}{4} - \frac{S_5}{8} + \frac{S_{-2}S_1 S_2}{4}$$
$$+\pi^2\left[\frac{S_{-2,1}}{8} - \frac{7S_{-3}}{48} - \frac{S_{-2}S_1}{12} + \frac{S_1 S_2}{48}\right]$$
$$+\zeta_3\left[-\frac{7S_{-1,1}}{4} + \frac{7S_{-2}}{8} + \frac{7S_{-1}S_1}{4} - \frac{S_2}{16}\right]$$
$$+\left[2\mathrm{Li}_4 - \frac{\pi^2\log^2 2}{12} + \frac{\log^4 2}{12}\right](S_{-1} - S_1) - \pi^4\left[\frac{2S_{-1}}{45} - \frac{S_1}{96}\right]$$
$$+\frac{\log^5 2}{60} - \frac{\pi^2\log^3 2}{36} - \frac{2\pi^4\log 2}{45} - \frac{\pi^2\zeta_3}{24} + \frac{49\zeta_5}{32} - 2\mathrm{Li}_5$$

Found from
iterative solution of QSC

Confirmed by an independent calculation by Caron-Huot, Herran

Numerical test

At $\Delta = 0$ and $\Delta = 0.45$ computed NNLO by fitting our numerical data

Reached 60 digits precision – perfect match with our analytic result!

$$\begin{aligned} S(\Delta = 0) \quad = \quad & -1 + 0.07023049277268287640893859949699700963287653244326254137743 44g^2 \\ & -84.0785668074649199122952886915360731240938374433839202117447g^4 \\ & -2543.0481651804494295352453425547790233427709977821342068768429587g^6 \end{aligned}$$

$\Delta = 0.45$

	value	error
N^2LO	10775.6358188471766379575931271924 5699592917094805765378342453322 9	10^{-61}
N^3LO	$-$366392.2052053917038937903507478 5445499355319593339191634038 36	10^{-56}
N^4LO	1.33273645568112691569404431036982 8561521940588979476878854 $\times 10^7$	10^{-51}
N^5LO	$-$4.92174012665791650091395555207507006072145095843655 9876 $\times 10^8$	10^{-47}

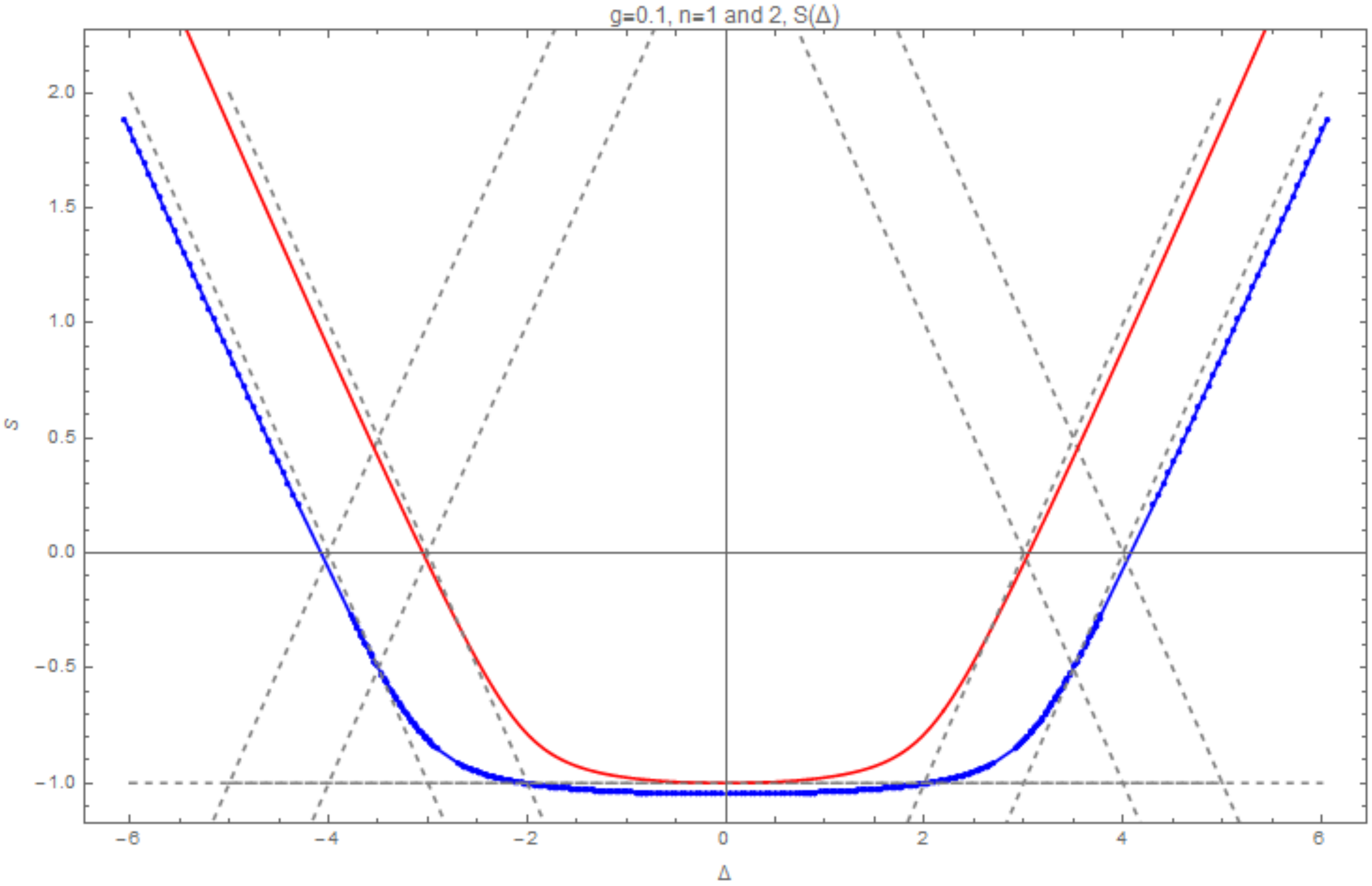
Non-zero conformal spin

$$\text{tr}Z(D_+)^{S_1}(\partial_\perp)^{S_2}Z + \text{permutations}\,.$$

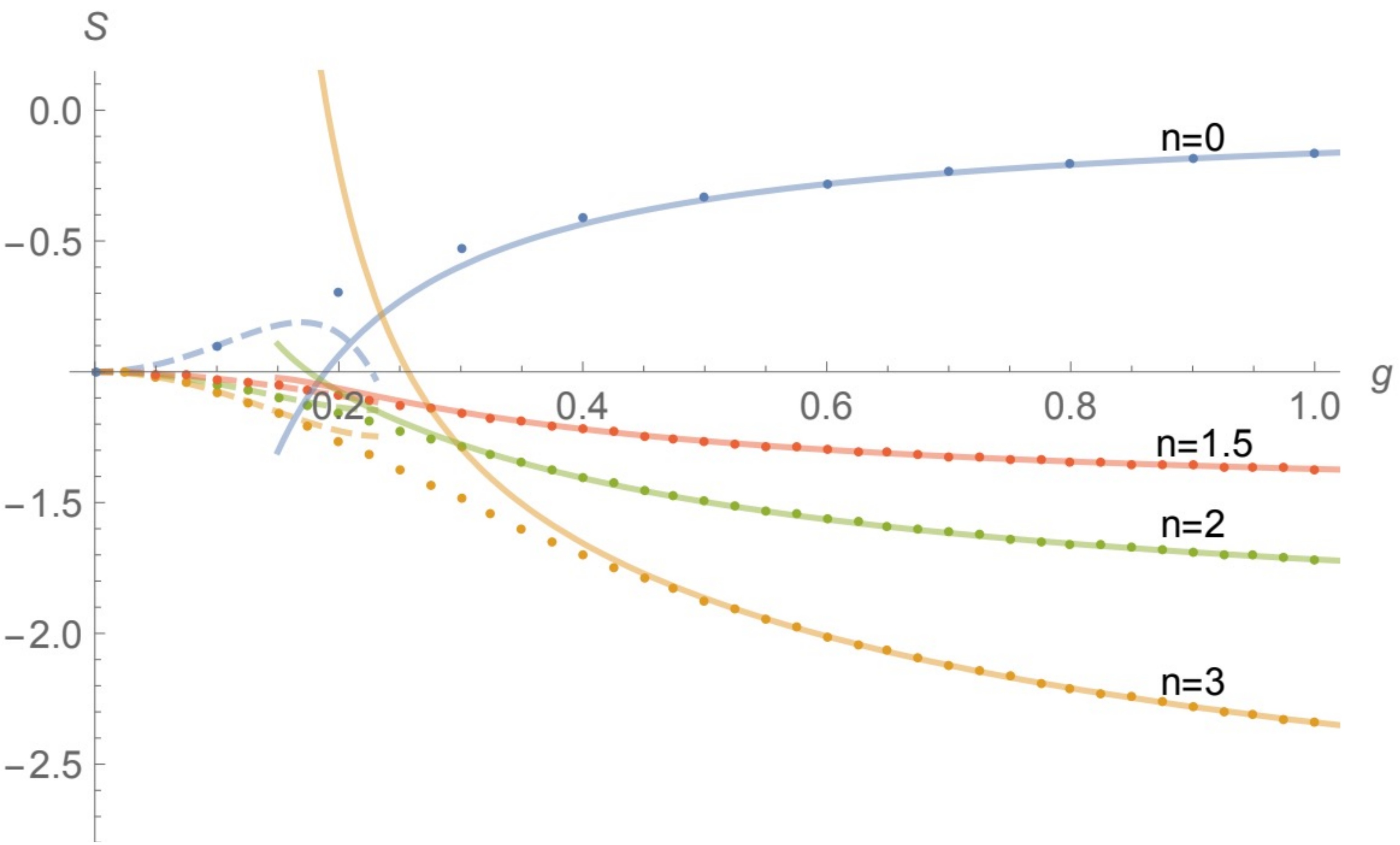
M. Alfimov, N.G., G. Sizov

Non-zero conformal spin means that one of the Conformal charges S2=n is nonzero
We analytically continue in S1=S finding S(Δ,n)

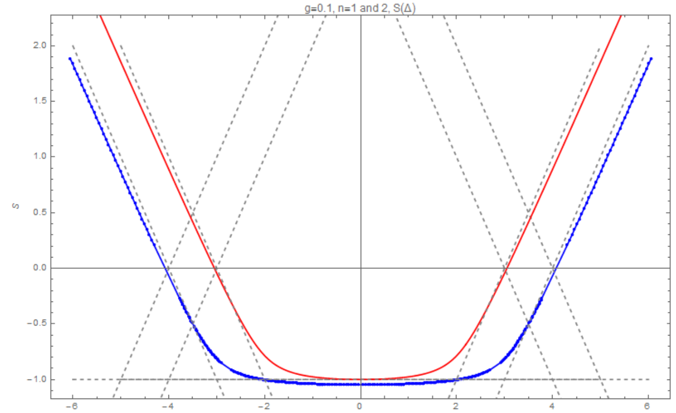
The Regge trajectory for g=1/10



- Intercept $S(0,n)$ as the function of the coupling constant g for conformal spins $n=0$, $n=3/2$, $n=2$ and $n=3$ (dots), weak coupling expansion of the intercept (dashed lines) and strong coupling expansion (continuous lines).



$$S(0,n) = -n + \frac{(n-1)(n+2)}{\lambda^{1/2}} - \frac{(n-1)(n+2)(2n-1)}{2\lambda} + \frac{(n-1)(n+2)(7n^2-9n-1)}{8\lambda^{3/2}} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)$$



Exact all loop result in the vicinity of n=1

$$\theta(g) = 1 + \frac{I_1(4\pi g) I_2(4\pi g)}{\sum_{k=1}^{+\infty} (-1)^k I_k(4\pi g) I_{k+1}(4\pi g)}$$

$$\theta = \frac{\partial S_1}{\partial S_2}\Big|_{\substack{\Delta=0 \\ S_2=1}} = \frac{\partial j}{\partial n}\Big|_{n=1}$$

- Similar calculations give the curvature function

$$\gamma(g) = \frac{1}{4\pi g^4 I_2^2} \oint_{-2g}^{2g} dv \left(\cosh_-^v v \Gamma[\cosh_-^u u](v) - \cosh_-^v v^2 \Gamma[\cosh_-^u](v)\right) +$$

$$+ \frac{1}{16\pi g^5 I_2} \oint_{-2g}^{2g} dv \left(\frac{v^3 \Gamma[\cosh_-^u](v) - 2v^2 \Gamma[\cosh_-^u u](v) + v \Gamma[\cosh_-^u u^2]}{x_v - \frac{1}{x_v}} \right),$$

where

$$\Gamma[h(v)](u) = \oint_{-2g}^{2g} \frac{dv}{2\pi i} \partial_u \log \frac{\Gamma[i(u-v)+1]}{\Gamma[-i(u-v)+1]} h(v).$$

$$\gamma(g) = 2\zeta_3 g^2 + \left(-\frac{2\pi^2}{3}\zeta_3 - 35\zeta_5\right) g^4 + \left(\frac{16\pi^4}{45}\zeta_3 + \frac{22\pi^2}{3}\zeta_5 + 504\zeta_7\right) g^6 +$$

$$+ \left(-\frac{28\pi^6}{135}\zeta_3 - \frac{8\pi^4}{3}\zeta_5 - 56\pi^2\zeta_7 - 6930\zeta_9\right) g^8 + \mathcal{O}(g^{10}).$$

M. Alfimov, N.G., G. Sizov

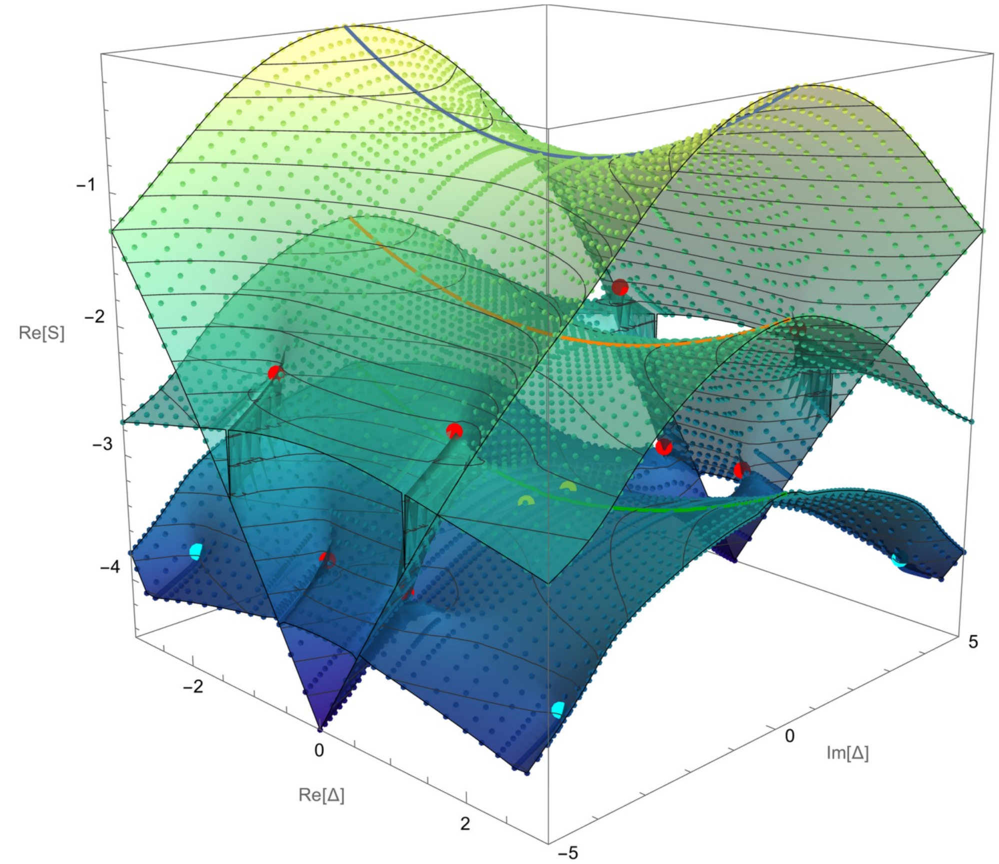
Intercept for any n

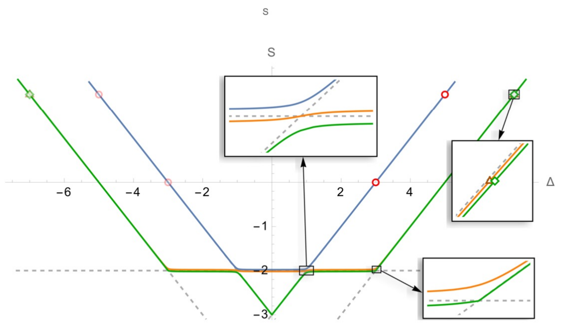
	-1
LO	$-8 g^2 S_1$
NLO	$g^4 \left(-(32 S_{-2,1}) + 16 S_{-3} + \frac{8\pi^2 S_1}{3} + 16 S_3\right)$
NNLO	$g^6 \left(128 S_{-4,1} - 128 S_{-3,2} + \pi^2 \left(\frac{64 S_{-2,1}}{3} - \frac{32 S_{-3}}{3}\right) + 128 S_{-2,3} + 256 S_{-3,1,1} + 256 S_{1,-3,1} + 256 S_{1,-2,2} - 512 S_{1,-2,1,1} - 192 S_{-5} - \frac{64\pi^4 S_1}{45} - 64 S_5\right)$
NNNLO	$g^8 \left(-(128 \zeta_3 S_{-3,1}) - 256 \zeta_3 S_{-2,2} + 640 \zeta_3 S_{1,-3} + 896 \zeta_5 S_{1,1} + 1920 \zeta_3 S_{1,3} + 512 \zeta_3 S_{2,-2} + 1536 \zeta_3 S_{2,2} + 1920 \zeta_3 S_{3,1} + 512 \zeta_3 S_{-2,1,1} - 256 \zeta_3 S_{1,-2,1} - 1024 \zeta_3 S_{1,1,-2} - 1536 \zeta_3 S_{1,1,2} - 1536 \zeta_3 S_{1,2,1} - 1536 \zeta_3 S_{2,1,1} + \pi^2 \left(\frac{128}{3} \zeta_3 S_{1,1} + \frac{128 S_{-4,1}}{3} + 128 S_{-3,2} - \frac{128}{3} S_{-2,-3} - \frac{128 S_{-2,3}}{3} + \frac{256 S_{1,-4}}{3} + \frac{128 S_{2,-3}}{3} - \frac{128 S_{4,1}}{3} - 256 S_{-3,1,1} + \frac{256}{3} S_{-2,-2,1} - \frac{1024}{3} S_{1,-3,1} - 256 S_{1,-2,2} - \frac{256}{3} S_{1,1,-3} - \frac{256}{3} S_{2,-2,1} + 512 S_{1,-2,1,1} + \frac{512}{3} S_{1,1,-2,1} - \frac{64 \zeta_3 S_2}{3} + \frac{128 S_{-5}}{3}\right) + \pi^4 \left(-\frac{1}{15}(256 S_{-2,1}) + \frac{128 S_{-3}}{15} - \frac{32 S_3}{15}\right) + \text{const} - 128 \zeta_3 S_{-4} - 1152 \zeta_3 S_4 - 448 \zeta_5 S_2 + \frac{112\pi^6 S_1}{135}\right)$

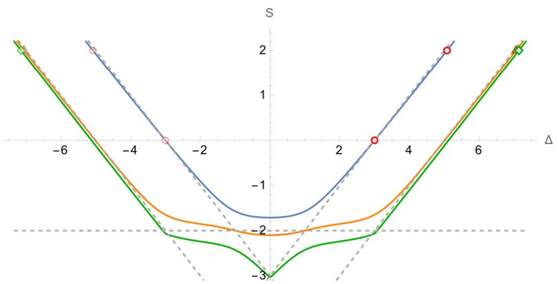
Argument of all the harmonic sums is (|n|-1)/2

R. Klabbers,M. Preti, I. Szecsenyi` 23

Higher twist trajectories







Intercept:

$$\alpha = -2 + 2g + 16\log 2\ g^2 - \frac{2\pi^2}{3}g^3 - 204.77377158292661g^4 + 136.29333638813g^5 + 4733.39078974g^6 - 6116.79585g^7 + \ldots,$$