Workshop: Overlap between QCD resummations

Centre Paul Langevin, *Aussois, France*
14-17 January 2024

Integrability of Planar N=4 SYM and BFKL

Vladimir Kazakov

Motivation and outline

- N =4 Super-Yang-Mills theory – a great non-perturbative playground for 4d gauge theories
- Emblematic, and well understood example of AdS/CFT duality
- N =4 SYM is integrable in the 't Hooft (planar, large N) limit

Gromov, V.K., Leurent, Volin '13, '14
V.K., Leurent, Volin '15

- Quantum Spectral Curve (QSC) – exact system Riemann-Hilbert equations and an efficient formalism for planar spectrum of anomalous dimensions in N =4 SYM at any comping
- QSC allows to study analytically a few non-trivial (nonperturbative) limits: strong coupling, fishnet CFT, Regge (BFKL), very high orders of PT, high precision numerics, etc

Gurdogan, V.K., '15

Plan of the talk

- Perturbative integrability
- Quantum Spectral Curve (general formulation)
- Regge limit from QSC and reproduction of LO BFKL spectrum

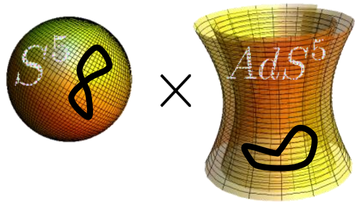
Alfimov, Gromov, V.K.,'12

- NLO, NNLO and much more - in Nikolay Gromov's talk (the force of QSC)

$\mathcal{N}$=4 SYM: planar integrability from AdS/CFT duality

$$\mathcal{S}_{SYM} = \frac{1}{\lambda}\int d^4x \, \mathrm{Tr}\left(F^2 + (\mathcal{D}\Phi)^2 + [\Phi,\Phi]^2\right) + \text{fermions}$$

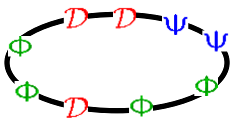
dual to



Maldacena '97
Gubser,Klebanov,Polyakov
Witten

super-conformal theory: PSU(2,2|4) symmetry
β-function=0, no massive particles

operators



$$\mathcal{O}(x) = \mathrm{Tr}\,[\mathcal{D}\mathcal{D}\Psi\Psi\Phi\Phi\mathcal{D}\Psi\ldots]\,(x)$$

Anomalous dimensions $\Delta_{\mathcal{O}}$

$$\mathcal{O}(\xi x) \to \xi^{\Delta_{\mathcal{O}}(\lambda)}\mathcal{O}(x)$$

Operator product expansion,
structure constants, correlators, amplitudes…

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k |x|^{-\Delta_i-\Delta_j+\Delta_k}\, C_{i,j}^k\, \mathcal{O}_k(0) + \ldots$$

1- and 2-loop integrability
Classical integrability
S-matrix, asymptotic Bethe ansatz

→ Y-system → Thermodynamical Bethe ansatz (TBA) → Quantum spectral curve (QSC)

Bena, Roiban, Polchinski 02'
Metsaev-Tseytlin 02'
Minahan, Zarembo 03'
Beisert, Kristjansen, Staudacher 03'
V.K.,Marshakov, Minahan, Zarembo 04'
Beisert, V.K.,Sakai, Zarembo 05'
Beisert,Staudacher 04'
Beisert 05'
Janik 05'
Beisert, Eden, Staudacher '06

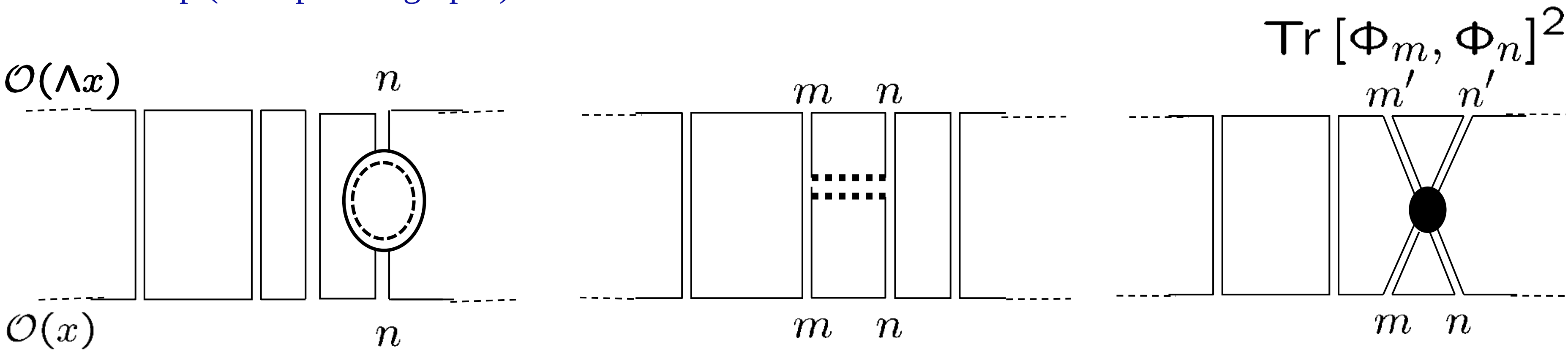
Gromov, V.K,, Vieira '09

Bombardelli,Fioravanti,Tateo '09
Gromov,V.K.,Kozak,Vieira '09
Arutyunov,Frolov '09
Cavaglia, Fioravanti, Tateo '09

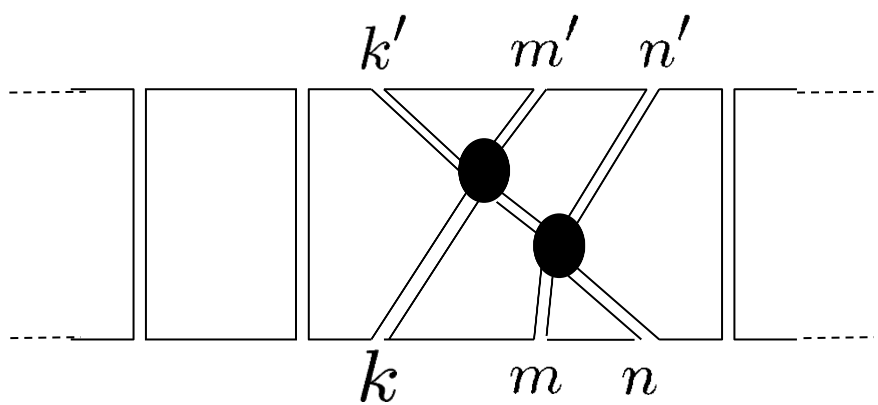
Gromov, V.K., Leurent, Volin '13, '14
V.K., Leurent, Volin '15

Weak coupling calculation from SYM

- Example: O(6) sector (scalar fields): $\mathcal{O}(x) = \mathrm{Tr}\,(\Phi_{n_1}\Phi_{n_2}\cdots\Phi_{n_L})$
- Tree level: $\Delta_0 = L$ - degeneracy (for scalars)
- 1-loop (examples of graphs):



- 2-loop:



nontrivial action on R-indices.

Examples: su(2) and sl(2) sectors at one loop

Notations: $Z=\Phi_1+i\Phi_2, \quad X=\Phi_3+i\Phi_4, \quad Y=\Phi_5+i\Phi_6$

su(2) operators: $\mathrm{Tr} Z^{L-J}X^J(x) + \text{permutations}$ $\quad -Z-Z-X-Z-Z-X-Z-$

- Dilatation operator - Heisenberg Hamiltonian, integrable by Bethe ansatz!

$$\hat{D} = L + \frac{\lambda}{16\pi^2}\sum_{l=1}^{L}\left(1-\sigma_l\cdot\sigma_{l+1}\right) + O(\lambda^2)$$

Minahan, Zarembo
Beisert,Kristjansen,Staudacher

Solution in terms of Baxter equation

$$T(u)Q(u) = \left(u+\frac{i}{2}\right)^L Q(u+i) + \left(u-\frac{i}{2}\right)^L Q(u-i)$$

where the function *T(u)* – transfer matrix eigenvalue (a polynomial)

It has with two solutions (Baxter functions): $Q_1(u)=\prod_{k=1}^{J}(u-u_k)\,, \qquad Q_2(u)=\prod_{k=1}^{L-J}(u-u_k)$

Equivalent, Wronskian equation: $\begin{vmatrix} Q_1(u+\frac{i}{2}) & Q_2(u+\frac{i}{2}) \\ Q_1(u-\frac{i}{2}) & Q_2(u-\frac{i}{2}) \end{vmatrix} = \text{Const}\ u^{L-1}$

Anomalous dimensions: $\Delta - L = \frac{\lambda}{8\pi^2}\,\partial_u \log\frac{Q(u+\frac{i}{2})}{Q(u-\frac{i}{2})}\Big|_{u=0} + \mathcal{O}(\lambda^2)$

with trace cyclicity condition $\quad Q\left(\frac{i}{2}\right) = Q\left(-\frac{i}{2}\right)$

sl(2) operators: $\mathrm{Tr} Z\nabla^S Z^{L-1}(x) + \text{permutations}$

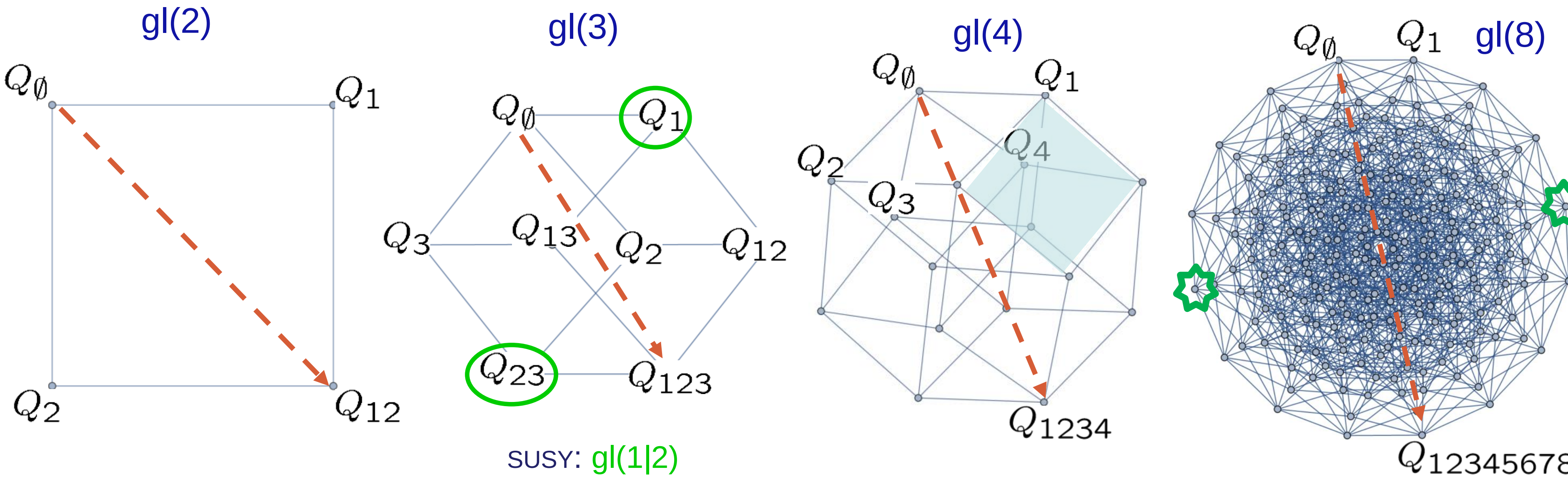
Baxter relation slightly change. Baxter functions are not necessarily polynomial

Quantum Spectral Curve of $\mathcal{N}$=4 SYM: algebraic structure

Gromov, V.K., Leurent, Volin ‘13,‘14
V.K., Leurent, Volin ‘15

- QSC eqs. close on finite set of Baxter functions of spectral parameter $Q_I(u)$
- gl(n): each Q placed at an edge of Hasse diagram - n-hypercube

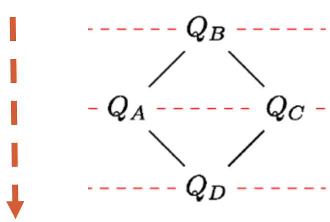
for $\mathcal{N}$=4 SYM



SUSY: gl(1|2)

SUSY: gl(8) → gl(4|4)

- Plücker QQ-relations on each face (“determinant flow”):



$$\Leftrightarrow \quad Q_B(u)Q_D(u) = \begin{vmatrix} Q_A(u+\frac{i}{2}) & Q_C(u+\frac{i}{2}) \\ Q_A(u-\frac{i}{2}) & Q_C(u-\frac{i}{2}) \end{vmatrix}$$

Grassmanian structure!

Krichever, Lupan, Wiegmann, Zabrodin, ’94
Tsuboi ‘13

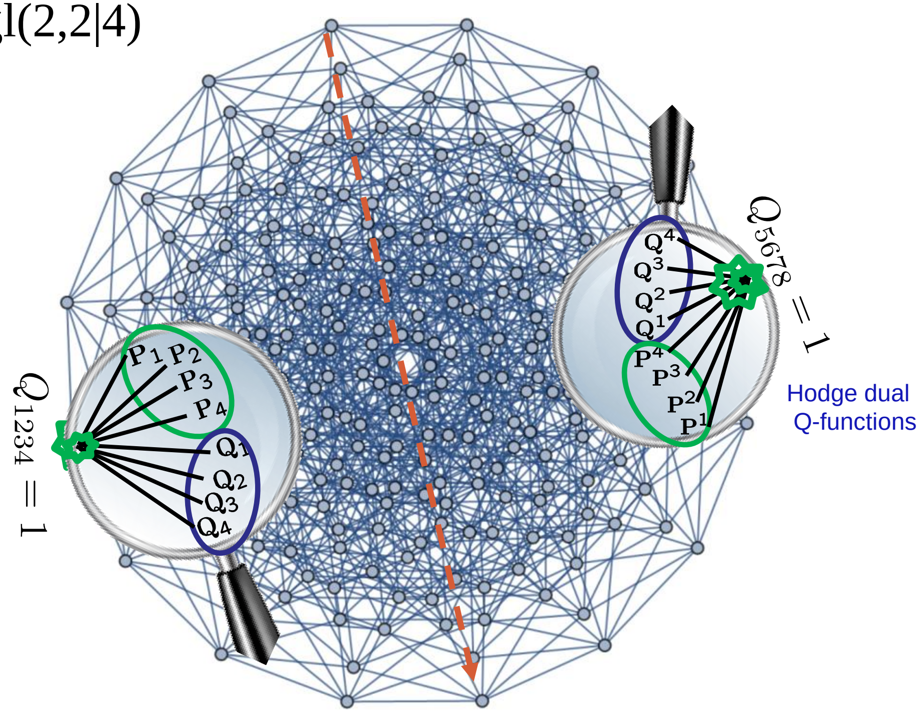
- gl(N) Heisenberg spin chain: $Q_0 = 1, \quad Q_j = \text{Polynomial}(u), \quad Q_{1,2,\ldots,N} \sim u^{\text{Length}}$
- gl(M|K) Heisenberg spin chain: $Q_{12\ldots M} = 1, \quad Q_{\text{neighbors}\, 12\ldots M} = \text{Polyn}(u) \quad Q_{M+1,M+2,\ldots K} \sim u^{\text{Length}}$

Gromov, V.K., Leurent, Volin '13,'14
V.K., Leurent, Volin '15

Quantum Spectral Curve of AdS_5/CFT_5: analytic structure

- Special 8 + 8 Q-functions with nice analyticity on physical sheet

gl(2,2|4)

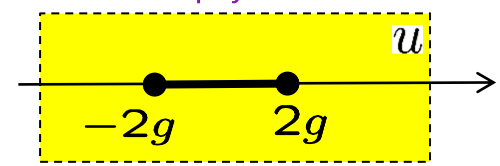


Large u asymptotics fixed by PSU(2,2|4)

Cartan charges $\{\Delta, S_1, S_2 | J_1, J_2, J_3\}$
SU(2,2) SU(4)

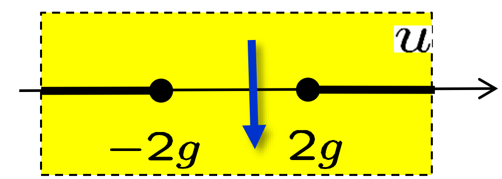
$$\mathbf{P}_b,\ \mathbf{P}^b \sim q_b^{\pm iu}\, u^{(\pm J_1 \pm J_2 \pm J_3)/2}$$

short cut on physical sheet



$$\mathbf{Q}_j,\ \mathbf{Q}^j \sim u^{(\pm\Delta \pm S_1 \pm S_2)/2}$$

long cut on physical sheet



- Various Q-functions are related by complex conjugation ("gluing" relations)

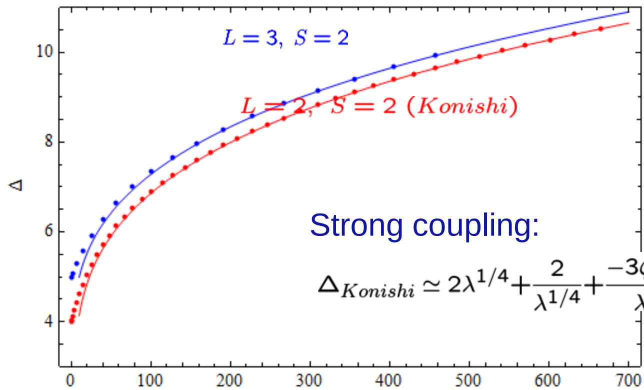
$$\mathbf{Q}_1 \propto \bar{\mathbf{Q}}^2\,, \quad \mathbf{Q}_2 \propto \bar{\mathbf{Q}}^1\,, \quad \mathbf{Q}_3 \propto \bar{\mathbf{Q}}^4\,, \quad \mathbf{Q}_4 \propto \bar{\mathbf{Q}}^3$$

- These Riemann-Hilbert conditions fix all physical solutions for Q-system and thus conformal dimensions $\Delta(g)$ with given PSU(2,2|4) charges

Dimensions of twist-2,3,… operators $\mathrm{Tr}\,(\Phi\,\nabla^S\,\Phi)$

- Numerics, weak and strong coupling from Quantum Spectral Curve;

Gromov,V.K.,Vieira '09
Frolov '10
Gromov,Valatka '12



$\lambda = 16\pi^2 g^2$

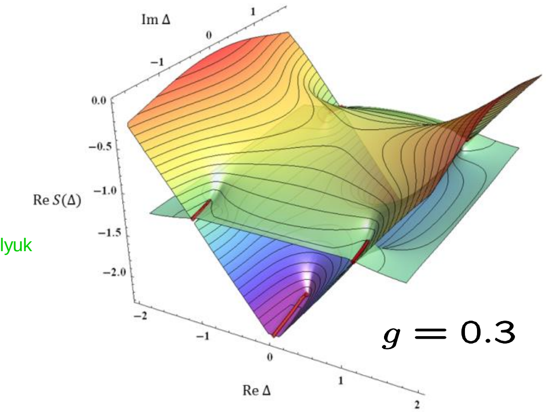
Strong coupling:

$$\Delta_{Konishi} \simeq 2\lambda^{1/4} + \frac{2}{\lambda^{1/4}} + \frac{-3\zeta_3 + \frac{1}{2}}{\lambda^{3/4}} + \frac{\frac{15}{2}\zeta_5 + 6\zeta_3 - \frac{1}{2}}{\lambda^{5/4}} + \ldots$$

Gromov, Valatka, Sizov, Levkovich-Maslyuk
Gromov,Shenderovich,
Serban, Volin
Roiban, Tseytlin
Vallilo, Mazzucato
Gubser, Klebanov, Polyakov

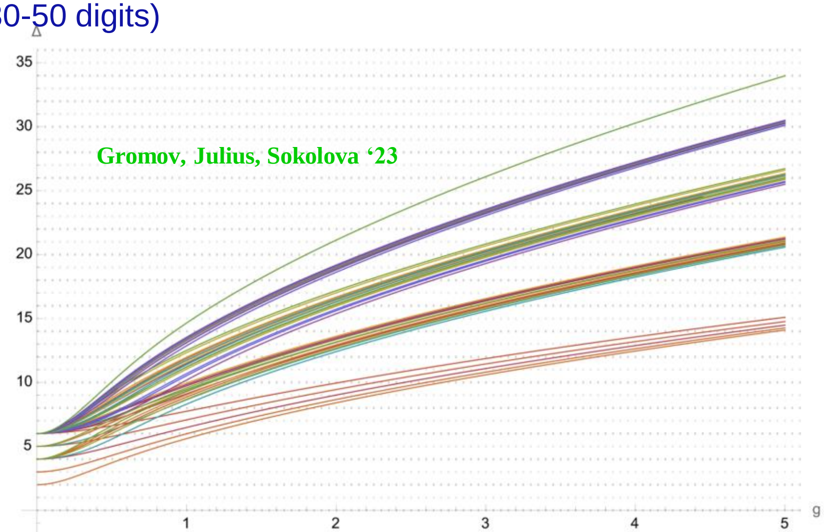
Function of complex conformal spin $\Delta(S, g)$

Gromov, Levkovich-Maslyuk, Sizov '15



$g = 0.3$

Recent results "unlimited" precision
(~30-50 digits)



Gromov, Julius, Sokolova '23

Weak couping (11 loops)

$$\gamma_{Konishi} = \sum_{j=1}^{\infty} g^{2j}\,\gamma_j$$

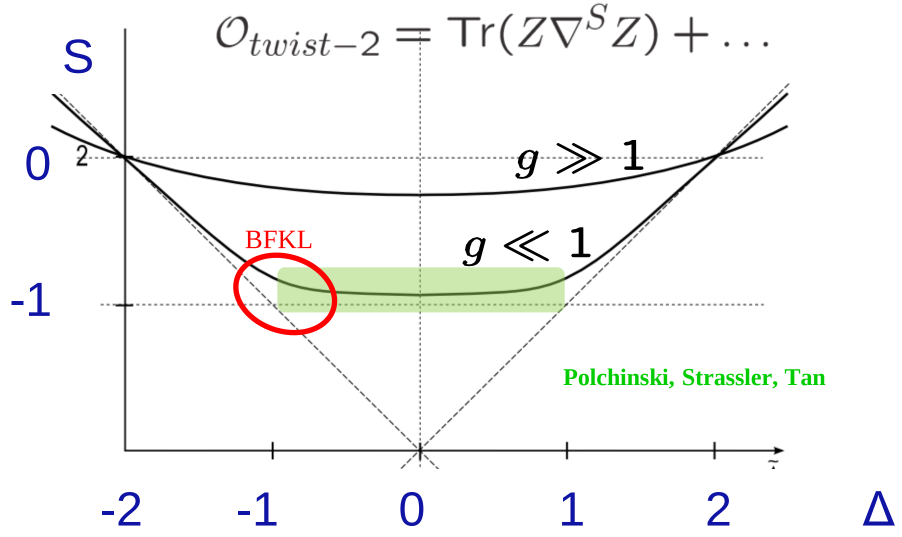
Gromov,VK,Leurent, Volin '13
Leurent, Serban, Volin '12
Volin, Marboe '18

$$\begin{aligned}
\gamma_{11} \;=\; & -242508705792 + 107663966208\zeta_3 + 70251466752\zeta_3^2 - 12468142080\zeta_3^3 \\
& +1463132160\zeta_3^4 - 71663616\zeta_3^5 + 180173002752\zeta_5 - 16655486976\zeta_3\zeta_5 \\
& -24628230144\zeta_3^2\zeta_5 - 2895575040\zeta_3^3\zeta_5 + 19278176256\zeta_5^2 - 9619845120\zeta_3\zeta_5^2 \\
& +2504494080\zeta_3^2\zeta_5^2 + \tfrac{882108048384}{175}\zeta_5^3 + 45602231040\zeta_7 + 14993482752\zeta_3\zeta_7 \\
& -12034759680\zeta_3^2\zeta_7 + 1406730240\zeta_3^3\zeta_7 + 30605033088\zeta_5\zeta_7 + 21217637376\zeta_3\zeta_5\zeta_7 \\
& -\tfrac{1309941061632}{275}\zeta_5^2\zeta_7 - 13215327552\zeta_7^2 - 4059901440\zeta_3\zeta_7^2 - 69762034944\zeta_9 \\
& +23284599552\zeta_3\zeta_9 - 3631889664\zeta_3^2\zeta_9 - 11032374528\zeta_5\zeta_9 - 6666706944\zeta_3\zeta_5\zeta_9 \\
& -23148129024\zeta_7\zeta_9 - 10024051968\zeta_9^2 - 54555179184\zeta_{11} + \tfrac{100485411184}{5}\zeta_3\zeta_{11} \\
& -726029568\zeta_3^2\zeta_{11} - 8975463552\zeta_5\zeta_{11} - 22529041920\zeta_7\zeta_{11} - \tfrac{1437993422496}{175}\zeta_{13} \\
& +\tfrac{1504385419392}{35}\zeta_3\zeta_{13} - 30324602880\zeta_5\zeta_{13} - \tfrac{151130039581392}{875}\zeta_{15} - 41375093760\zeta_3\zeta_{15} \\
& -\tfrac{196484147423712}{275}\zeta_{17} + 309361358592\zeta_{19} - 1729880064Z_{11}^{(2)} - \tfrac{1620393984}{5}\zeta_3 Z_{11}^{(2)} \\
& -131383296\zeta_5 Z_{11}^{(2)} + \tfrac{138107420928}{175}Z_{13}^{(2)} + \tfrac{3543865344}{35}\zeta_3 Z_{13}^{(2)} - \tfrac{5716780416}{7}Z_{13}^{(3)} \\
& -\tfrac{674832384}{7}\zeta_3 Z_{13}^{(3)} + \tfrac{48227088384}{175}Z_{15}^{(2)} + \tfrac{3581880576}{25}Z_{15}^{(3)} + 754974720Z_{15}^{(4)} \\
& -\tfrac{854924544}{11}Z_{17}^{(2)} + \tfrac{4963244544}{55}Z_{17}^{(3)} + \tfrac{818159616}{275}Z_{17}^{(4)} + \tfrac{175363688448}{1925}Z_{17}^{(5)}\,. \qquad \text{(A.}
\end{aligned}$$

- QSC for ABJM, AdS3/CFT2

Cavaglia, Fioravanti,Gromov,Tateo '14
Cavaglia, Gromov, Stefanski,Torielli '21

BFKL Dimension from Quantum Spectral Curve



- QSC allows for analytic continuation of exact dimension $\Delta(S,g)$ to continuous spins $-1 < S < \infty$
  We need to find the appropriate analytic continuation of Q-functions.

  Janik Gromov, V.K.
  Gromov, Levkovich-Maslyuk, Sizov, Valatka

- BFKL is a double scaling limit:

$$w = S+1 \to 0, \quad g \to 0, \quad \Lambda = \frac{g^2}{S+1} \quad - \text{fixed}$$

- We will restore from QSC the leading order (LO) BFKL approximation for $\Delta(S,g)$
  (already known up to NLO from direct summation of Feynman graphs)

Balitsky, Fadin, Kuraev, Lipatov

$$\frac{S+1}{4g^2} = \Psi(\Delta) + g^2\delta(\Delta) + \mathcal{O}(g^4) \qquad \text{where} \qquad \Psi(\Delta) = -\psi\left(\frac{1+\Delta}{2}\right) - \psi\left(\frac{1-\Delta}{2}\right) + 2\psi(1)$$

$$\delta(\Delta) = 4\Psi''(\Delta) + 6\zeta_3 + 2\zeta_2\Psi(\Delta) - \frac{\pi^3}{\cos\frac{\pi\Delta}{2}} - 4\Phi(\tfrac{1}{2}-\tfrac{\Delta}{2}) - 4\Phi(\tfrac{1}{2}+\tfrac{\Delta}{2}), \qquad \Phi(x) = \sum_{k=0}^{\infty}\frac{(-)^k}{(x+k)^2}[\psi(k+1+x)-\psi(1)]$$

- In particular, near the Regge pole

$$\Delta - 1 \simeq \frac{-8g^2}{w} + w\zeta_3\left(\frac{-4g^2}{w}\right)^3 + \mathcal{O}\left(\left(\frac{g^2}{w}\right)^4\right)$$

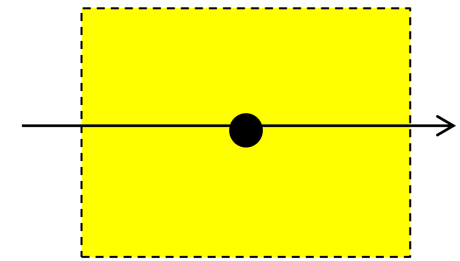
- BFKL is an excellent test for the whole AdS/CFT integrability: it sums up “wrapped” graphs omitted in asymptotic Bethe ansatz

Kotikov, Lipatov, Rej, Staudacher
Bajnok, Janik, Lukowsky
Lukowski, Rej, Velizhanin, Orlova

Alfimov Gromov V.K. 2014

P-functions at LO BFKL

- We can split **P** (**or** μ) into regular and singular parts



$$\mathbf{P} = \frac{\tilde{\mathbf{P}} + \mathbf{P}}{2} + \sqrt{u^2 - 4g^2}\left(\frac{\tilde{\mathbf{P}} - \mathbf{P}}{2\sqrt{u^2 - 4g^2}}\right)$$

- In the regime $g \ll |u| \ll 1$ singular part gives poles at $u = 0$

$$\sqrt{u^2 - 4g^2} \equiv \sqrt{u^2 - 4\Lambda w} = u - \frac{2\Lambda}{u} w - \frac{2\Lambda^2}{u^3} w^2 + O(w^3)$$

reminder:
$w = S + 1$
$\Lambda = \frac{g^2}{S+1}$

- We can uniformize P by Zhukovsky map $u = \sqrt{\Lambda w}(x + 1/x)$

$$\mathbf{P}_a = \sum_{n=-1}^{\infty} \frac{c_{a,n}}{[x(u)]^n} \qquad c_{a,n}(\Lambda, w) = (\sqrt{\Lambda w})^{n-4} \sum_{k=0}^{+\infty} c_{a,n}^{(k)} w^k$$

- Pμ-equations $\tilde{\mathbf{P}}_a = \mu_{ab}\mathbf{P}^b$ and $\mu_{ab}(u+i) - \mu_{ab}(u) = \mathbf{P}_a\tilde{\mathbf{P}}_b - \mathbf{P}_b\tilde{\mathbf{P}}_a$

ansatz for μ : $\mu_{ab} = \frac{1}{w^2}\mathrm{Polyn}_{ab}(u)\cosh^2(\pi u)$

- Asymptotics $\mathbf{P}_1 = \frac{1}{u},,$ $\quad \mathbf{P}_2 = \frac{1}{u^2},$ $\quad \mathbf{P}_3 = A_3^{(0)} u + \frac{c_{3,1}^{(1)}}{\Lambda u},$ $\quad \mathbf{P}_4 = A_4^{(0)}$

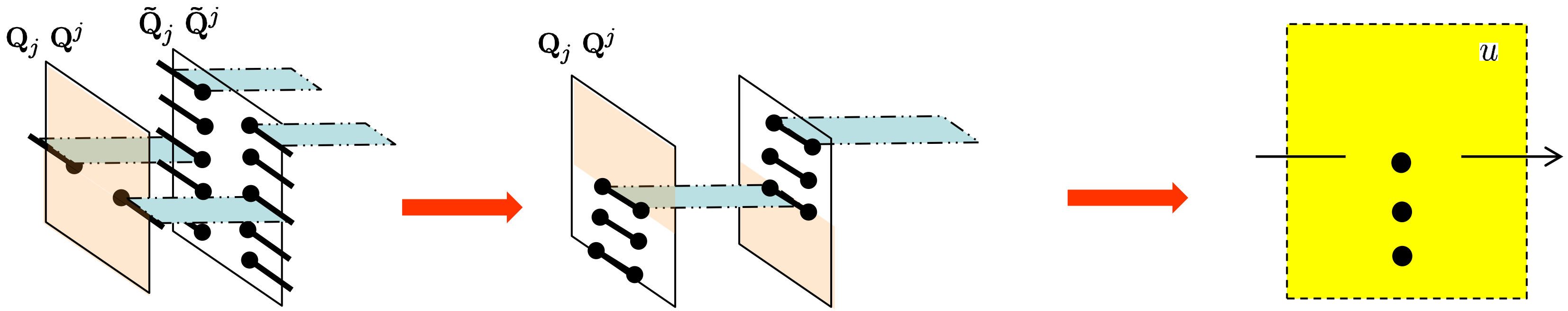
where $A_3^{(0)} = -\frac{i(\Delta^2 - 1)(\Delta^2 - 9)}{32},$ $\quad A_4^{(0)} = -\frac{i(\Delta^2 - 1)(\Delta^2 - 25)}{96},$ $\quad c_{3,1}^{(1)} = -\frac{i(\Delta^2 - 1)^2}{96}$

Analytic properties of Q-functions

- Natural objects for BFKL limit are $\mathbf{Q}$-functions: asymptotics contain conformal charges, including Δ

$$\begin{pmatrix}\mathbf{Q}_1\\ \mathbf{Q}_2\\ \mathbf{Q}_3\\ \mathbf{Q}_4\end{pmatrix} \sim \begin{pmatrix} u^{\frac{\Delta+1-w}{2}} \\ u^{\frac{\Delta-3+w}{2}} \\ u^{\frac{-\Delta+1-w}{2}} \\ u^{\frac{-\Delta-3+w}{2}} \end{pmatrix}$$

- Re-gluing sheets: from long to short cuts



- In weak coupling limit "ladder" of cuts generates poles at $u = i\mathbb{Z}_-$

- From purely algebraic relations of Q-system we get a 4-th order finite difference equation with 4 solutions giving all 4 $\mathbf{Q}$-functions:

$$0 = \mathbf{Q}^{[+4]} d_0 - \mathbf{Q}^{[+2]}\left[d_1 - \mathbf{P}_a^{[+2]}\mathbf{P}^{a[+4]} d_0\right] + \frac{1}{2}\mathbf{Q}\left[d_3 + \mathbf{P}_a\mathbf{P}^{a[+4]} d_0 + \mathbf{P}_a\mathbf{P}^{a[+2]} d_1\right] + \text{c.c.}$$

The coefficients depend only on $\mathbf{P}$-functions: $d_m = \det_{1\le a,k\le 4} (\mathbf{P}^a)^{[4-2k+2\delta_{k,m}]}$

$f^{[n]} := f\left(u + n\frac{i}{2}\right)$

- Plugging here the LO $\mathbf{P}$-functions we get an equation factorized as follows

$$\left[D + D^{-1} - 2 - \frac{1-\Delta^2}{4u^2}\right]\mathbf{Q} = 0$$

$D = e^{i\partial_u}$

- 2-nd order equation is the Faddeev-Korchemsky-Baxter eq. for BFKL pomeron !

Finding the BFKL Dimension

- On the other hand, from the explicit knowledge of NLO we find the 4'th order NLO equation for $\mathbf{P}$ which factorizes again, to give for j=1,3

reminder

$$w = S+1$$

$$\Lambda = \frac{g^2}{S+1}$$

$$\mathbf{Q}_j\left(\frac{\Delta^2-1-8u^2}{4u^2}+w\frac{\left(\Delta^2-1\right)\Lambda-u^2}{2u^4}\right)+\mathbf{Q}_j^{--}\left(1-\frac{iw/2}{u-i}\right)+\mathbf{Q}_j^{++}\left(1+\frac{iw/2}{u+i}\right)=0$$

Explicit $\mathbf{Q}_1,\ \mathbf{Q}_3$ – a linear combination of solutions with $\pm\sqrt{w}$ with correct asymptotics:

$$\mathbf{Q}=\frac{\sqrt{w}(u^2-2\Lambda w)}{iu-\frac{w}{4}-i\sqrt{2\Lambda w}}\frac{\Gamma\left(iu-\frac{w}{4}+i\sqrt{2\Lambda w}\right)}{\Gamma\left(-iu-\frac{w}{4}-i\sqrt{2\Lambda w}\right)}\ {}_3F_2\left(\frac{1-\Delta}{2},\frac{1+\Delta}{2},-iu-\frac{w-i\sqrt{32\Lambda w}}{4};-\frac{w}{2},2i\sqrt{2\Lambda w}+1;1\right)$$

- Values at the pole $$\frac{\mathbf{Q}_j^{(1)}(u)}{\mathbf{Q}_j^{(0)}(u)}=+\frac{iw}{2u}+\mathcal{O}(u^0)\ ,\quad j=1,3$$

- To extract the dimension we have to compute these Q-functions on the 2nd sheet via the monodromy $\tilde{\mathbf{Q}}_j=\omega_{jk}\mathbf{Q}_k$ from the LO $\mathbf{Q}$. This gives

$$\mathbf{Q}_3(u)\simeq 2iw\Lambda\mathbf{Q}_3(0)\frac{\Psi(\Delta)}{u}+\mathrm{regular}(u)+\mathcal{O}(w^2)$$

where $\Psi(\Delta)=-\psi\left(\frac{1+\Delta}{2}\right)-\psi\left(\frac{1-\Delta}{2}\right)+2\psi(1)$

- This allows to fix the dimension and restore the BFKL formula

$$\frac{S+1}{4g^2}=-\psi\left(\frac{1}{2}-\frac{\Delta}{2}\right)-\psi\left(\frac{1}{2}+\frac{\Delta}{2}\right)+2\psi(1)+\mathcal{O}(g^2)$$

Conclusions and Future Directions

- Quantum Spectral Curve (QSC) is now a mature method for the study of anomalous dimensions in N =4 SYM in various perturbative and non-perturbative regimes

  Gromov, V.K., Leurent, Volin '13, '14

- Regge approximation for dimensions is under the full control in planar N =4 Super-Yang-Mills LO, NLO, NNLO are analytically computed (see talk of Nikolay)

- Structure constants (3-point correlators) of N =4 SYM are still an open problem, in intensive study. BFKL 3-point correlator is computed in LO by traditional PT methods

  Balitsky,V.K., Sobko '13, '15

- An interesting Fishnet limit of gamma-deformed N =4 SYM described via twisted QSC. Certain two dimensional generalization of Fishnet CFT is also described by Lipatov (BFKL) Hamiltonian.

  Gurdogan, V.K., '15
  Gromov, V.K., Korchemsky, Negro, Sizov '16
  Alfimov, Ferrando, V.K., Olivucci '23

- Comparing the BFKL limit in N =4 SYM and QCD: From SYM NNLO to QCD NNLO ?