Double Logs in Regge Limit

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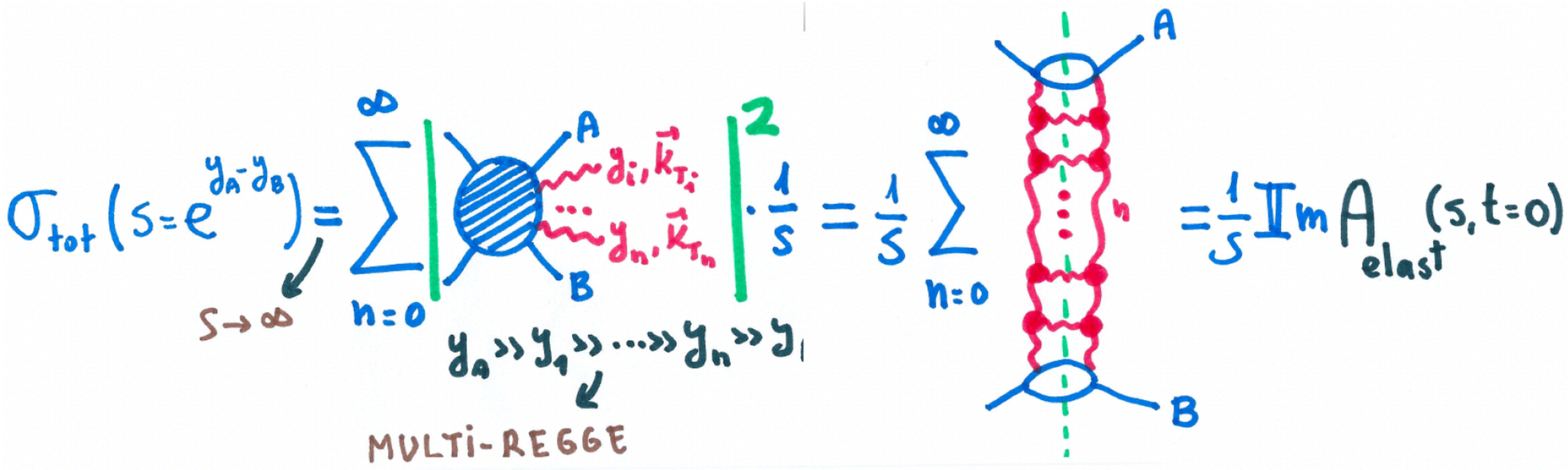
Workshop on overlap between QCD resummations

14-17 Jan 2024
Aussois, Centre Paul Langevin

Intro & a DIS calculation
Analytic vs Monte Carlo: Integrability
Matrix Hamiltonian

Intro & a DIS calculation

In BFKL formalism we can describe multi particle production and diffractive events



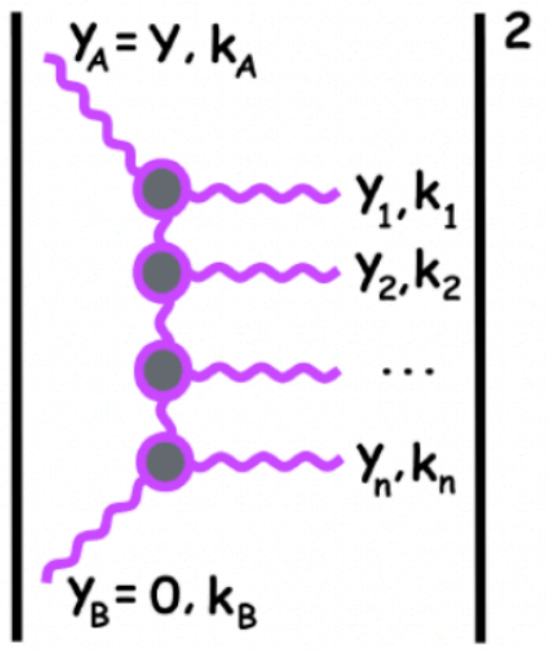
At LL and NLL we have a factorisation into impact factors and Green's function

$$\sigma(Q_1, Q_2, Y) = \int d^2\vec{k}_A d^2\vec{k}_B \underbrace{\phi_A(Q_1, \vec{k}_A)\,\phi_B(Q_2, \vec{k}_B)}_{\text{PROCESS-DEPENDENT}} \underbrace{f(\vec{k}_A, \vec{k}_B, Y)}_{\text{UNIVERSAL}}$$

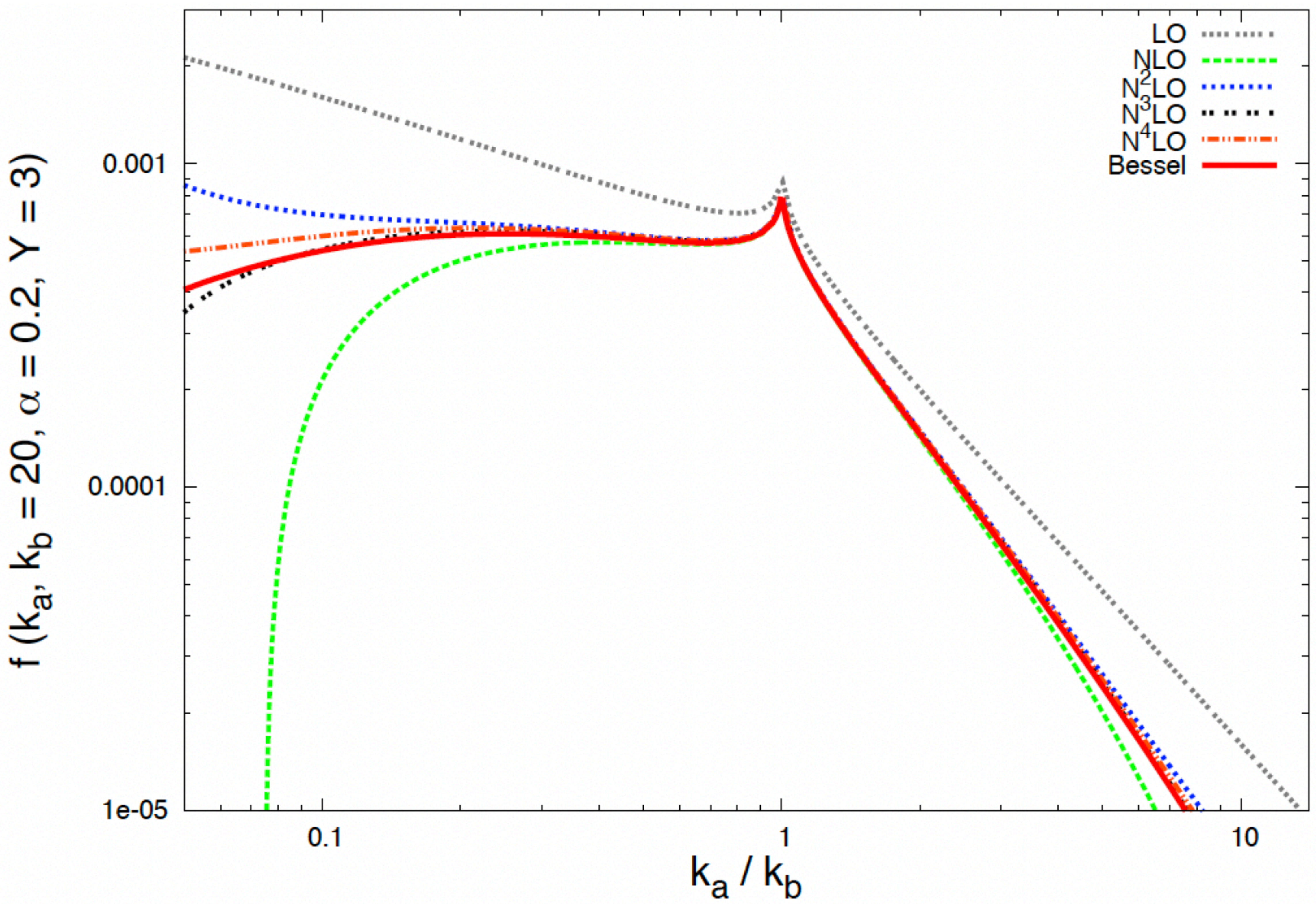
At NLO the collinear contributions are very large when moving away from Multi-Regge kinematics

Green's function written in terms of no-emission factors due to gluon reggeization

$$f\left(\vec{k}_A, \vec{k}_B, Y\right) = \sum_n \left| \quad \right|^2 =$$



$$e^{\omega\left(\vec{k}_A\right)Y}\left\{\delta^{(2)}\left(\vec{k}_A - \vec{k}_B\right) + \sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\alpha_s N_c}{\pi}\int d^2\vec{k}_i \frac{\theta\left(k_i^2 - \lambda^2\right)}{\pi k_i^2} \times \int_0^{y_{i-1}} dy_i e^{\left(\omega\left(\vec{k}_A + \sum_{l=1}^{i}\vec{k}_l\right) - \omega\left(\vec{k}_A + \sum_{l=1}^{i-1}\vec{k}_l\right)\right)y_i}\delta^{(2)}\left(\vec{k}_A + \sum_{l=1}^{n}\vec{k}_l - \vec{k}_B\right)\right\}$$



Need to resum them to all orders
[Ciafaloni, Colferai, Salam, Stasto ...]

Their leading structure is simple in pT space [SV]

$$\sum_{n=1}^{\infty}\frac{\left(-\bar{\alpha}_s\right)^n}{2^n n!(n+1)!}\ln^{2n}\frac{\vec{k}_A^2}{\left(\vec{k}_A + \vec{k}_i\right)^2}$$

We can attempt to use it for DIS structure functions at small x

$$\frac{d^2\sigma}{dxdQ^2} = \frac{2\pi\alpha^2}{xQ^4}\{[1+(1-y)^2]F_2(x,Q^2) - y^2 F_L(x,Q^2)\}$$

$$F_I(x,Q^2) = \frac{1}{(2\pi)^4}\int\frac{d^2\mathbf{q}_\perp}{q^2}\int\frac{d^2\mathbf{p}_\perp}{p^2}\Phi_I(q,Q^2) \times \Phi_P(p,Q_0^2)\mathcal{F}(s,q,p),$$

Original BFKL calculated for MRK, for rapidities

$$\mathcal{F}(s,q,p) = \frac{1}{2\pi qp}\int\frac{d\omega}{2\pi i}\int\frac{d\gamma}{2\pi i}\left(\frac{q^2}{p^2}\right)^{\gamma-(1/2)} \times \left(\frac{s}{qp}\right)^{\omega}\frac{1}{\omega - \bar{\alpha}_s\chi_0(\gamma)},$$

In DIS the evolution variable is Bjorken x

$$\mathcal{F}(s,q,p) = \frac{1}{2\pi q^2}\int\frac{d\omega}{2\pi i}\int\frac{d\gamma}{2\pi i}\left(\frac{q^2}{p^2}\right)^{\gamma}\left(\frac{s}{q^2}\right)^{\omega} \times \frac{1}{\omega - \bar{\alpha}_s\chi_0(\gamma - \frac{\omega}{2})}.$$

This generates unphysical terms at higher orders which would be cancelled if we evaluate the kernel order by order

We can obtain the leading collinear ones by shifting in omega from the beginning [Salam …]

$$\omega = \bar{\alpha}_s\left(2\psi(1) - \psi\left(\gamma + \frac{\omega}{2}\right) - \psi\left(1 - \gamma + \frac{\omega}{2}\right)\right)$$

The leading structure can be found [SV]

$$\omega \simeq \int_0^1 \frac{dx}{1-x}\left\{\left(x^{\gamma-1} + x^{-\gamma}\right)\sqrt{\frac{2\bar{\alpha}_s}{\ln^2 x}}J_1\left(\sqrt{2\bar{\alpha}_s\ln^2 x}\right) - 2\bar{\alpha}_s\right\}$$

A complete calculation requires many more details [Hentschinki, Salas, SV]

The kernel should have all the NLO terms

$$\chi(\gamma) = \bar{\alpha}_s \chi_0(\gamma) + \bar{\alpha}_s^2 \chi_1(\gamma) - \frac{1}{2}\bar{\alpha}_s^2 \chi_0'(\gamma)\chi_0(\gamma) + \chi_{\mathrm{RG}}(\bar{\alpha}_s, \gamma, a, b)$$

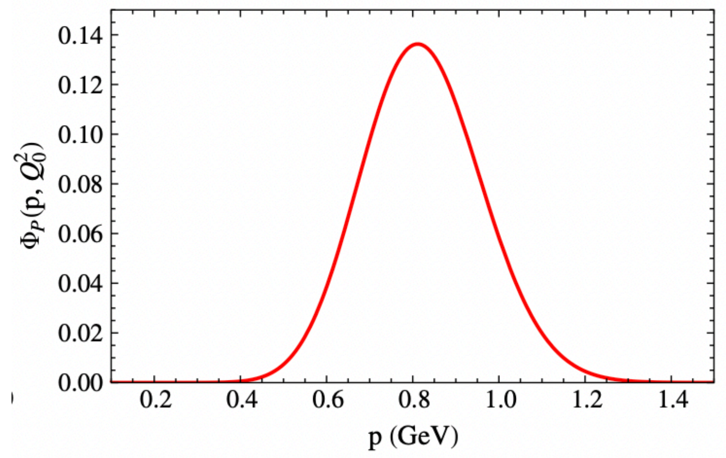
The collinear resummation should not double count and also include subleading collinear poles

$$\chi_{\mathrm{RG}}(\bar{\alpha}_s, \gamma, a, b) = \bar{\alpha}_s(1 + a\bar{\alpha}_s)[\psi(\gamma) - \psi(\gamma - b\bar{\alpha}_s)] - \frac{\bar{\alpha}_s^2}{2}\psi''(1-\gamma) - b\bar{\alpha}_s^2 \frac{\pi^2}{\sin^2(\pi\gamma)} + \frac{1}{2}\sum_{m=0}^{\infty}\left(\gamma - 1 - m + b\bar{\alpha}_s - \frac{2\bar{\alpha}_s(1+a\bar{\alpha}_s)}{1-\gamma+m} + \sqrt{(\gamma - 1 - m + b\bar{\alpha}_s)^2 + 4\bar{\alpha}_s(1+a\bar{\alpha}_s)}\right)$$

$$a = \frac{5}{12}\frac{\beta_0}{N_c} - \frac{13}{36}\frac{n_f}{N_c^3} - \frac{55}{36}, \qquad b = -\frac{1}{8}\frac{\beta_0}{N_c} - \frac{n_f}{6N_c^3} - \frac{11}{12}$$

LO virtual photon impact factor & with exact kinematics [Kwiecinski, Martin, Stasto]

A simple gaussian model for the coupling to the proton



More uncertainty when dealing with the running of the coupling. Interesting when going to low Q.

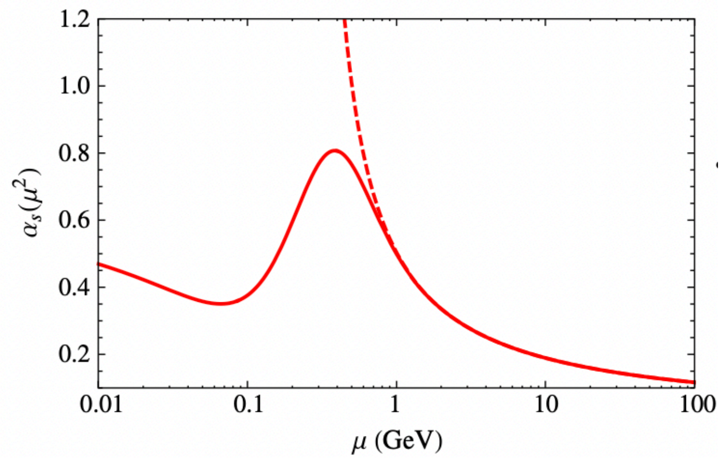
$$F_I(x, Q^2) = \frac{1}{(2\pi)^4}\int \frac{d^2\mathbf{q}_\perp}{q^2}\int \frac{d^2\mathbf{p}_\perp}{p^2}\Phi_I(q, Q^2) \times \Phi_P(p, Q_0^2)\mathcal{F}(s, q, p),$$

$$F_I(x, Q^2) \propto \int d\nu\, x^{-\chi\left(\frac{1}{2}+i\nu\right)}\Gamma\left(\delta - \frac{1}{2} - i\nu\right)\left[1 + \frac{\bar{\alpha}_s^2\beta_0\chi_0\left(\frac{1}{2}+i\nu\right)}{8N_c}\log\left(\frac{1}{x}\right) \times \left(i(\pi\coth(\pi\nu) - 2\pi\tanh(\pi\nu) - M_I(\nu)) - \psi\left(\delta - \frac{1}{2} - i\nu\right)\right)\right]\left(\frac{Q^2}{Q_0^2}\right)^{\frac{1}{2}+i\nu} c_I(\nu) \quad (11)$$

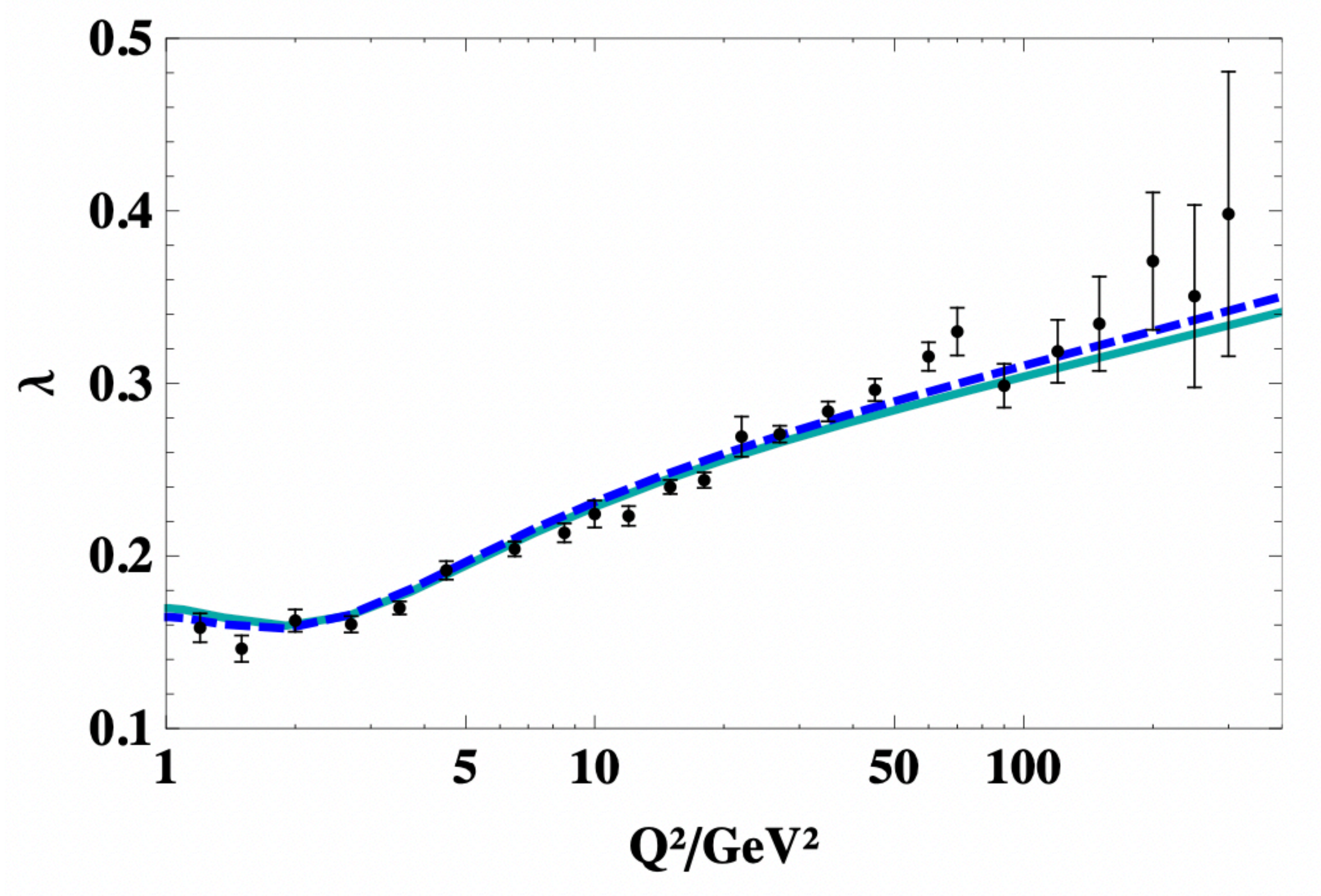
BLM optimal scale setting, MOM scheme. [Brodsky, Fadin, Kim, Lipatov, Pivovarov]

$$\tilde{\alpha}_s(QQ_0, \gamma) = \frac{4N_c}{\beta_0\left[\log\left(\frac{QQ_0}{\Lambda^2}\right) + \frac{1}{2}\chi_0(\gamma) - \frac{5}{3} + 2\left(1 + \frac{2}{3}Y\right)\right]}$$

Analytic running coupling from global jet observables [Webber]

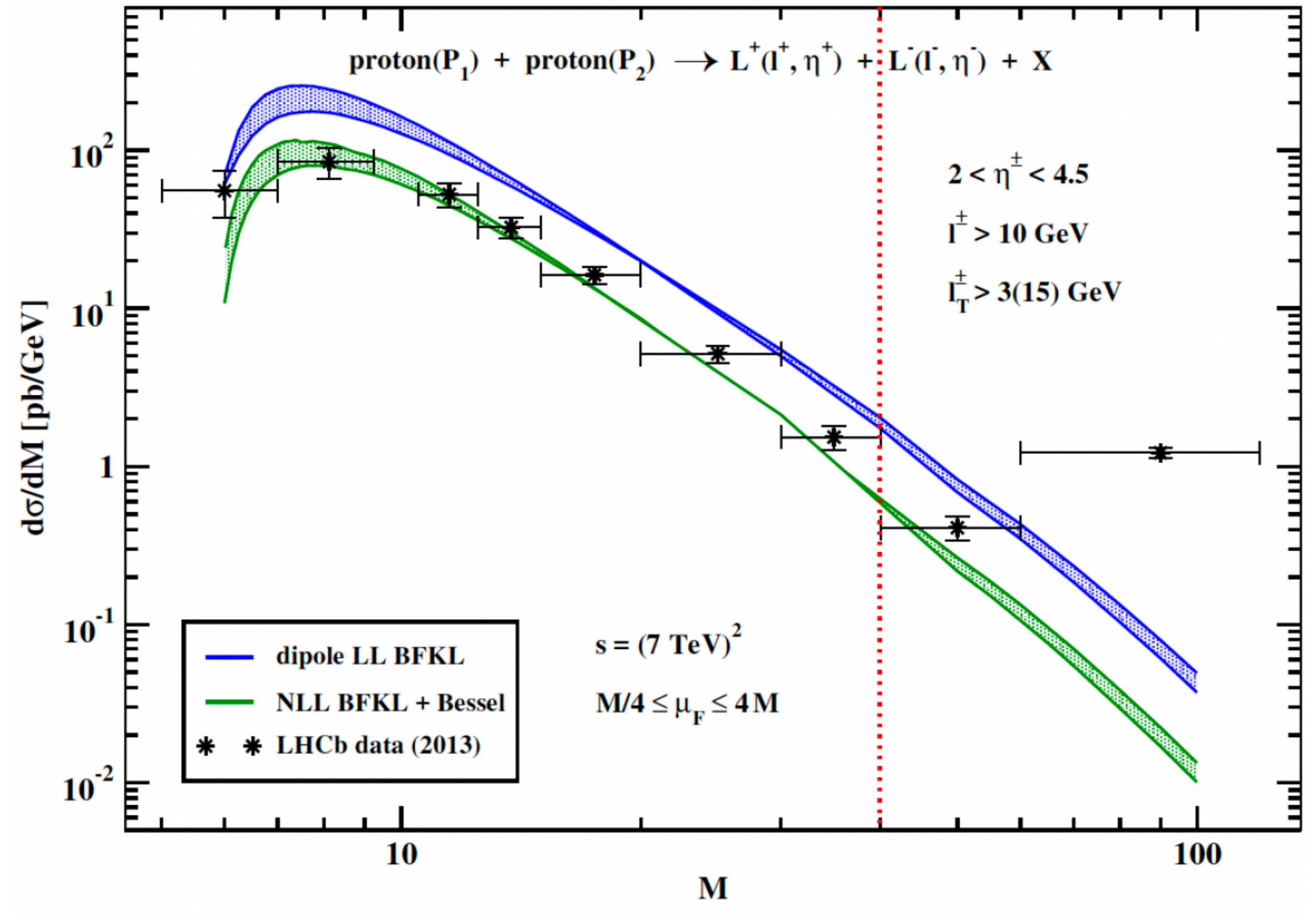


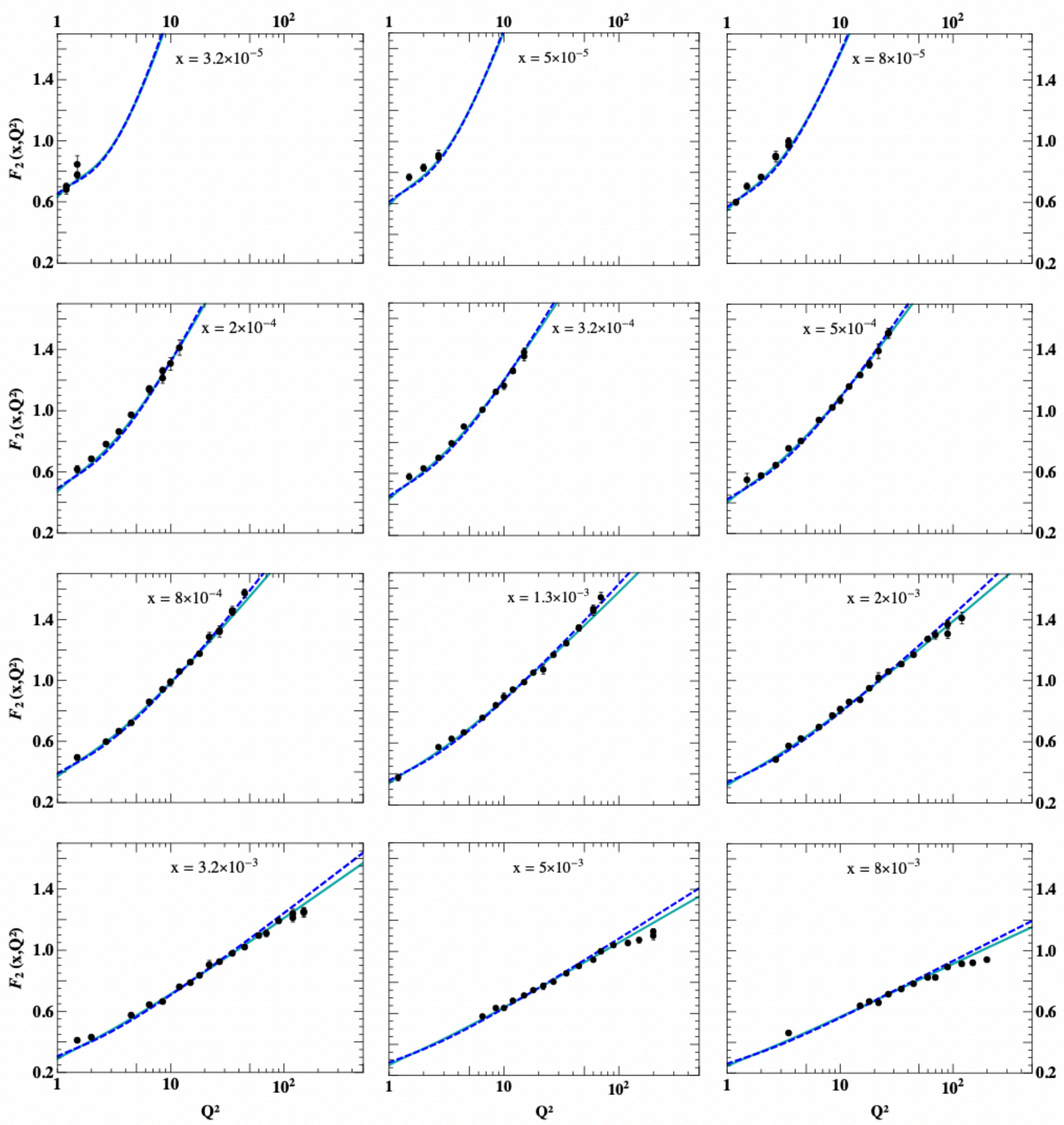
DIS at HERA $F_2(x, Q^2) \simeq x^{-\lambda(Q^2)}$

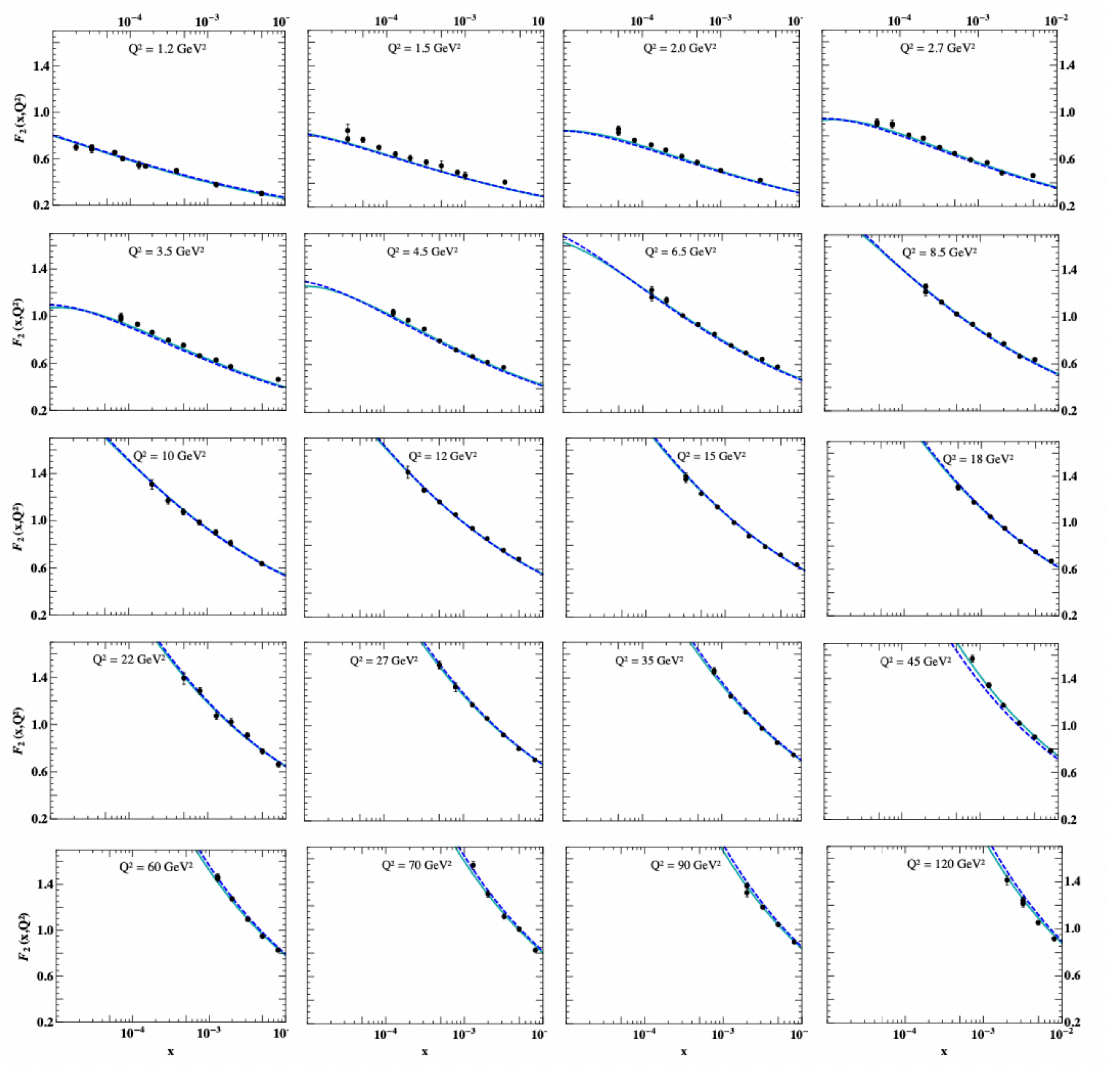


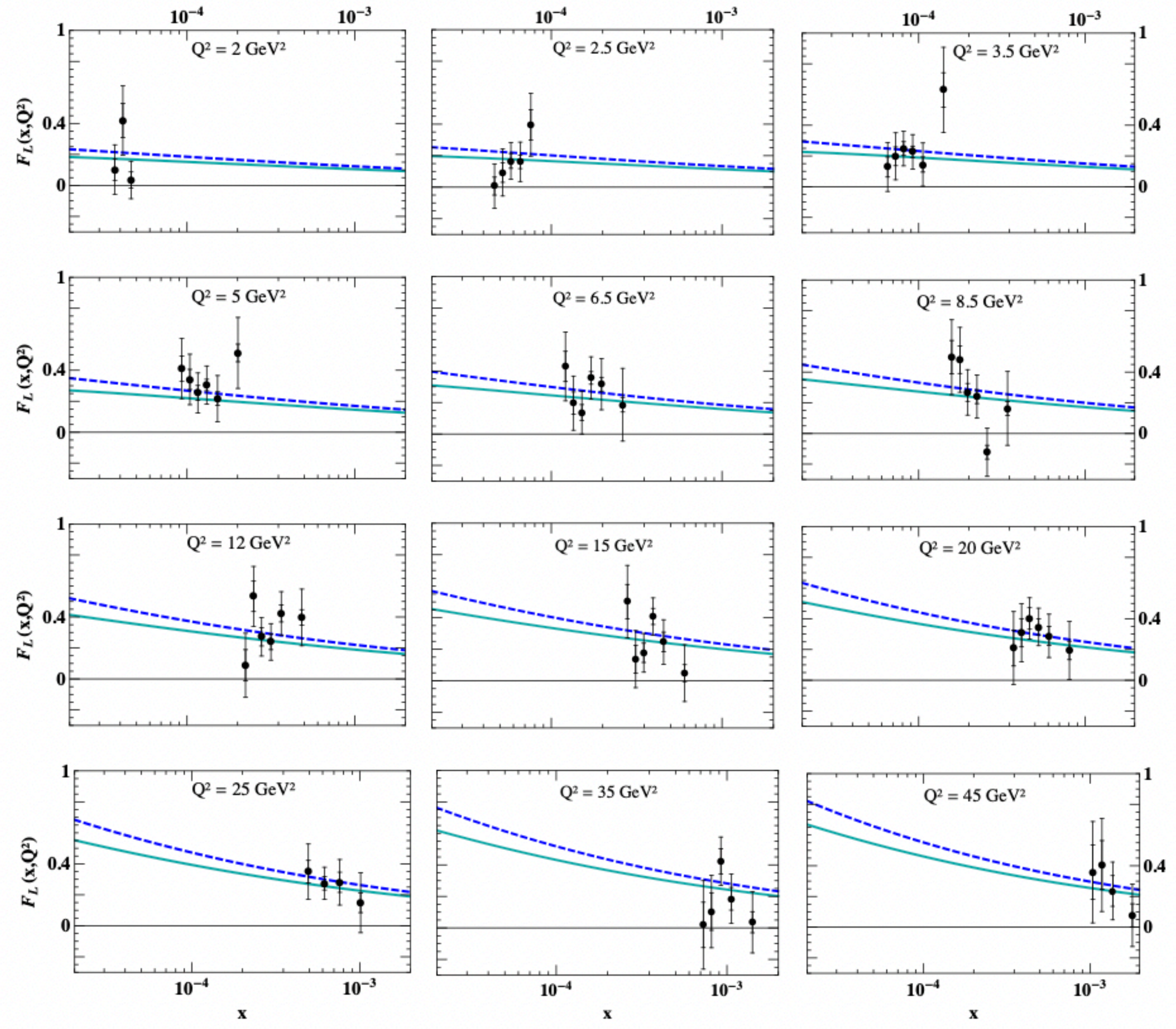
Interesting transition from large to small Q

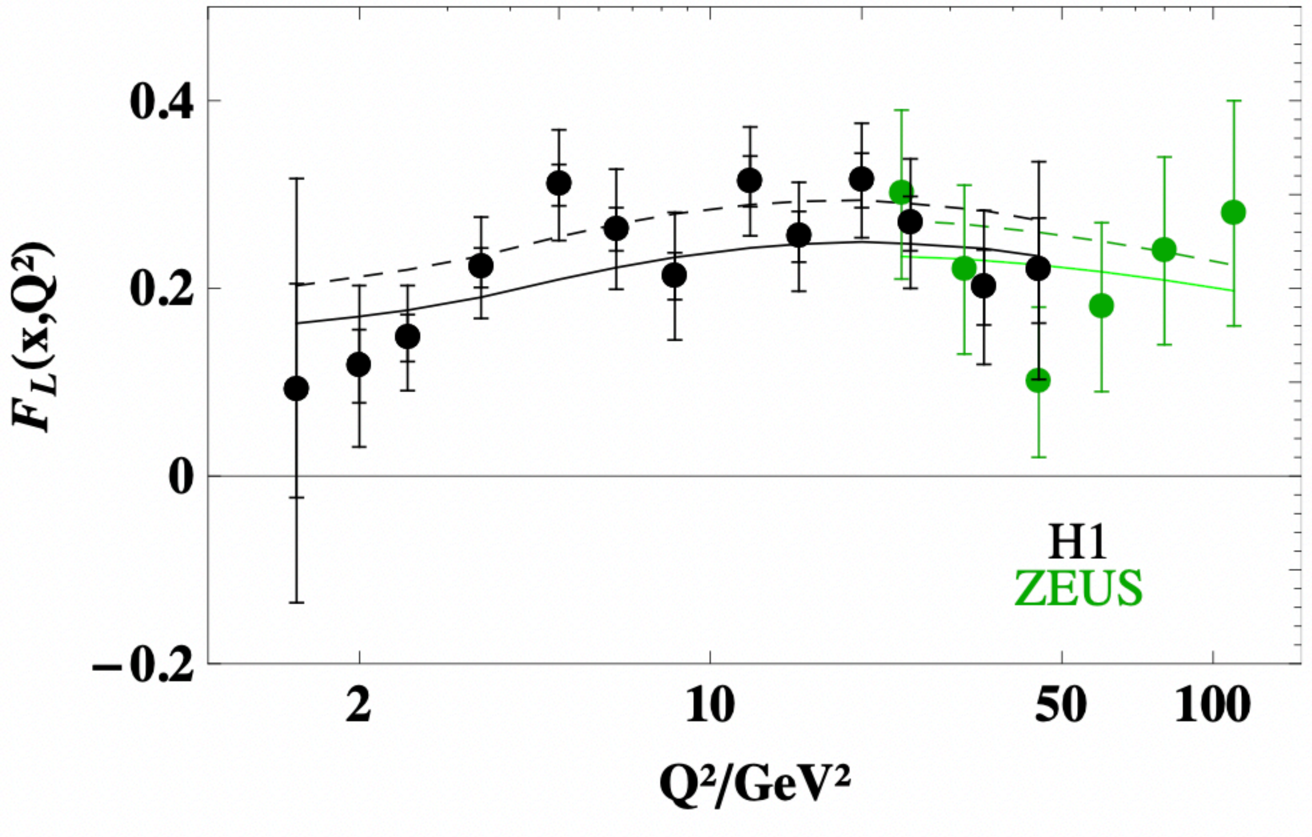
It also describes Forward Drell-Yan at LHC











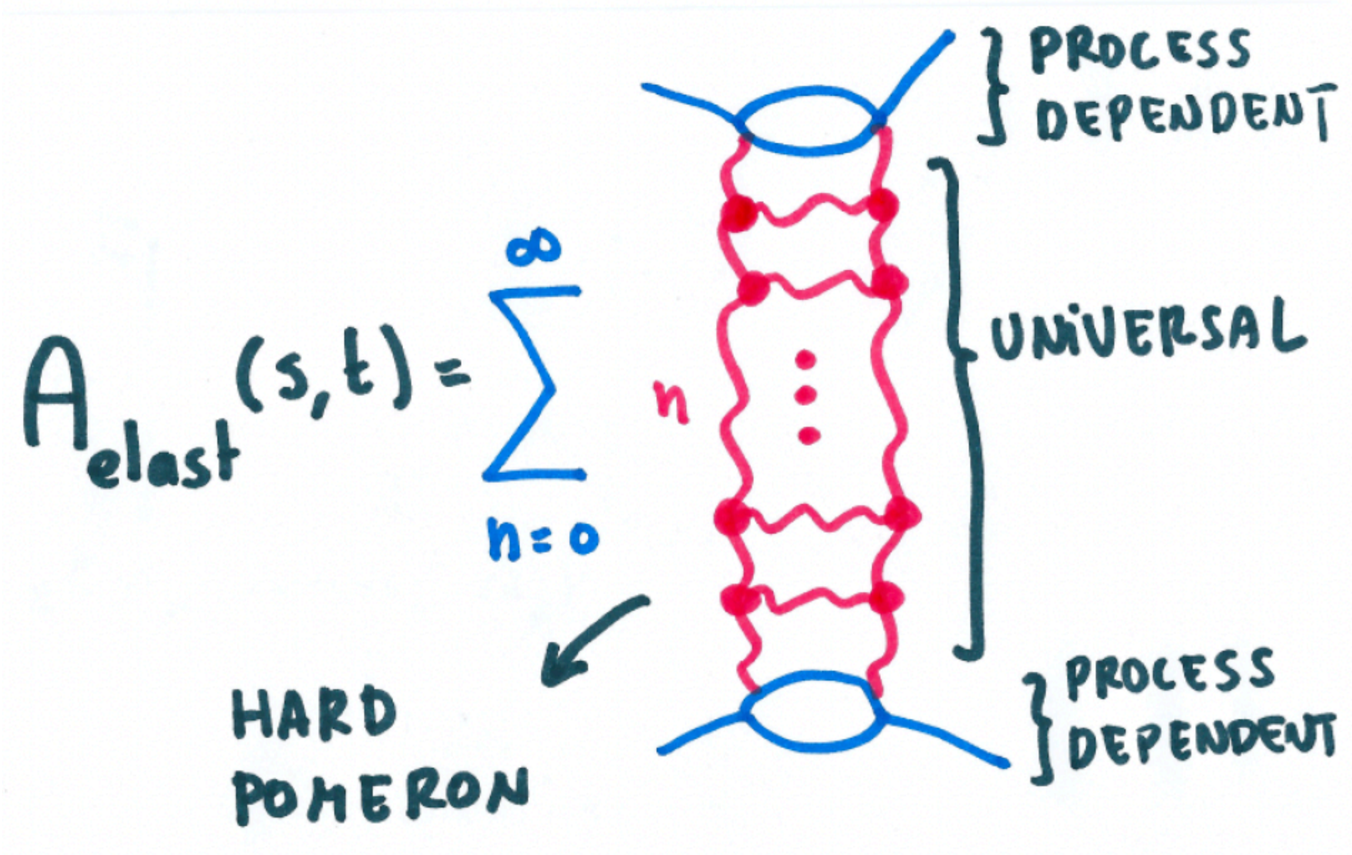
LO BFKL [Brzeminski, Motyka, Sadzikowski, Stebel]

NLO BFKL + Bessel [Celiberto-Gordo-SV]

Analytic vs Monte Carlo: Integrability

Non-forward BFKL equation

$$\omega f_\omega(k_a,k_b,q) = \frac{\delta^{(2)}(k_a-k_b)}{k_a^2(q-k_a)^2} + \frac{\bar\alpha_s}{2\pi}\int d^2k \left\{ \left[\frac{(q-k)^2}{(k-k_a)^2(q-k_a)^2} + \frac{k^2}{(k-k_a)^2k_a^2} - \frac{q^2}{k_a^2(q-k_a)^2} \right] f_\omega(k,k_b,q) - \left[\frac{k_a^2}{k^2+(k_a-k)^2} + \frac{(q-k_a)^2}{(q-k)^2+(k_a-k)^2} \right] \frac{f_\omega(k_a,k_b,q)}{(k-k_a)^2} \right\}$$



It can be solved by iteration and Monte Carlo integration [Chachamis-SV]

$$f(k_a,k_b,q,Y) = \left(\frac{\lambda^2}{k_a^2}\frac{\lambda^2}{(k_a-q)^2}\right)^{\frac{\bar\alpha_s}{2}Y} \left\{ \frac{\delta^{(2)}(k_a-k_b)}{k_a^2(q-k_a)^2} + \sum_{n=1}^{\infty}\prod_{i=1}^{n} \times \int d^2k_i \frac{\theta\left(k_i^2-\lambda^2\right)}{\pi k_i^2}\xi\left(k_a+\sum_{l=1}^{i-1}k_l,k_i,q\right)\int_0^{y_{i-1}}dy_i \left(\frac{\left(k_a+\sum_{l=1}^{i-1}k_l\right)^2}{\left(k_a+\sum_{l=1}^{i}k_l\right)^2}\right)^{\frac{\bar\alpha_s}{2}y_i} \times \left(\frac{\left(k_a+\sum_{l=1}^{i-1}k_l-q\right)^2}{\left(k_a+\sum_{l=1}^{i}k_l-q\right)^2}\right)^{\frac{\bar\alpha_s}{2}y_i} \frac{\delta^{(2)}\left(\sum_{l=1}^{n}k_l+k_a-k_b\right)}{\left(k_a+\sum_{l=1}^{n}k_l\right)^2\left(k_a+\sum_{l=1}^{n}k_l-q\right)^2} \right\}$$

Revisit the work of Lipatov, Navelet-Peschanski

New analytic representations for Pomeron Green's function

$$\omega(\nu,n) = 2\bar{\alpha}_s\left(\gamma_E - \Re e\,\psi\left(\frac{|n|+1}{2}+i\nu\right)\right)$$

$$f_\omega(k,k',q) = \frac{-1}{2\pi^5|q|^6}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}d\nu\frac{\frac{4^{2i\nu}\Gamma\left(\frac{(-1)^n+1}{4}+i\nu\right)^2}{\Gamma(2i\nu)^2\Gamma\left(\frac{(-1)^n+3}{4}-i\nu\right)^2}\frac{\left|\Gamma\left(\frac{2+|n|}{2}+i\nu\right)\Gamma\left(\frac{2-|n|}{2}+i\nu\right)\right|^2}{\left|\Gamma\left(\frac{3+n}{2}+i\nu\right)\Gamma\left(\frac{3-n}{2}+i\nu\right)\right|^2}}{(\omega-\omega(\nu,n))((-1)^n+\cosh(2\pi\nu))}$$

$$\times\int dz dz^*|z|e^{iz\left(2\frac{k^*}{q^*}-1\right)}e^{iz^*\left(2\frac{k}{q}-1\right)}\mathbf{J}_{\frac{i\nu}{2},-n}\left(\frac{z^2}{16\pi^2}\right)$$

$$\times\int dv dv^*|v|e^{-iv(2\frac{k'^*}{q^*}-1)}e^{-iv^*(2\frac{k'}{q}-1)}\mathbf{J}_{-\frac{i\nu}{2},n}\left(\frac{v^2}{16\pi^2}\right)$$

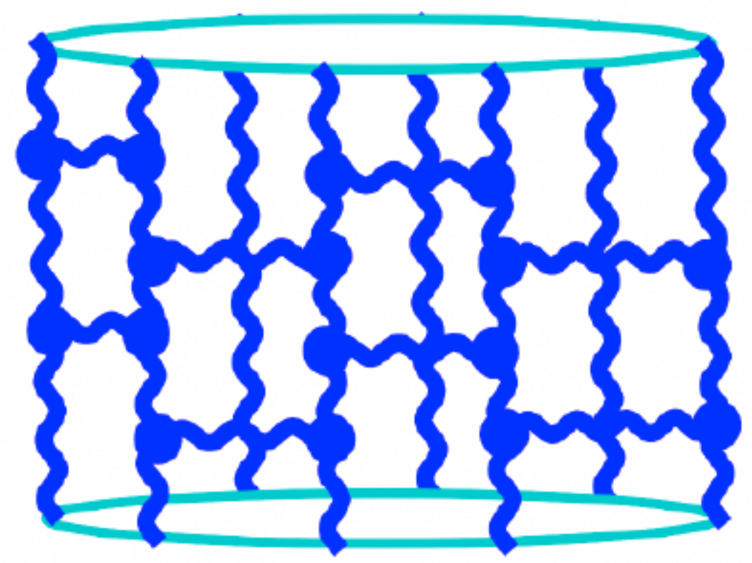
$$\left(z^2\frac{\partial^2}{\partial z^2}+z\frac{\partial}{\partial z}+z^2-\left(\frac{n}{2}-i\nu\right)^2\right)\mathbf{J}_{\frac{i\nu}{2},-n}\left(\frac{z^2}{16\pi^2}\right) = 0$$

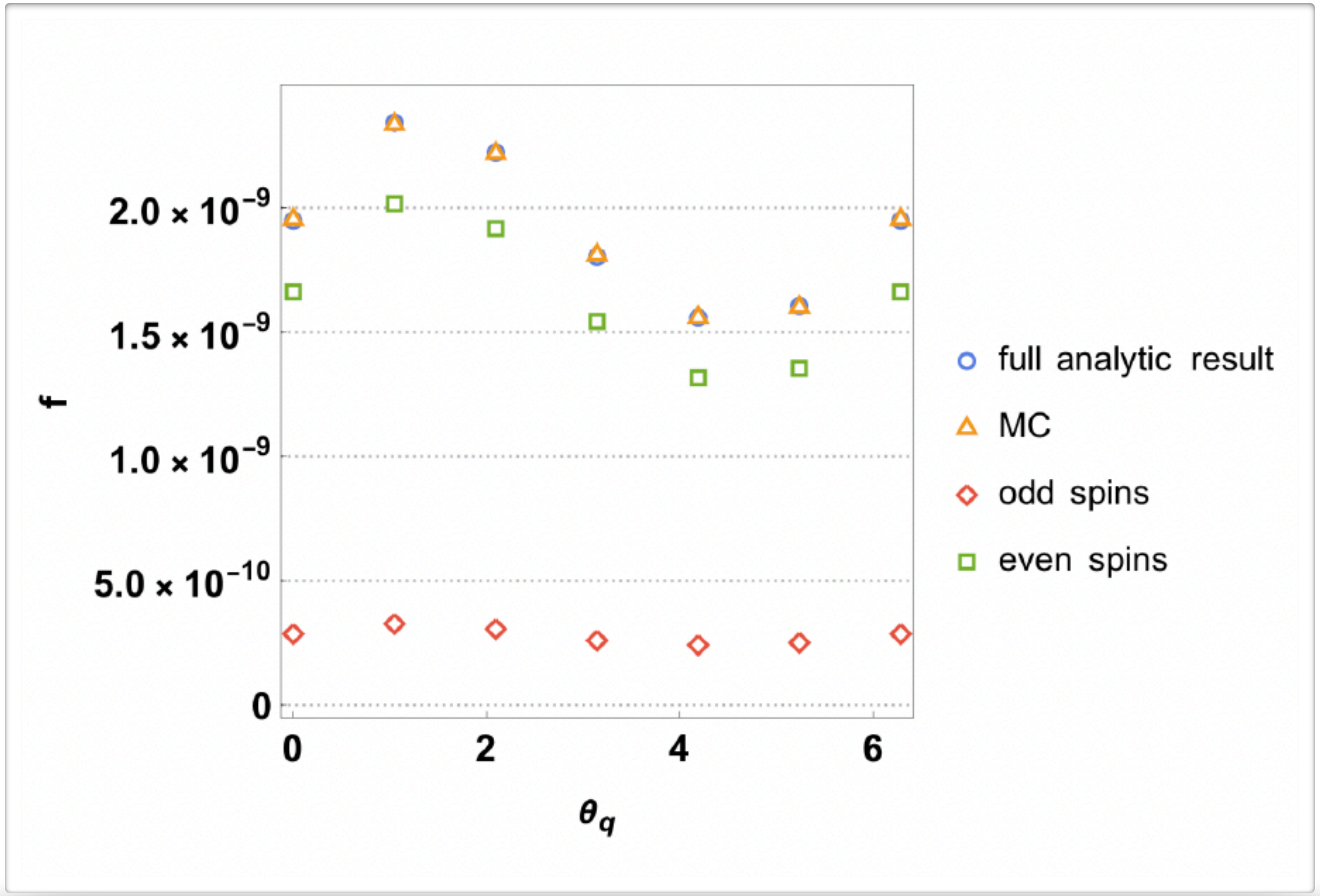
$$\left(z^{*2}\frac{\partial^2}{\partial z^{*2}}+z^*\frac{\partial}{\partial z^*}+z^{*2}-\left(\frac{n}{2}+i\nu\right)^2\right)\mathbf{J}_{\frac{i\nu}{2},-n}\left(\frac{z^2}{16\pi^2}\right) = 0$$

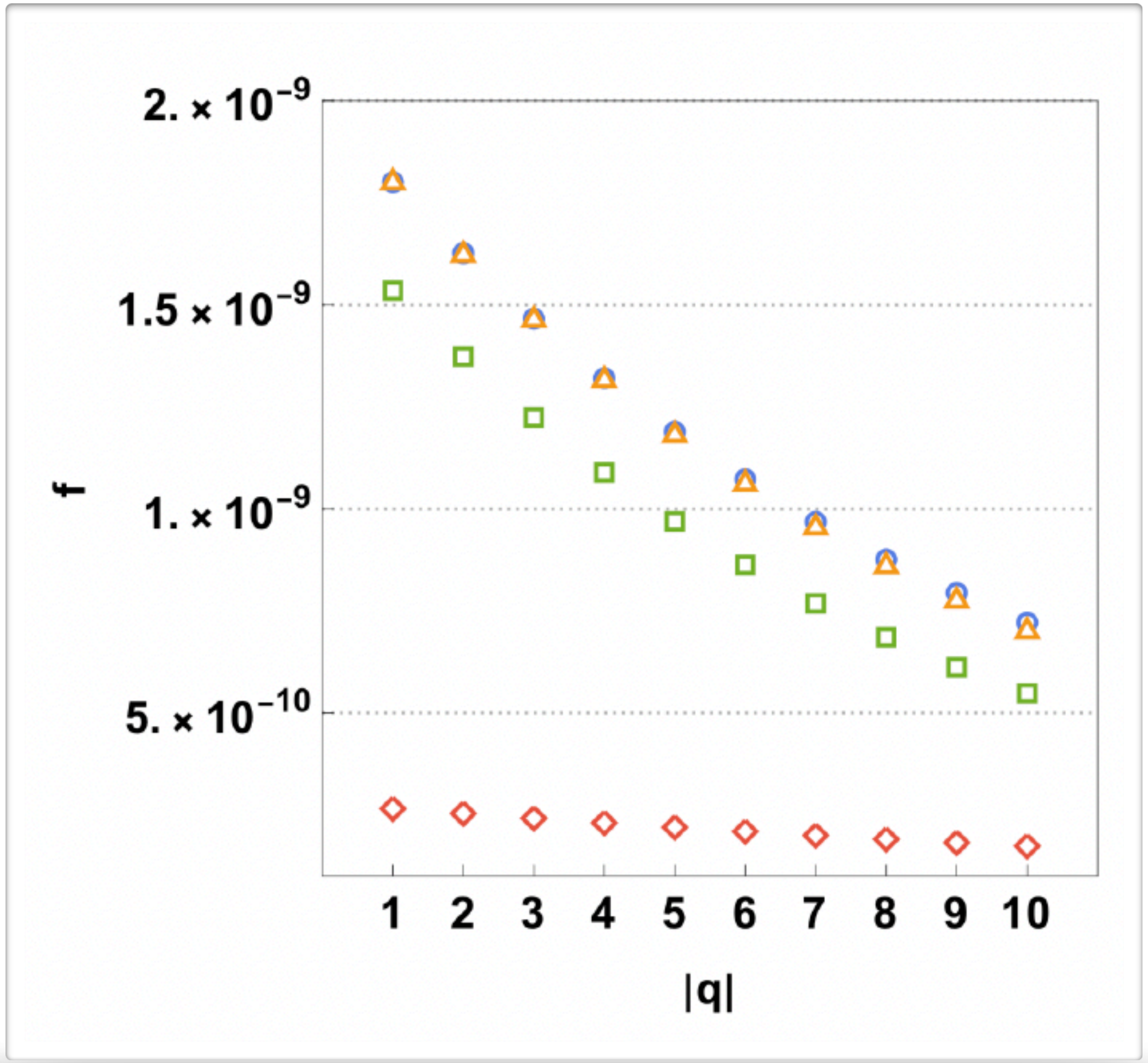
Both odd and even conformal spins contribute to the Pomeron wave function

This is equivalent to a two-site Heisenberg spin chain

Analytic solution in terms of conformal blocks agrees with the Monte Carlo solution







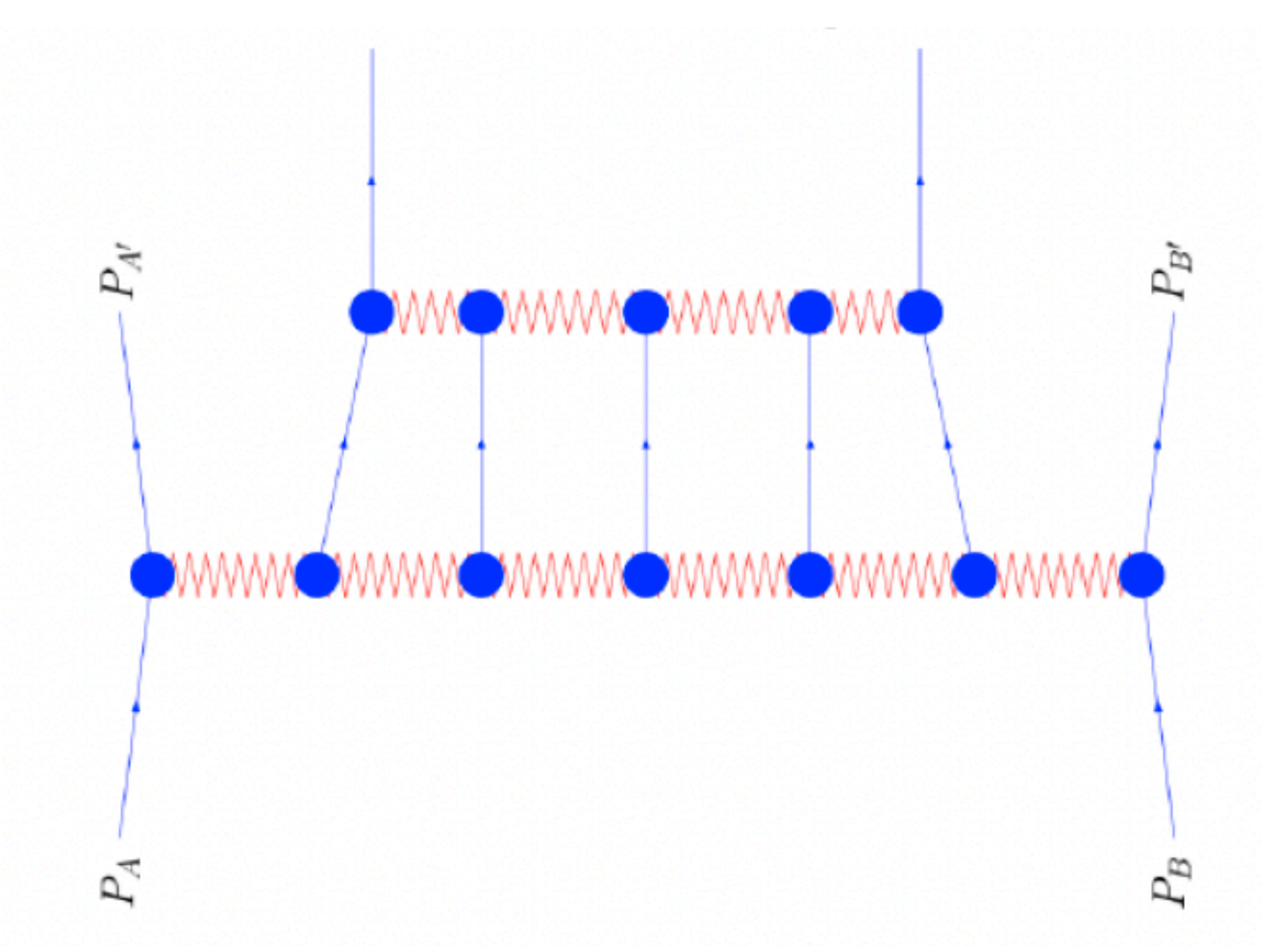
These structures are very important in N=4 SYM, MHV, planar amplitudes

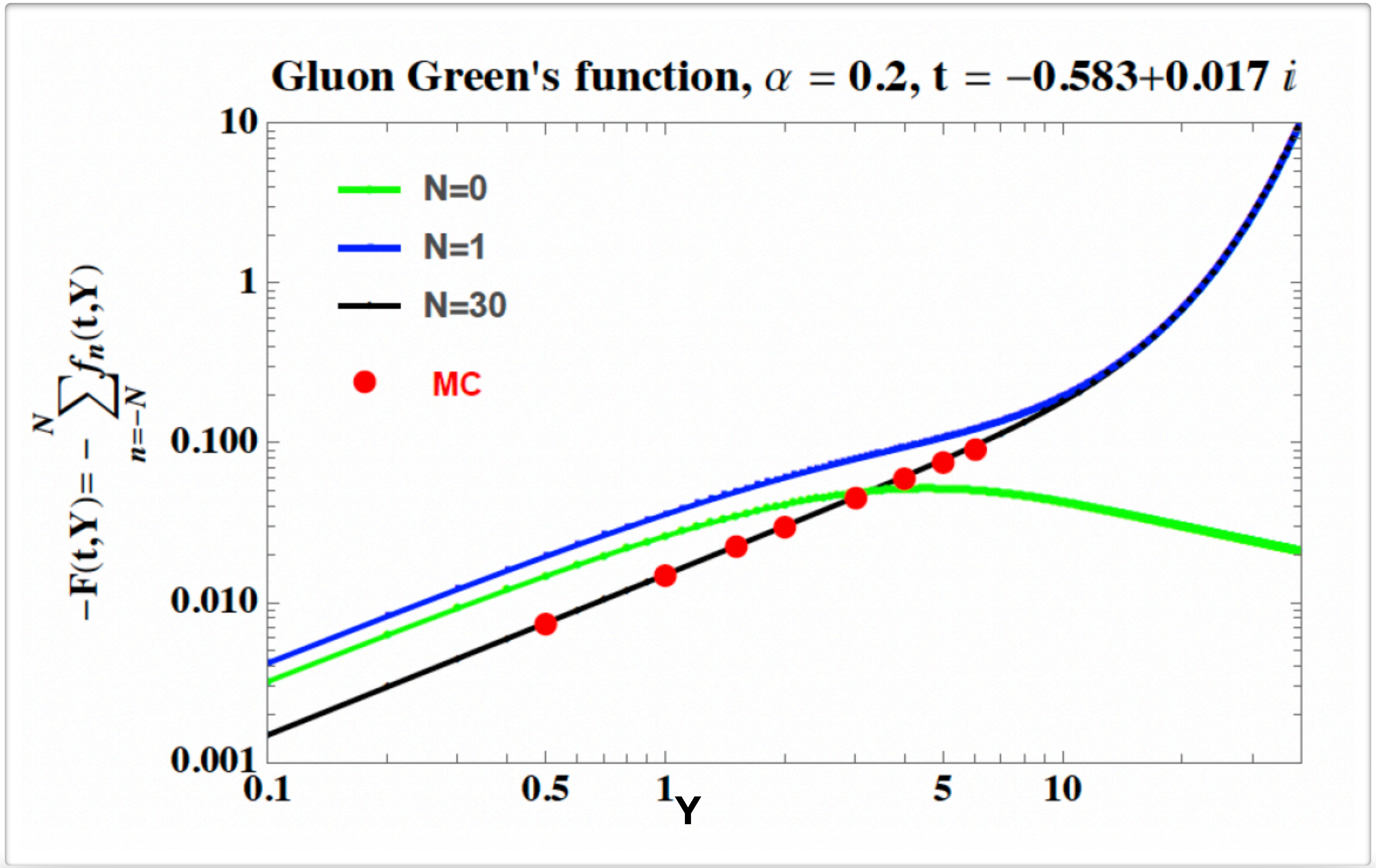
Agreement analytic / Monte Carlo for the Regge cut improving the Bern-Dixon-Smirnov 2 to 4 amplitude

$$\frac{M_{2\to 4}}{\Gamma(t_1)\Gamma(t_3)} = \underbrace{e^{i\pi\frac{\gamma_K(a)}{4}\left(-\frac{1}{\epsilon}+\log\Omega\right)}}_{\text{Regge cut}}\left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)}\Gamma_{RRP}\left(\frac{-s_2}{\mu^4}\right)^{\omega(t_2)}\Gamma_{RRP}\left(\frac{-s_3}{\mu^2}\right)^{\omega(t_3)}$$

[Bartels-Lipatov-SV]

Two-site "open" spin-chain [Chachamis-SV]





Agreement between two very different representations in the singlet (close) & adjoint (open) two-site spin chains

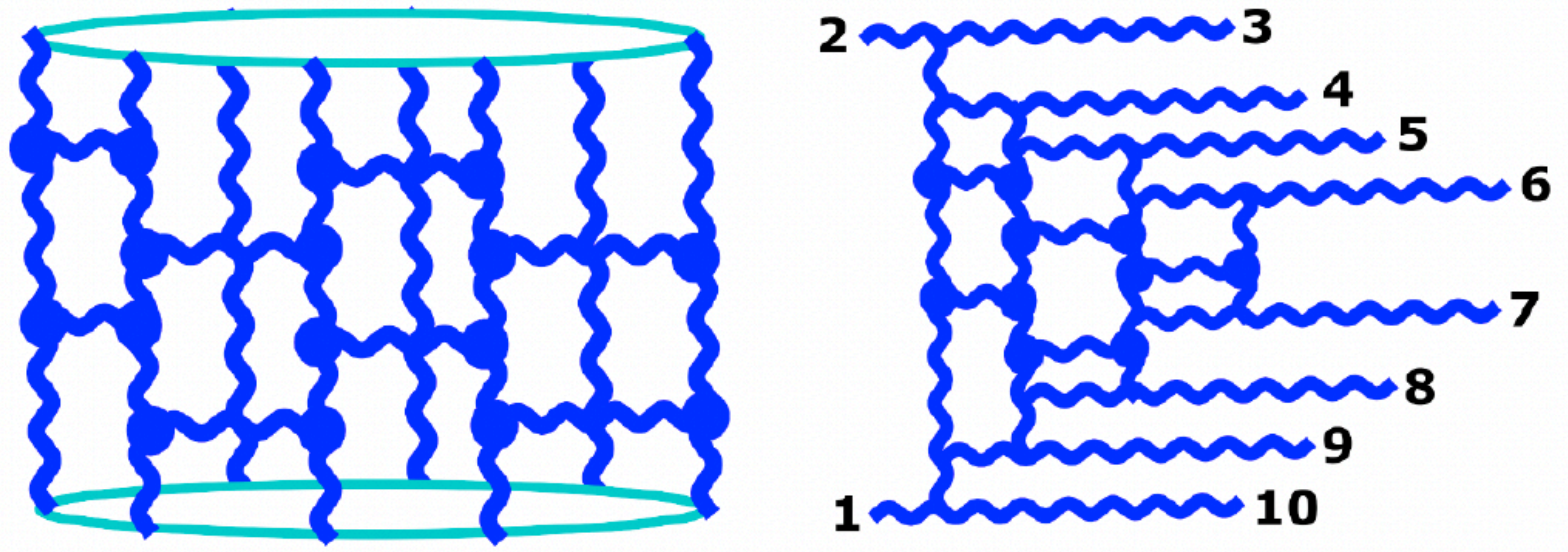
$$
\begin{aligned}
f\left(k_a, k_b, q, Y\right) \;=\;& \left(\frac{\lambda^2}{k_a^2}\frac{\lambda^2}{\left(k_a-q\right)^2}\right)^{\frac{\bar{\alpha}_s}{2}Y}\left\{\frac{\delta^{(2)}\left(k_a-k_b\right)}{k_a^2(q-k_a)^2}+\sum_{n=1}^{\infty}\prod_{i=1}^{n}\right.\\
&\times\int d^2k_i\frac{\theta\left(k_i^2-\lambda^2\right)}{\pi k_i^2}\xi\left(k_a+\sum_{l=1}^{i-1}k_l, k_i, q\right)\int_0^{y_{i-1}}dy_i\left(\frac{\left(k_a+\sum_{l=1}^{i-1}k_l\right)^2}{\left(k_a+\sum_{l=1}^{i}k_l\right)^2}\right)^{\frac{\bar{\alpha}_s}{2}y_i}\\
&\times\left.\left(\frac{\left(k_a+\sum_{l=1}^{i-1}k_l-q\right)^2}{\left(k_a+\sum_{l=1}^{i}k_l-q\right)^2}\right)^{\frac{\bar{\alpha}_s}{2}y_i}\frac{\delta^{(2)}\left(\sum_{l=1}^{n}k_l+k_a-k_b\right)}{\left(k_a+\sum_{l=1}^{n}k_l\right)^2\left(k_a+\sum_{l=1}^{n}k_l-q\right)^2}\right\}
\end{aligned}
$$

$$
f(k_a,k_b,q,Y)=\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}d\nu\,\mathcal{M}^{(n,\nu)}(k_a,q,Y)(\mathcal{M}^{(n,\nu)}(k_b,q,Y))^*
$$

$$
\begin{aligned}
\mathcal{M}^{(n,\nu)}(k_a,q,Y) \;=\;& \frac{e^{Y\tilde{\omega}(\nu,n)}}{2\pi|k_a|^3}\frac{\left|\Gamma\left(\frac{2+n}{2}+i\nu\right)\Gamma\left(\frac{2-n}{2}+i\nu\right)\right|}{\left|\Gamma\left(\frac{3+n}{2}+i\nu\right)\Gamma\left(\frac{3-n}{2}+i\nu\right)\right|}\left[\left(\frac{k_a^*q}{k_aq^*}\right)^{\frac{n}{2}}\left(\frac{|q|}{4|k_a|}\right)^{2i\nu}\right.\\
&\times\frac{\Gamma\left(\frac{3+n}{2}+i\nu\right)}{\Gamma\left(\frac{2+n}{2}+i\nu\right)}\,{}_2F_1\left(\frac{3+n}{2}+i\nu,\frac{1+n}{2}+i\nu;1+n+i2\nu;\frac{q}{k_a}\right)\\
&\times\left.\frac{\Gamma\left(\frac{3-n}{2}+i\nu\right)}{\Gamma\left(\frac{2-n}{2}+i\nu\right)}\,{}_2F_1\left(\frac{3-n}{2}+i\nu,\frac{1-n}{2}+i\nu;1-n+i2\nu;\frac{q^*}{k_a^*}\right)-\text{c.c.}\right]
\end{aligned}
$$

The key point to connect with integrability is the correct representation of several delta functions for n-site spin chains when Y=0

$$
f(k_a,k_b,q,Y=0)=\frac{\delta^{(2)}(k_a-k_b)}{k_a^2(k_a-q)^2}
$$



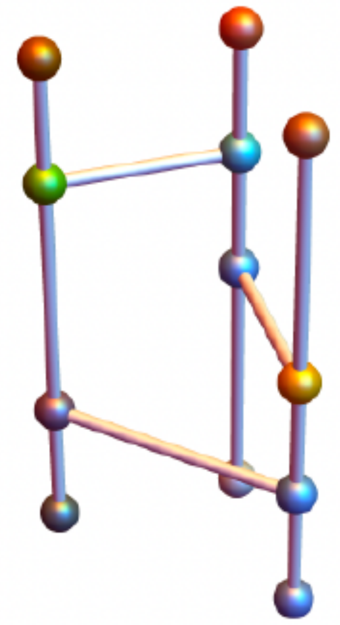
Work in progress …

Work in progress to extract Bartels-Lipatov-Vacca and Faddeev-Korchemsky solutions from exact numerical result

We can only compare at the level of Green's function, not spectrum or eigenfunctions

The simplest case is the three-site close spin chain or Odderon

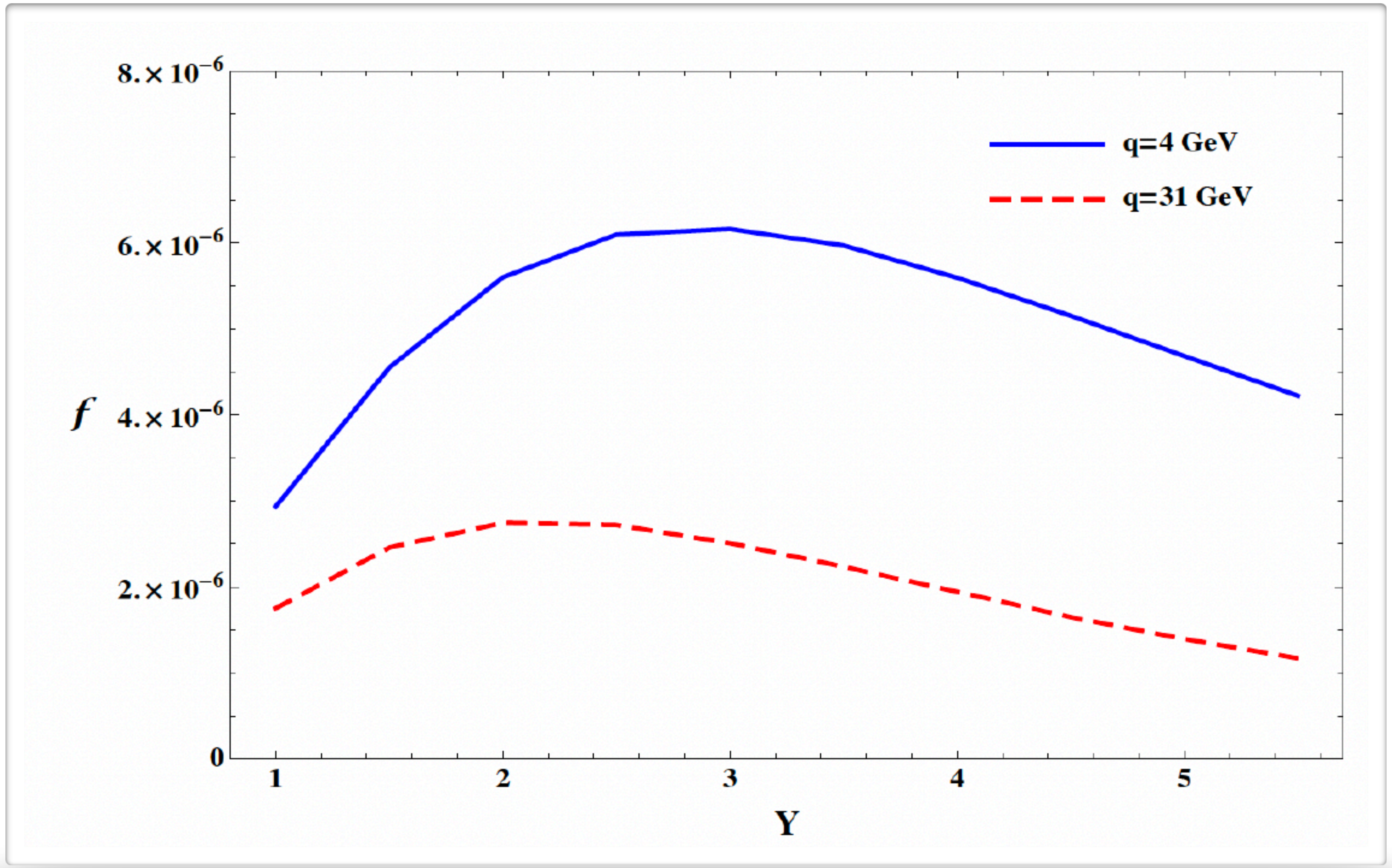
The Bartels-Kwiecinski-Praszalowicz equation



$$
\begin{aligned}
(\omega - \omega(\mathbf{p}_1) - \omega(\mathbf{p}_2) - \omega(\mathbf{p}_3))\, f_\omega(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \; = \\
\delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_4)\, \delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_5)\, \delta^{(2)}(\mathbf{p}_3 - \mathbf{p}_6) \\
+ \int d^2\mathbf{k}\, \xi(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{k})\, f_\omega(\mathbf{p}_1 + \mathbf{k}, \mathbf{p}_2 - \mathbf{k}, \mathbf{p}_3) \\
+ \int d^2\mathbf{k}\, \xi(\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_1, \mathbf{k})\, f_\omega(\mathbf{p}_1, \mathbf{p}_2 + \mathbf{k}, \mathbf{p}_3 - \mathbf{k}) \\
+ \int d^2\mathbf{k}\, \xi(\mathbf{p}_1, \mathbf{p}_3, \mathbf{p}_2, \mathbf{k})\, f_\omega(\mathbf{p}_1 + \mathbf{k}, \mathbf{p}_2, \mathbf{p}_3 - \mathbf{k})
\end{aligned}
$$

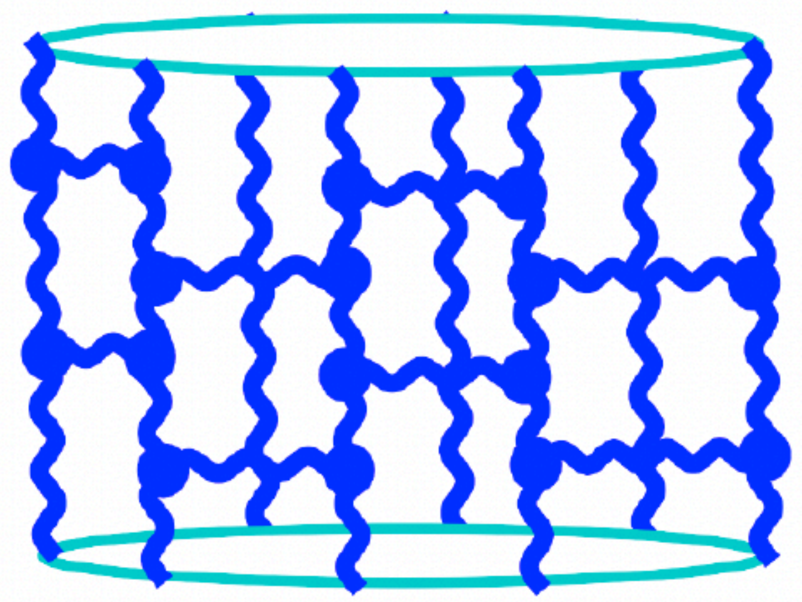
$$\omega(\mathbf{p}) = -\frac{\bar{\alpha}_s}{2} \ln \frac{\mathbf{p}^2}{\lambda^2}$$

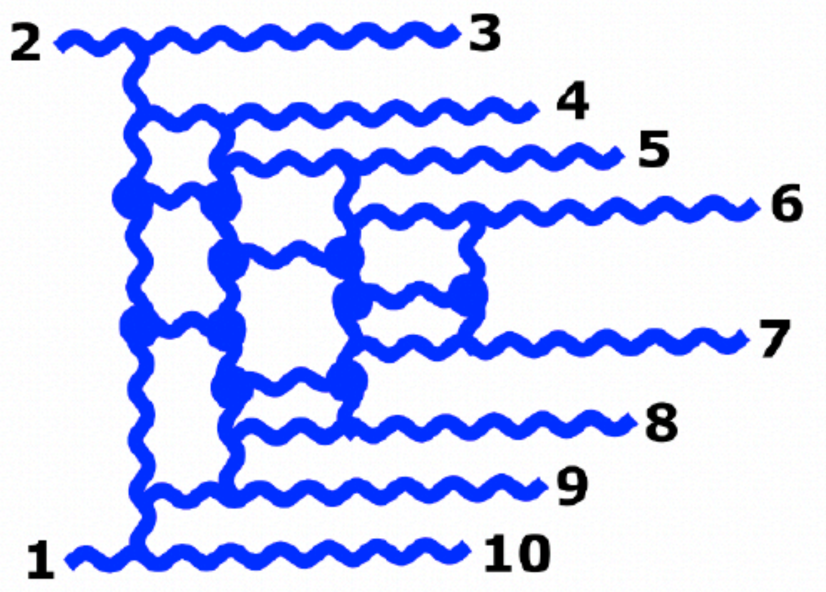
$$\xi(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{k}) \; = \frac{\alpha_s N_c}{4} \frac{\theta(\mathbf{k}^2 - \lambda^2)}{\pi^2 \mathbf{k}^2} \left(1 + \frac{(\mathbf{p}_1 + \mathbf{k})^2 \mathbf{p}_2^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 \mathbf{k}^2}{\mathbf{p}_1^2 (\mathbf{k} - \mathbf{p}_2)^2}\right)$$

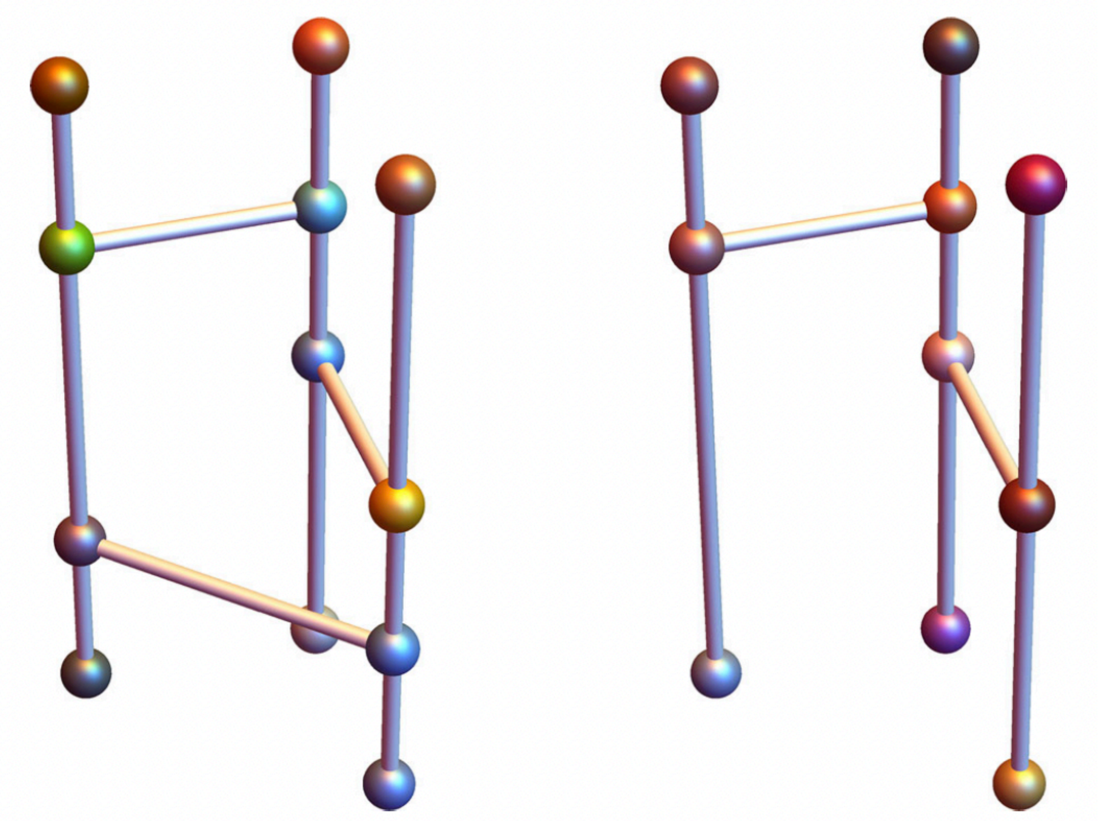


The same spin-chain appears in N=4 SYM MHV amplitudes with many external gluons

This time the spin chain is OPEN

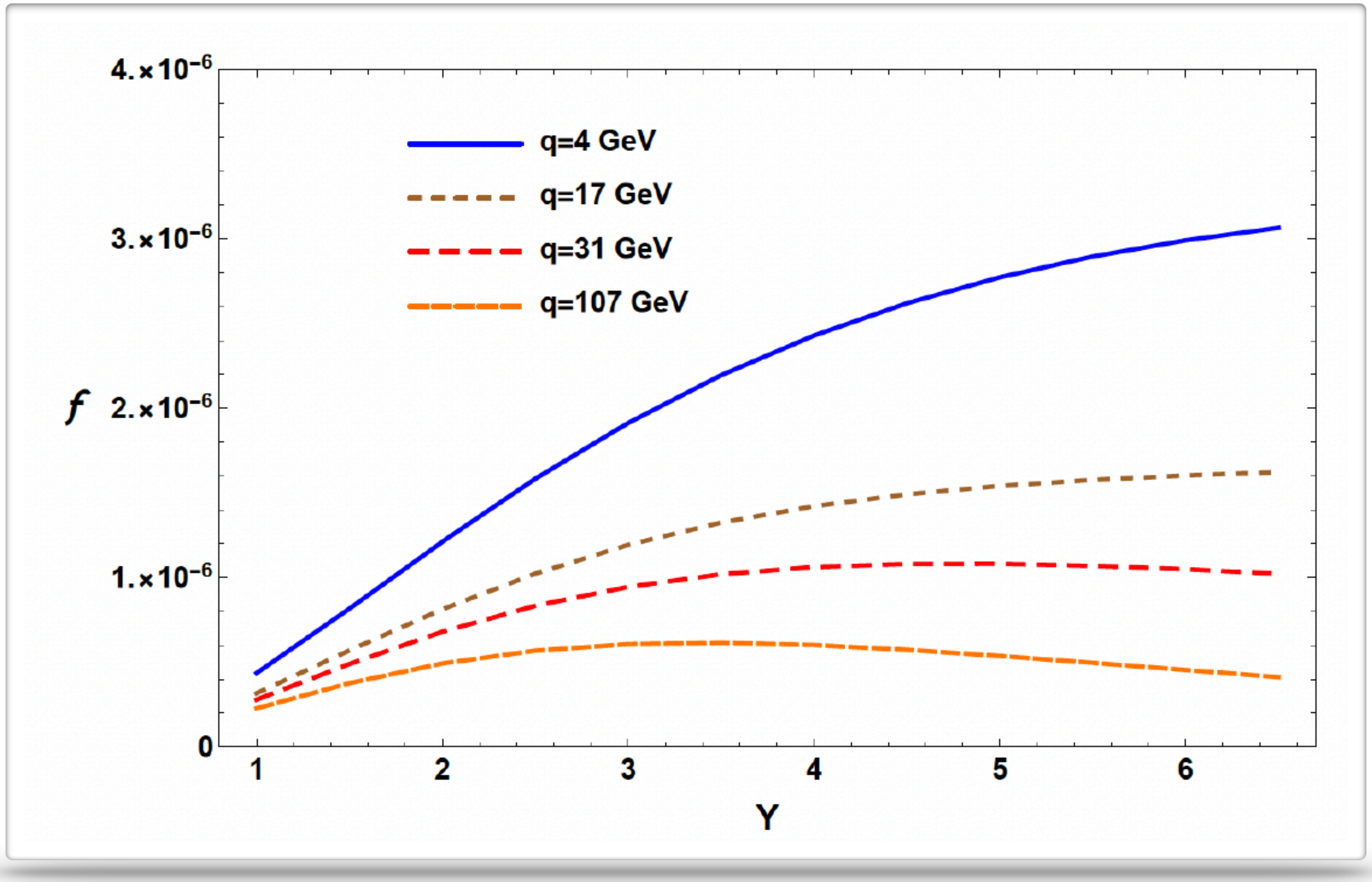






The Monte Carlo solution is available for the three-site open spin chain

$$
\begin{aligned}
(\omega - \omega(\mathbf{p}_1) - \omega(\mathbf{p}_2) - \omega(\mathbf{p}_3))\, f_\omega(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \\
= \delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_4)\, \delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_5)\, \delta^{(2)}(\mathbf{p}_3 - \mathbf{p}_6) \\
+ \int d^2\mathbf{k}\, \xi(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{k})\, f_\omega(\mathbf{p}_1 + \mathbf{k}, \mathbf{p}_2 - \mathbf{k}, \mathbf{p}_3) \\
+ \int d^2\mathbf{k}\, \xi(\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_1, \mathbf{k})\, f_\omega(\mathbf{p}_1, \mathbf{p}_2 + \mathbf{k}, \mathbf{p}_3 - \mathbf{k}).
\end{aligned}
$$



Working towards the analytic representation in terms of conformal blocks

Matrix Hamiltonian

Forward BFKL equation, with azimuthal angle average

$$\frac{\partial \varphi(Q^2,Y)}{\alpha \partial Y} = \int_0^\infty \frac{dq^2}{|q^2-Q^2|}\left(\varphi(q^2,Y) - 2\frac{\min(q^2,Q^2)}{q^2+Q^2}\varphi(Q^2,Y)\right)$$

Discretize virtuality of t-channel reggeized gluons $q^2 = n\,\Delta \quad Q^2 = N\,\Delta \quad \phi_n = \varphi(n\Delta, Y)$

In matrix form

$$\frac{\partial}{\alpha\partial Y}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \vdots\end{pmatrix} = \begin{pmatrix} -2h(0)\phi_1 & \frac{\phi_2}{1} & \frac{\phi_3}{2} & \frac{\phi_4}{3} & \cdots \\ \frac{\phi_1}{1} & -2h(1)\phi_2 & \frac{\phi_3}{1} & \frac{\phi_4}{2} & \cdots \\ \frac{\phi_1}{2} & \frac{\phi_2}{1} & -2h(2)\phi_3 & \frac{\phi_4}{1} & \cdots \\ \frac{\phi_1}{3} & \frac{\phi_2}{2} & \frac{\phi_3}{1} & -2h(3)\phi_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$h(N) = \sum_{l=1}^{N}\frac{1}{l} = \psi(N+1) - \psi(1)$$

It can be truncated by N x N matrix [Bethencourt-Chachamis-Hentschinski-Romagnoni-SV]

$$\frac{\partial}{\alpha\partial Y}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \vdots\\ \phi_N\end{pmatrix} = \begin{pmatrix} -2h(0) & 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{N-1} \\ 1 & -2h(1) & 1 & \frac{1}{2} & \cdots & \frac{1}{N-2} \\ \frac{1}{2} & 1 & -2h(2) & 1 & \cdots & \frac{1}{N-3} \\ \frac{1}{3} & \frac{1}{2} & 1 & -2h(3) & \cdots & \frac{1}{N-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N-1} & \frac{1}{N-2} & \frac{1}{N-3} & \frac{1}{N-4} & \cdots & -2h(N-1) \end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \vdots\\ \phi_N\end{pmatrix}$$

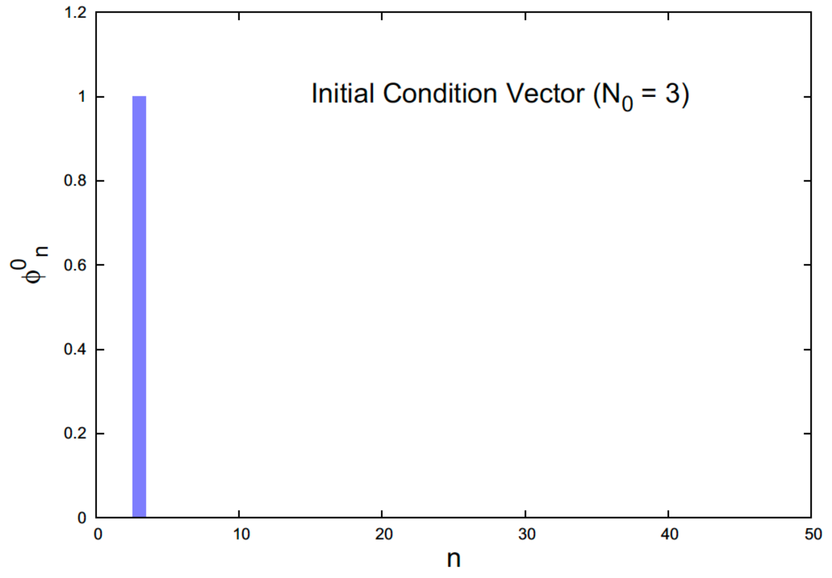
$$\frac{\partial}{\alpha\partial Y}\left|\phi^{(N)}\right\rangle = \hat{\mathcal{H}}_N^{\square}\left|\phi^{(N)}\right\rangle$$

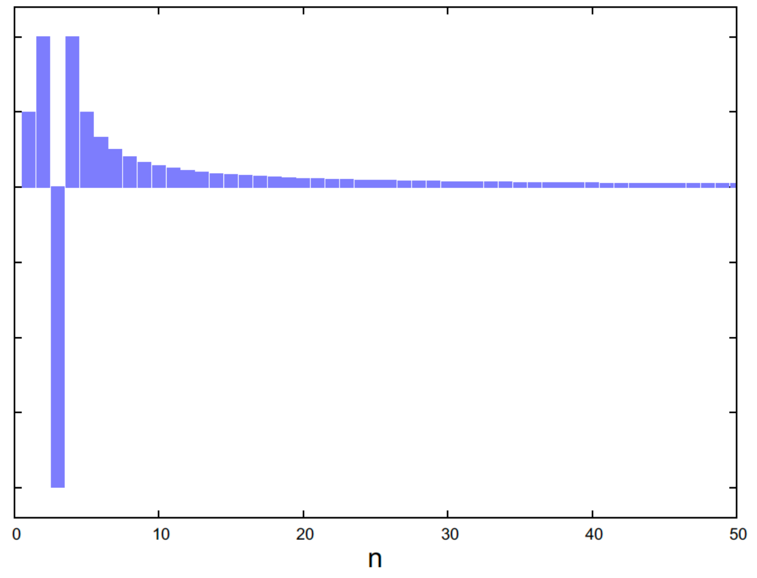
$$(\hat{\mathcal{H}}_N^{\square})_{i,j} = \sum_{n=1}^{N-1}\frac{\delta_i^{j+n}}{n} + \sum_{n=1}^{N-1}\frac{\delta_{i+n}^{j}}{n} - 2h(i-1)\delta_i^j$$

Spectral decomposition and Green's function in terms of eigenvectors

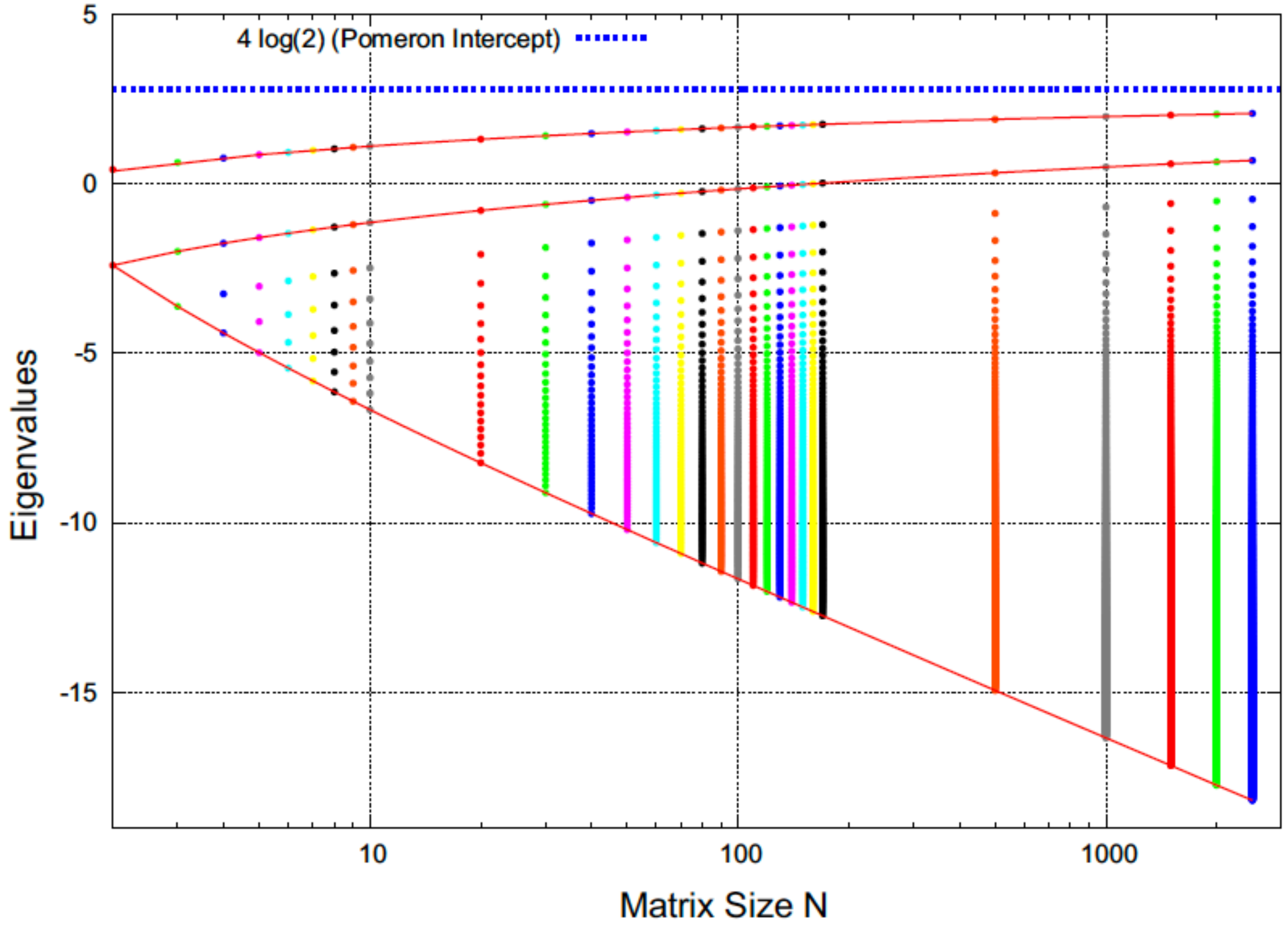
$$\hat{\mathcal{H}}_N^{\square} = \sum_{L=1}^{N}\lambda_L^{(N)}\left|\psi_L^{(N)}\right\rangle\left\langle\psi_L^{(N)}\right|$$

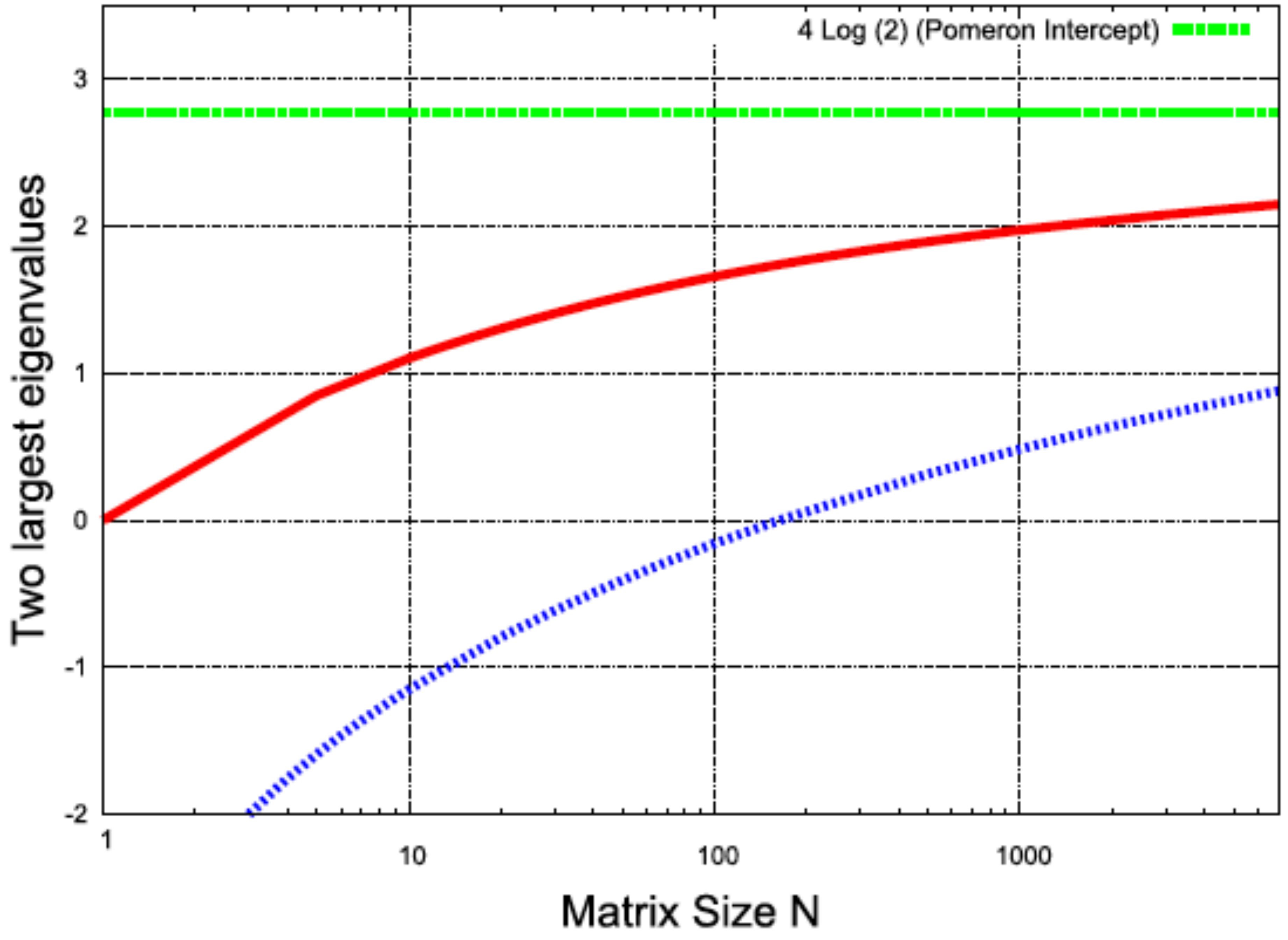
$$\left|\phi^{(N)}\right\rangle = e^{i\lambda Y\hat{\mathcal{H}}_N^{\square}}\left|\varphi_0^{(N_0)}\right\rangle = \sum_{L=1}^{N} c_L^{(N_0)} e^{i\lambda Y\lambda_L^{(N)}}\left|\psi_L^{(N)}\right\rangle$$





Largest positive eigenvalue dominates asymptotics





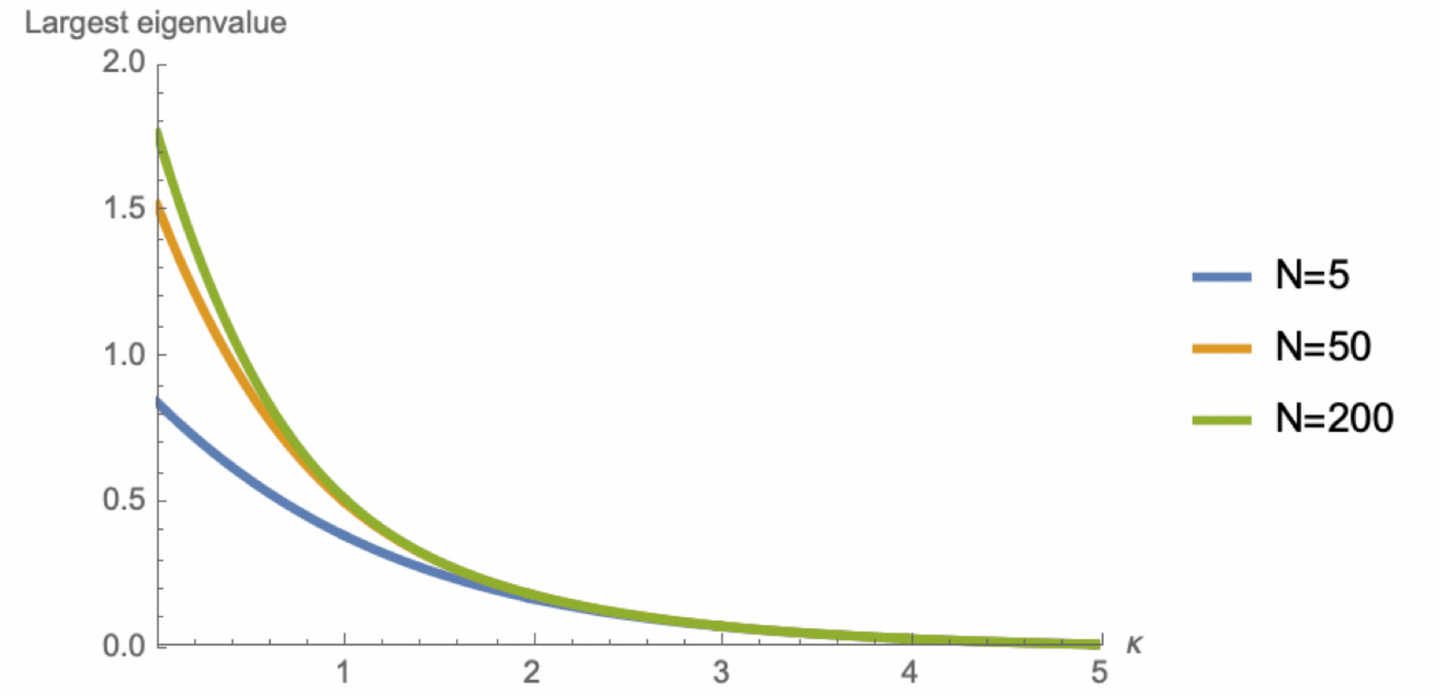
N dimensional Hilbert space of discretised virtualities [Chachamis-Hentschinski-SV]

Density matrix $$\hat{\rho}_{\rm pure}^{(N,N_0)}(Y) = \left|\phi^{(N)}(Y)\right\rangle \left\langle \phi^{(N)}(Y)\right|$$

Pure states transform into pure states $$\mathrm{Tr}\left(\hat{\rho}_{\rm pure}^{(N,N_0)}(Y)\right) = \sum_{L,M=1}^{N} \left(\hat{\rho}_{\rm pure}^{(N,N_0)}(Y)\right)_{L,M} \left\langle \psi_M^{(N)} \middle| \psi_L^{(N)} \right\rangle = \sum_{L=1}^{N} \left| c_L^{(N_0)} \right|^2 = 1$$

Zero von Neumann entropy $$\mathcal{S}_{\rm vN}^{(N,N_0)}(Y) = -\mathrm{Tr}\left(\hat{\rho}_{\rm pure}^{(N,N_0)}(Y) \log_2 \hat{\rho}_{\rm pure}^{(N,N_0)}(Y)\right) = -\lambda_{\hat{\rho}_{\rm pure}^{(N,N_0)}}(Y) \log_2 \lambda_{\hat{\rho}_{\rm pure}^{(N,N_0)}}(Y)$$

Unitarization corrections break the IR/UV symmetry in the evolution

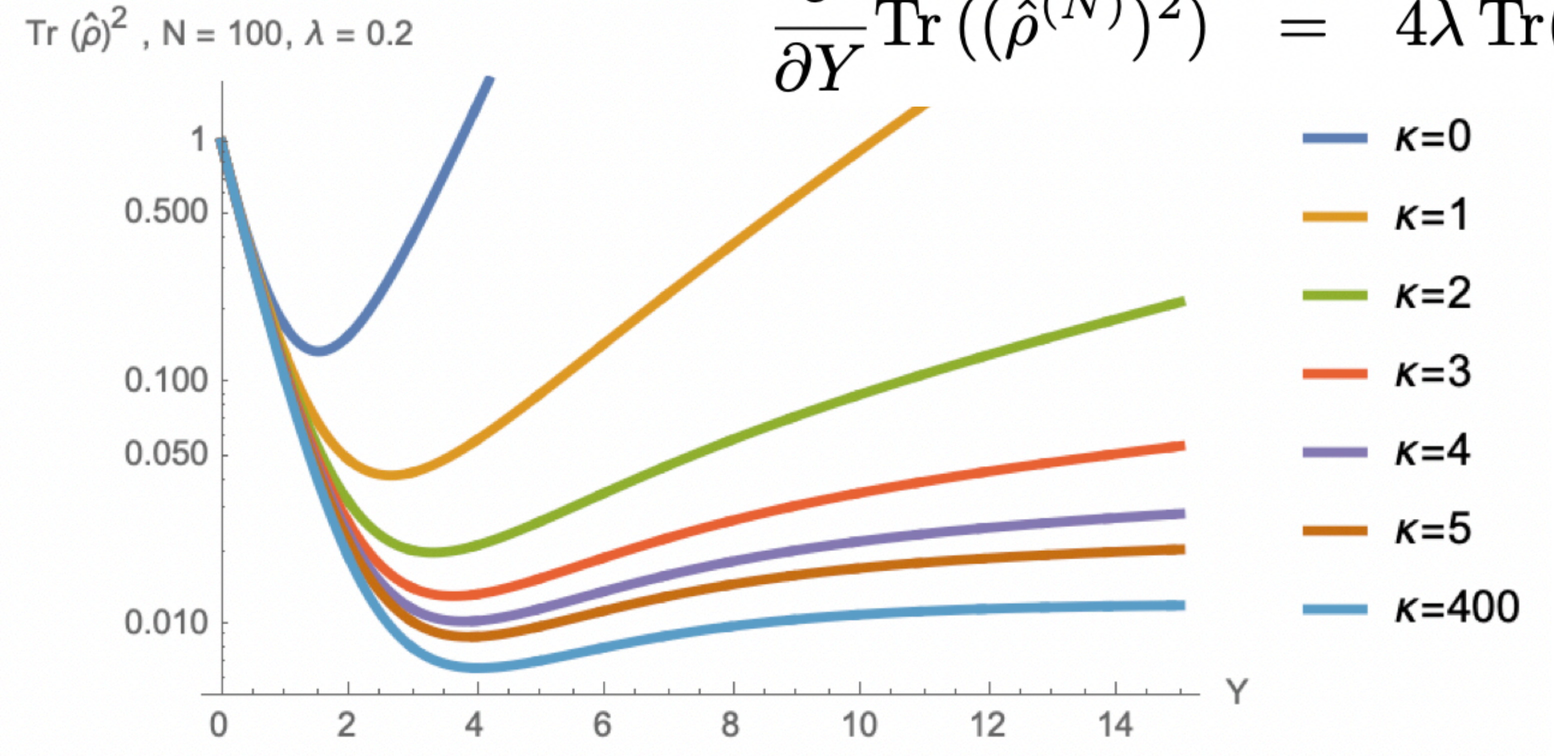


Can we study it in this simple matrix representation?

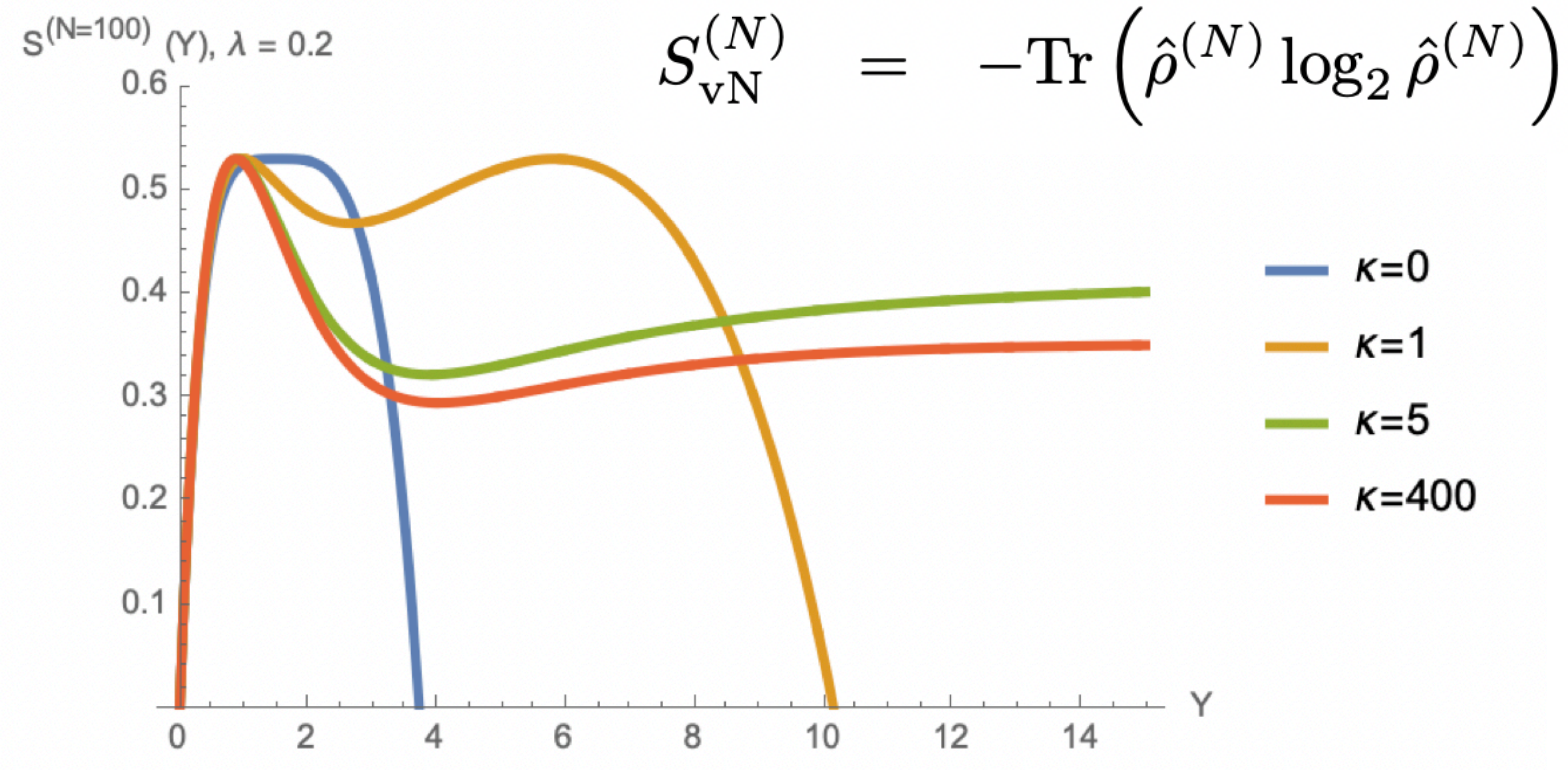
$$(\hat{\mathcal{H}}_N^{\text{dressed}})_{i,j} = \sum_{n=1}^{N-1} \left(\frac{j}{i}\right)^{\kappa} \frac{\delta_i^{j+n}}{n} + \sum_{n=1}^{N-1} \frac{\delta_{i+n}^{j}}{n} - 2h(i-1)\delta_i^j$$

Purity & von Neumann entropy become physical for large enough suppression of IR modes

$$\frac{\partial}{\partial Y}\text{Tr}\left((\hat{\rho}^{(N)})^2\right) = 4\lambda\, \text{Tr}(\hat{\rho}^{(N)})\text{Tr}(\hat{H}_N^{+}\hat{\rho}^{(N)})$$



$$S_{\text{vN}}^{(N)} = -\text{Tr}\left(\hat{\rho}^{(N)} \log_2 \hat{\rho}^{(N)}\right)$$



Non-hermitian Hamiltonian

$$\hat{\mathcal{H}}_N^{\text{dressed}} = \hat{H}_N^{+} + \hat{H}_N^{-}\,,$$

$$\hat{H}_N^{+} = (\hat{H}_N^{+})^T = \frac{1}{2}\left(\hat{\mathcal{H}}_N^{\text{dressed}} + (\hat{\mathcal{H}}_N^{\text{dressed}})^T\right),$$

$$\hat{H}_N^{-} = -(\hat{H}_N^{-})^T = \frac{1}{2}\left(\hat{\mathcal{H}}_N^{\text{dressed}} - (\hat{\mathcal{H}}_N^{\text{dressed}})^T\right)$$

$$\frac{\partial}{\partial Y}\text{Tr}(\hat{\rho}^{(N)}) = 2\lambda\, \text{Tr}(\hat{H}_N^{+}\hat{\rho}^{(N)})$$

Quantum averages of operators. Normalized density. $\langle \mathcal{O}\rangle_Y = \text{Tr}\left(\hat{\mathcal{O}}\,\hat{\Omega}^{(N)}\right)$

$$\hat{H}_N^{+} = \hat{L}_N^T \hat{L}_N$$

$$\hat{\Omega}^{(N)} \equiv \frac{\hat{\rho}^{(N)}}{\text{Tr}\left(\hat{\rho}^{(N)}\right)}$$

$$\frac{\partial}{\partial Y}\hat{\Omega}^{(N)} = \lambda\left(\hat{H}_N^{-} + \hat{L}_N^T\hat{L}_N\right)\hat{\Omega}^{(N)} + \text{h.c.} - 2\lambda\,\hat{\Omega}^{(N)}\text{Tr}\left(\hat{L}_N\,\hat{\Omega}^{(N)}\,\hat{L}_N^T\right)$$

Open system - Lindblad

[Armesto, Domínguez, Kovner, Lublinsky, Skokov]

Spin chain for one-loop anomalous dimensions of N=4 SYM operators of spin S-1 in planar limit of sl(2) sector [Beisert]

Nearest-neighbor one-loop Hamiltonian for XXX s= -1/2 chain

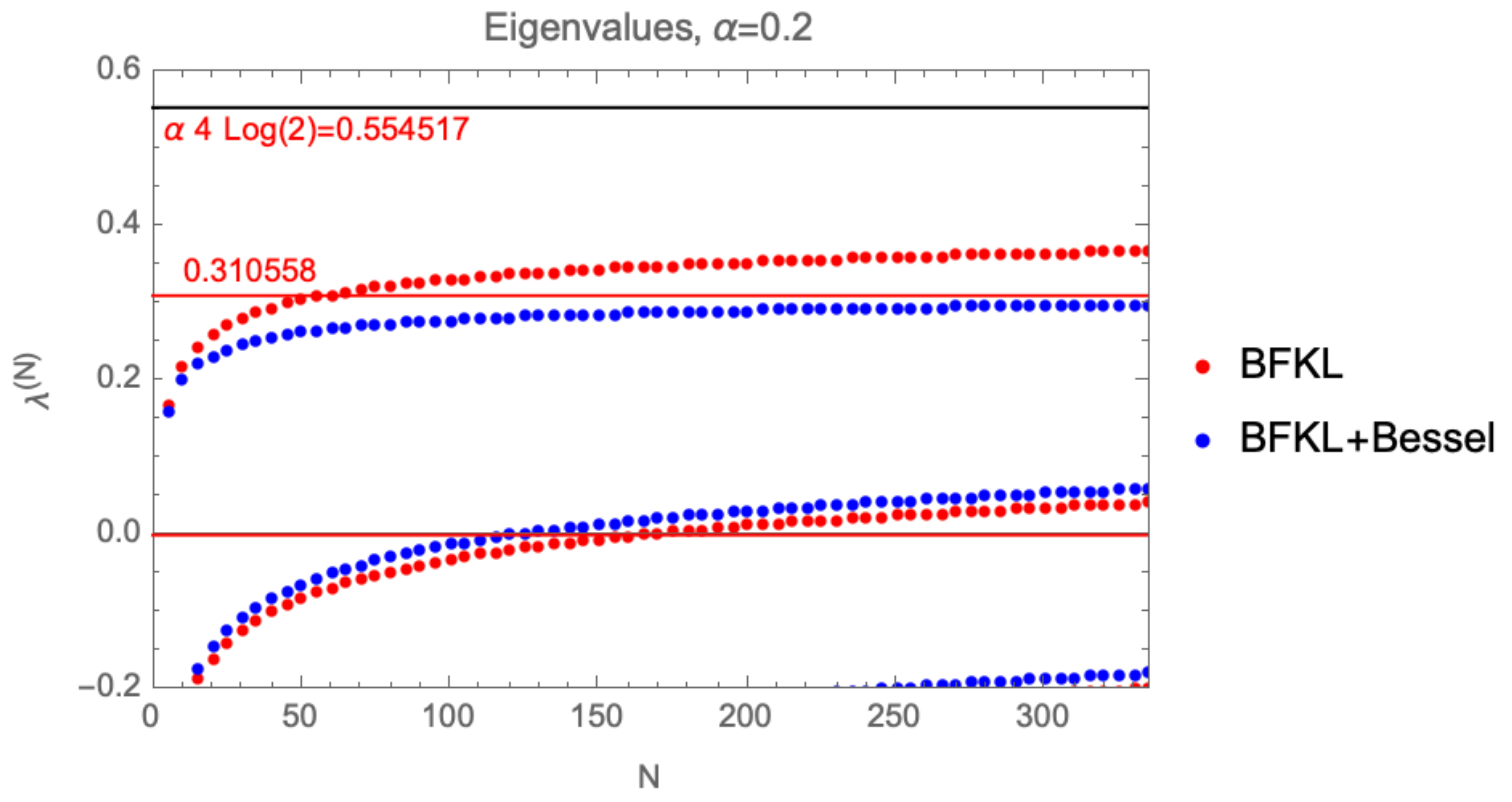
$$\mathcal{H}_{1,2}^{sl(2)}\theta(S-N)\left(a_1^\dagger\right)^{N-1}\left(a_2^\dagger\right)^{S-N}|00\rangle$$
$$= -\lambda\sum_{l=1}^{\infty}\left(\frac{(1-\delta_l^N)}{|l-N|} - (h(N-1) + h(S-N))\,\delta_l^N\right)\theta(S-l)\left(a_1^\dagger\right)^{l-1}\left(a_2^\dagger\right)^{S-l}|00\rangle$$

S=2N-1

$$\mathcal{H}_{1,2}^{\mathbf{sl}(2)}(a_1^+)^{N-1}(a_2^+)^{N-1}|00\rangle = -\lambda\sum_{l=1}^{2N-1}\left(\frac{(1-\delta_l^N)}{|l-N|} - 2h(N-1)\delta_l^N\right) \times(a_1^+)^{l-1}(a_2^+)^{2N-1-l}|00\rangle .$$

What is the connection?

To come back to Double Logs, the spectrum of the matrix when the Bessel function is included:



Agreement with the solution to the omega shift in gamma space …

Intro & a DIS calculation
Analytic vs Monte Carlo: Integrability
Matrix Hamiltonian