THREE-LOOP GLUON REGGE TRAJECTORY IN QCD

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Workshop on overlap between QCD resummations

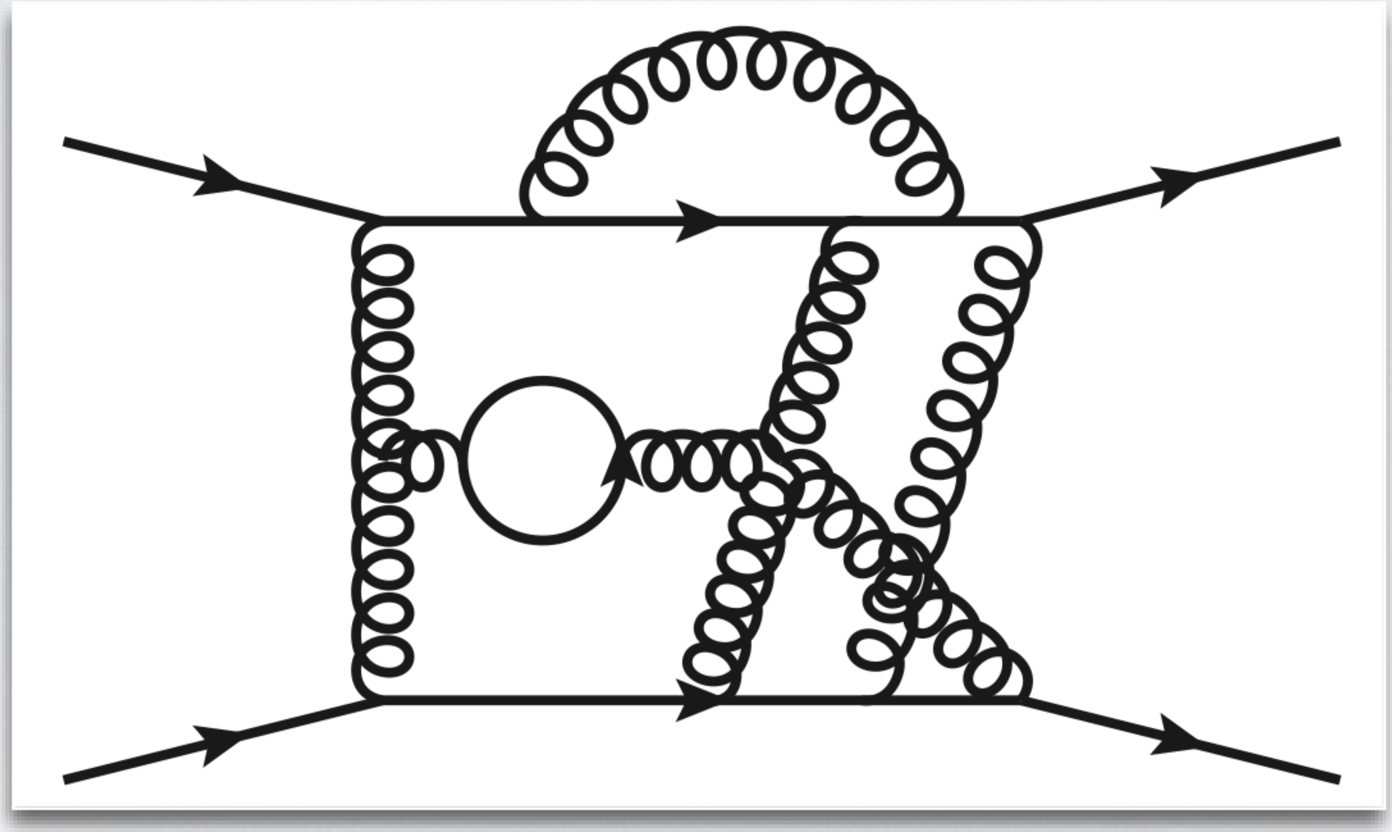
16 Jan 2024, Aussois, Centre Paul Langevin

OUTLINE

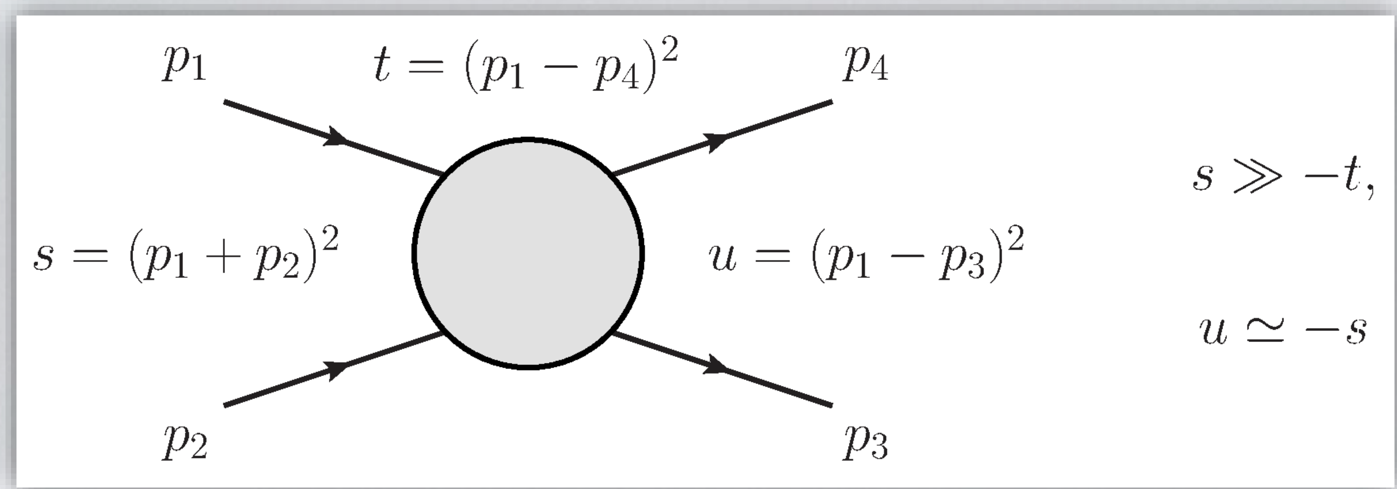
- **Factorisation of amplitudes in the high-energy limit**
- **Amplitudes by iterated solution of the BFKL equation**
- **The Regge trajectory at three loops**

- *JHEP 1706 (2017) 016, [arXiv:1701.05241], with S. Caron-Huot and E. Gardi,*
- *JHEP 1803 (2018) 098, [arXiv:1711.04850], with S. Caron-Huot, E. Gardi, and J. Reichel,*
- *JHEP 08 (2020) 116, [arXiv:2006.01267], with S. Caron-Huot, E. Gardi and J. Reichel,*
- *JHEP 03 (2022), 053, [arXiv:2111.10664], with G. Falcioni, E. Gardi, N. Maher and C. Milloy,*
- *Phys. Rev. Lett. 128, (2022) no.13, [arXiv:2112.11098], with G. Falcioni, E. Gardi, N. Maher, C. Milloy.*

FACTORISATION OF AMPLITUDES IN THE HIGH-ENERGY LIMIT



TWO-PARTON SCATTERING AMPLITUDES



- Expansion in the strong coupling and in towers of (large) logarithms:

$$\mathcal{M}_{ij\to ij} = \mathcal{M}^{(0)} + \frac{\alpha_s}{\pi}\log\frac{s}{-t}\,\mathcal{M}^{(1,1)} + \frac{\alpha_s}{\pi}\,\mathcal{M}^{(1,0)}$$

$$+ \left(\frac{\alpha_s}{\pi}\right)^2 \log^2\frac{s}{-t}\,\mathcal{M}^{(2,2)} + \left(\frac{\alpha_s}{\pi}\right)^2 \log\frac{s}{-t}\,\mathcal{M}^{(2,1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \mathcal{M}^{(2,0)} + \ldots$$

LL NLL NNLL

- Goal: develop a theory to calculate systematically the tower of logarithms at any order in the strong coupling expansion.

HIGH-ENERGY LIMIT

- Very interesting theoretical problem:
  - toy model for full amplitude, yet
    → retain rich dynamic in the 2D transverse plane,
    → non-trivial function spaces;
  - Understand the high-energy QCD asymptotic in terms of Regge poles and cuts;
  - predict amplitudes and other observables in overlapping limits:
    → soft limit, infrared divergences.

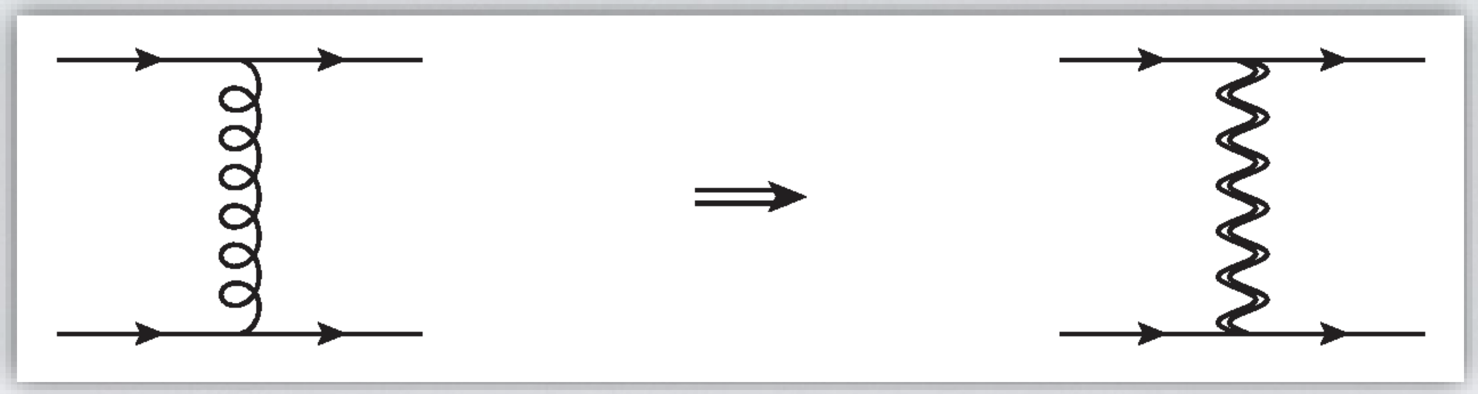
MRK in N=4 SYM:
Dixon, Pennington, Duhr, 2012;
Del Duca, Dixon, Pennington, Duhr, 2013;
Del Duca, Druc, Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek 2019

- Relevant for phenomenology at the LHC and future colliders:
  - perturbative phenomenology of forward scattering, e.g.
    → Deep inelastic scattering/saturation (small x = Regge, large Q^2 = perturbative),
    → Mueller-Navelet: pp → X+2jets, forward and backward.

See e.g. Andersen, Smillie, 2011; Andersen, Medley Smillie, 2016; Andersen, Hapola, Maier, Smillie, 2017; …

TWO-PARTON AMPLITUDES: LL

- LL tower: one-Reggeon exchange in the t-channel:



$$\mathcal{M}^{\rm tree}_{ij\to ij} = g_s^2\,\frac{2s}{t}\,(T_i^b)_{a_1a_4}(T_j^b)_{a_2a_3}\,\delta_{\lambda_1\lambda_4}\delta_{\lambda_2\lambda_3}\,, \qquad \Rightarrow \qquad \mathcal{M}^{\rm LL}_{ij\to ij} = \left(\frac{s}{-t}\right)^{C_A\,\alpha_g(t,\mu^2)} \mathcal{M}^{\rm tree}_{ij\to ij}\,,$$

where the Regge trajectory at one loop reads

Regge, Gribov ~ 1960;
Lipatov; Fadin,Kuraev,Lipatov 1976

$$\alpha_g(t,\mu^2) = \sum_n \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n \alpha_g^{(n)}(t,\mu^2), \qquad \alpha_g^{(1)}(t,\mu^2) = \frac{r_\Gamma}{2\epsilon}\left(\frac{\mu^2}{-t}\right)^{\epsilon}, \qquad r_\Gamma = e^{\epsilon\gamma_E}\,\frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}.$$

- What happens beyond LL?

TWO-PARTON AMPLITUDES: TOOLS

- We need some tools. 1) Color: amplitude is a vector in color space:

$$\mathcal{M}_{ij\to ij} = \sum_k c_{ij}^{[k]}\, \mathcal{M}_{ij\to ij}^{[k]}.$$

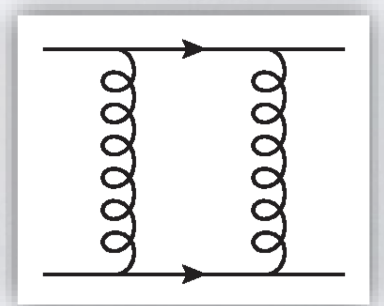
- Decompose the amplitude on a orthonormal color basis in the *t*-channel:

$$\begin{aligned} qq: &\quad 3\otimes\bar{3} &\to&\quad 1\oplus 8, \\ qg: &\quad 8\otimes 8 &\to&\quad 1\oplus 8_s\oplus 8_a, \\ gg: &\quad 8\otimes 8 &\to&\quad 1\oplus 8_s\oplus 8_a\oplus(10+\overline{10})\oplus 27\oplus 0. \end{aligned}$$

- Tree (LL) amplitude involves the exchange of a gluon (Reggeon) in the *t*-channel, thus

$$\mathcal{M}_{ij\to ij}^{\rm LL} = c_{ij}^{[8_{(a)}]}\, \mathcal{M}_{ij\to ij}^{\rm LL\,[8_{(a)}]}.$$

- Beyond LL we expect all components to contribute.

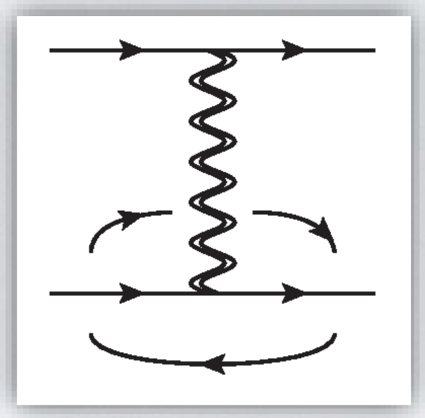


TWO-PARTON AMPLITUDES: TOOLS

- 2) Signature: in the high-energy limit $u \approx -s$; amplitude acquires an effective quantum number, describing the symmetry property w.r.t the exchange $s \leftrightarrow u$:

$$\mathcal{M}(s,t) = \mathcal{M}^{(+)}(s,t) + \mathcal{M}^{(-)}(s,t),$$

$$\mathcal{M}^{(\pm)}(s,t) = \frac{1}{2}\Big(\mathcal{M}(s,t) \pm \mathcal{M}(-s-t,t)\Big).$$



- At LL accuracy

$$\mathcal{M}^{\rm LL}_{ij\to ij} = \mathcal{M}^{\rm LL\,(-)}_{ij\to ij}.$$

- Beyond LL we expect both signature components to contribute.

- For $gg \to gg$ amplitude, Bose symmetry requires the color components to have definite symmetry under the signature:

$$\text{odd}: \quad 8_a, \quad 10+\overline{10},$$

$$\text{even}: \quad 1, \quad 8_s, \quad 27, \quad 0.$$

TWO-PARTON AMPLITUDES: N(N)LL

- Expand the amplitude in terms of the signature-symmetric logarithm:

$$L \equiv \log\left(\frac{s}{-t}\right) - \frac{i\pi}{2} = \frac{1}{2}\left[\log\left(\frac{-s-i0}{-t}\right) + \log\left(\frac{-u-i0}{-t}\right)\right],$$

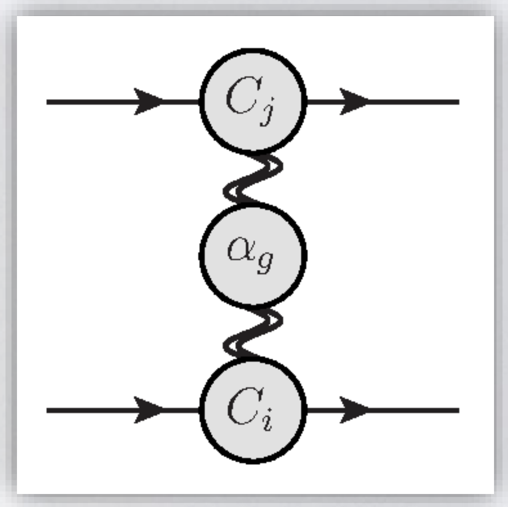
such that

$$\mathcal{M}^{(\pm)}_{ij\to ij} = \sum_{n=0}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^n \sum_{m=0}^{n} L^m \mathcal{M}^{(\pm,n,m)}_{ij\to ij}.$$

It ensures that $M^{(-,n,m)}$ is purely real, while $M^{(+,n,m)}$ is purely imaginary.

- The odd component at NLL is still given in terms of a single Reggeon exchange:

$$\mathcal{M}^{(-),\,\mathrm{NLL}}_{ij\to ij} = \mathcal{M}^{(-),\,\mathrm{SR}}_{ij\to ij} = e^{C_A\alpha_g(t)L}\, C_i(t)\, C_j(t)\, \mathcal{M}^{\mathrm{tree}}_{ij\to ij}\,,$$



Fadin, Fiore, Kozlov, Reznichenko, 2006; Ioffe, Fadin, Lipatov, 2010; Fadin, Kozlov, Reznichenko, 2015

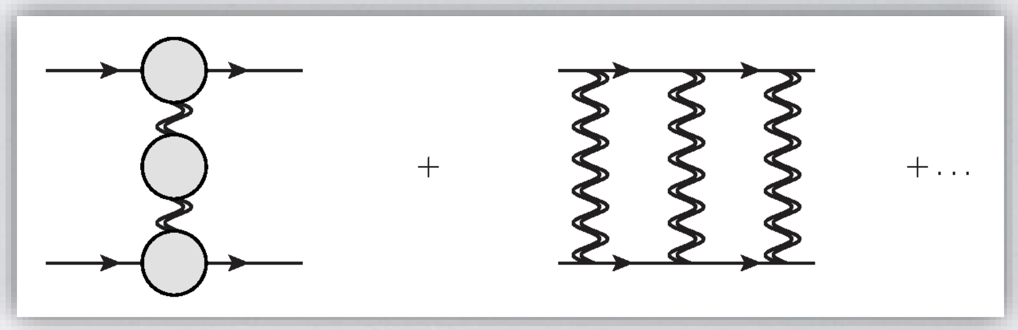
where the Regge trajectory is taken at two loops, and the impact factors $C_{i/j}$ at one loop.

TWO-PARTON AMPLITUDES: N(N)LL

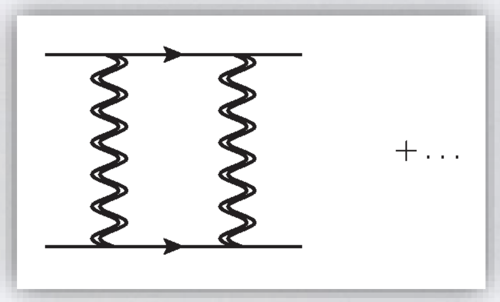
- In general, we expect higher logarithmic terms to be described in terms of multi-Reggeon states:

Fadin,Kuraev,Lipatov 1975-77; Balitsky,Lipatov 1978

$$\mathcal{M}^{(-)} = \mathcal{M}^{(-),\,\mathrm{SR}} + \mathcal{M}^{(-),\,\mathrm{MR}}\,,$$



$$\mathcal{M}^{(+)} = \mathcal{M}^{(+),\,\mathrm{MR}}\,.$$



- Task 1: develop a framework to calculate quantitatively multi-Reggeon exchanges.

MULTI-REGGEON STATES

- Multiple Reggeon exchange contribution in scattering amplitudes elusive, until recently.
- First evidence of violation of Regge-pole factorization in

  Del Duca, Glover 2001;

- Interplay with the infrared factorization theorem investigated in

  Del Duca, Duhr, Gardi, Magnea, White 2011; Del Duca, Falcioni, Magnea, LV, 2013, 2014;

- High-energy scattering via Wilson lines:

  Korchemskaya, Korchemsky, 1994,1996; Balitsky 1995; Babansky, Balitsky 2002;

- Two-parton scattering from rapidity evolution of Wilson lines

  Caron-Huot, 2013; Caron-Huot, Gardi, LV, 2017; Caron-Huot, Gardi, Reichel, LV, 2017, 2020; Falcioni, Gardi, Milloy, LV, 2020; Falcioni, Gardi, Maher, Milloy, LV, 2021,2022.

  → This talk

- SCET-based formulation in

  Rothstein, Stewart 2016; Ridgway, Moult, Stewart, 2019, 2020.

- Calculation of multiple Reggeon exchanges within QCD also obtained in

  Fadin, Lipatov 2017; Fadin 2019, 2020.

REGGE POLES AND CUTS

- The Regge trajectory is related to a Regge pole in the complex angular momentum plane.
- Multi-Reggeon contributions are expected to be related to Regge cuts.
- Write the amplitude as a dispersion relation

$$\mathcal{M}(s,t) = \frac{1}{\pi}\int_0^\infty \frac{d\hat{s}}{\hat{s}-s-i0}\, D_s(\hat{s},t) + \frac{1}{\pi}\int_0^\infty \frac{d\hat{u}}{\hat{u}+s+t-i0}\, D_u(\hat{u},t),$$

where D_s and D_u are discontinuities of M in the s- and u-channels. They are real (spectral density of positive energy states propagating in the s- and u-channels). Parametrize them as a sum of power laws by means of a Mellin transformation:

$$a_j^s(t) = \frac{1}{\pi}\int_0^\infty \frac{d\hat{s}}{\hat{s}}\, D_s(\hat{s},t)\left(\frac{\hat{s}}{-t}\right)^{-j} .$$

- Substituting the inverse transform into the dispersive representation and integrating over $\hat{s}$ and $\hat{u}$, one obtains a Mellin representation of the amplitude:

$$\mathcal{M}(s,t) = \frac{-1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{\sin(\pi j)}\left(a_j^s(t)\left(\frac{-s-i0}{-t}\right)^j + a_j^u(t)\left(\frac{s+t-i0}{-t}\right)^j\right) .$$

Caron-Huot, Gardi, Vernazza, 2017.

REGGE POLES AND CUTS

- In particular:

$$\mathcal{M}^{(+)}(s,t) = i \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{\sin(\pi j)} \cos\left(\frac{\pi j}{2}\right) a_j^{(+)}(t)\, e^{jL}\,,$$

$$\mathcal{M}^{(-)}(s,t) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{\sin(\pi j)} \sin\left(\frac{\pi j}{2}\right) a_j^{(-)}(t)\, e^{jL}\,,$$

  where $a_j^{(\pm)}(t) \equiv \frac{1}{2}(a_j^s(t) \pm a_j^u(t))$.
- At leading power in *t/s* the Mellin variable *j* is identical to the spin *j* which enters conventional partial wave expansion.
- Simplest asymptotic behavior: pure power law, whose Mellin transform is a Regge pole:

$$a_j^{(-)}(t) \simeq \frac{1}{j-1-\alpha(t)}, \quad \Rightarrow \quad \mathcal{M}^{(-)}(s,t)|_{\text{Regge pole}} \simeq \frac{\pi}{\sin\frac{\pi\,\alpha(t)}{2}} \frac{s}{t}\, e^{L\,\alpha(t)} + \ldots,$$

  a(t) is interpreted as the gluon Regge trajectory.

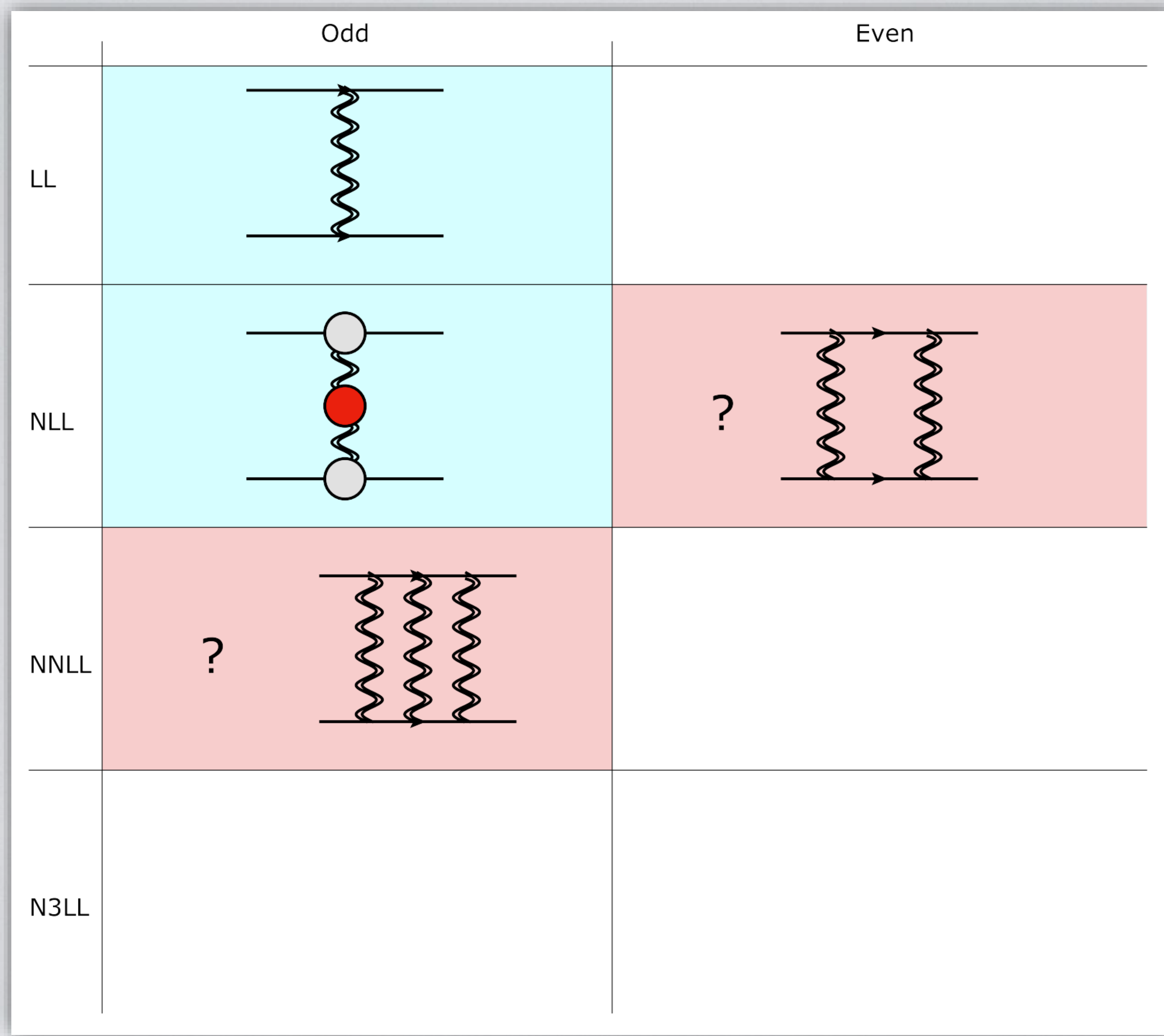
Caron-Huot, Gardi, Vernazza, 2017.

- A Regge cut arises e.g. from

$$a_j^{(-)}(t) \simeq \frac{1}{(j-1-\alpha(t))^{1+\beta(t)}}, \quad \Rightarrow \quad \mathcal{M}^{(-)}(s,t)|_{\text{Regge cut}} \simeq \frac{\pi}{\sin\frac{\pi\,\alpha(t)}{2}} \frac{s}{t} \frac{1}{\Gamma\left(1+\beta(t)\right)} L^{\beta(t)}\, e^{L\,\alpha(t)} + \ldots.$$

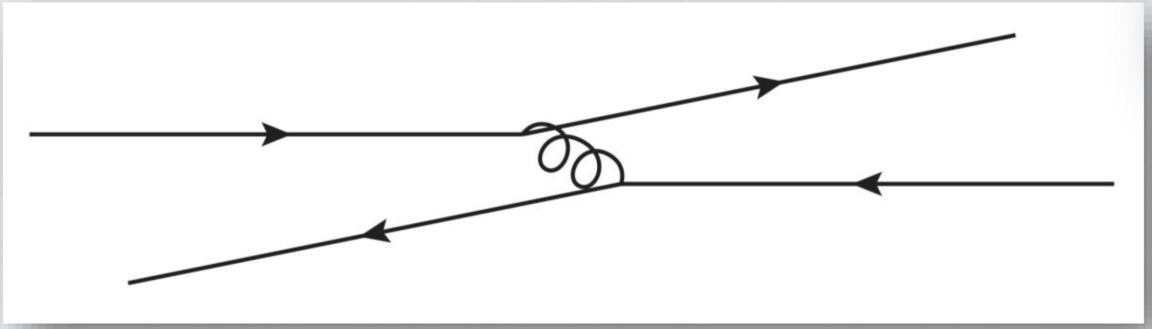
- Task 2: relate single- and multi-Reggeon states to Regge poles and cuts in perturbation theory.

TASK 1: A MULTI-REGGEON EFFECTIVE THEORY



FROM BALITSKY-JIMWLK TO AMPLITUDES

- The physical picture: high-energy limit = forward scattering:



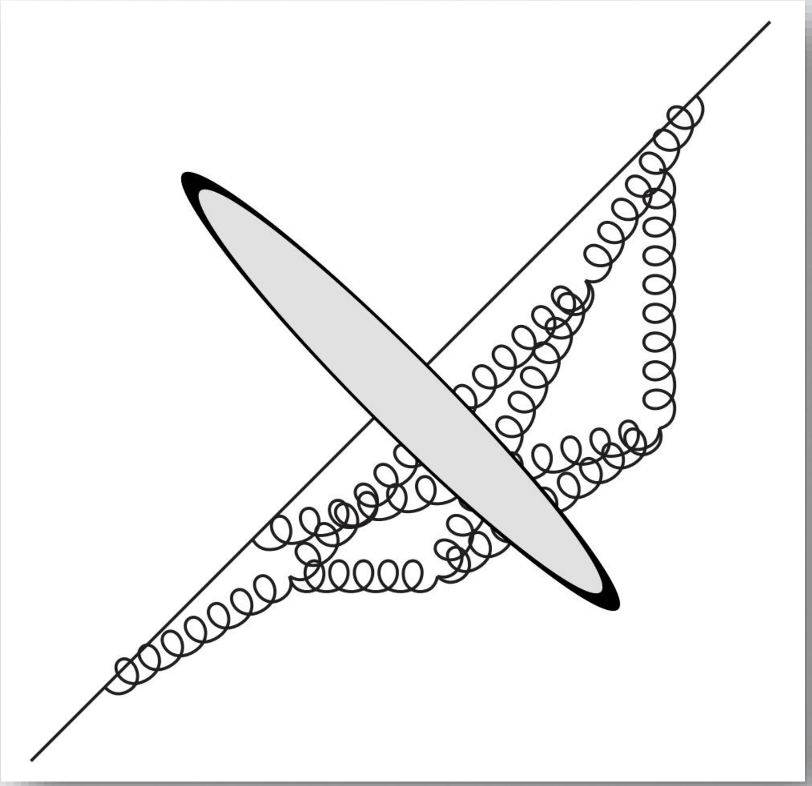
Korchemskaya, Korchemsky, 1994, 1996; Babansky, Balitsky, 2002; Caron-Huot, 2013

- To leading power, the fast projectile and target described in terms of Wilson lines:

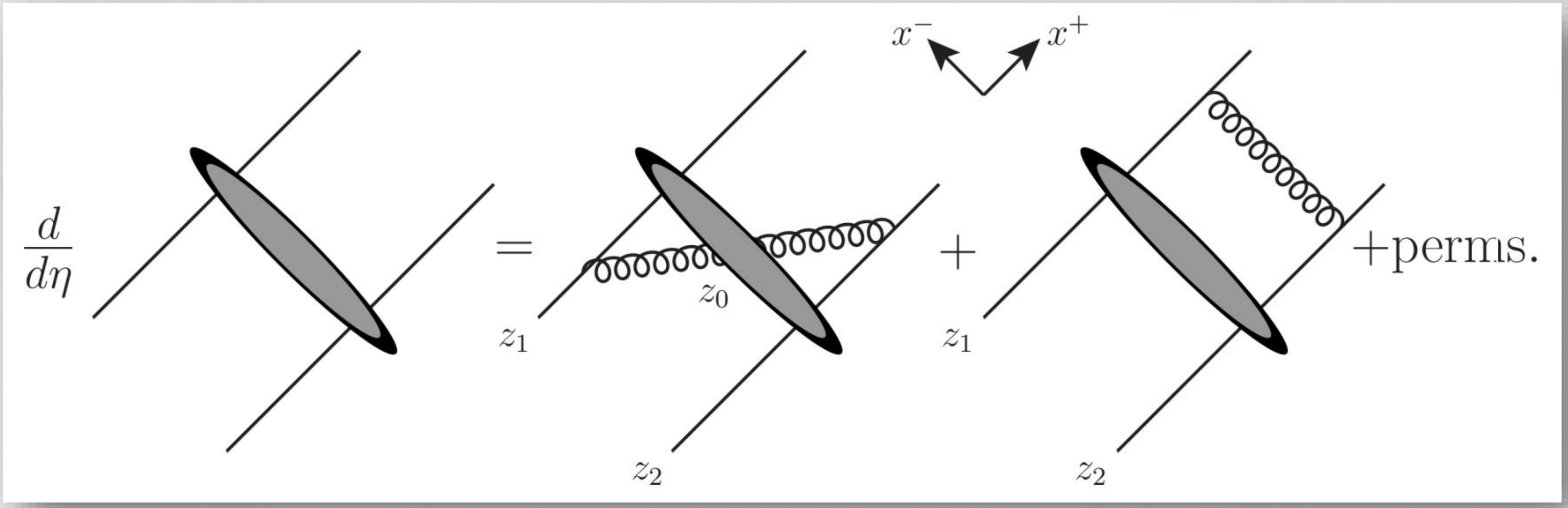
$$U(z_\perp) = \mathcal{P} \exp \left[i g_s \int_{-\infty}^{+\infty} A_+^a(x^+, x^- {=} 0, z_\perp) dx^+ T^a \right].$$

- Upon evolution in energy (rapidity), emitted radiation gives additional Wilson lines!

$$\eta = L \equiv \log \left| \frac{s}{t} \right| - i \frac{\pi}{2}.$$



FROM BALITSKY-JIMWLK TO AMPLITUDES



- This is expressed by the (nonlinear!) Balitsky-JIMWLK evolution equation:

$$\frac{d}{d\eta}UU \sim g_s^2 \int d^2 z_0\, K(z_0, z_1, z_2) \Big[U(z_0)UU - UU\Big].$$

***NLL:* Balitsky Chirilli, 2013; Kovner, Lublinsky, Mulian, 2013, 2014, 2016; *(some) NNLL:* Caron-Huot, Gardi, Vernazza, 2017.**

- Shock = Lorentz-contracted target;
- 45° lines = fast projectile partons;
- Each parton crossing the shock gets a Wilson line
- Evolution in rapidity resums the high-energy log: $\eta = L \equiv \log\left|\frac{s}{t}\right| - i\frac{\pi}{2}.$

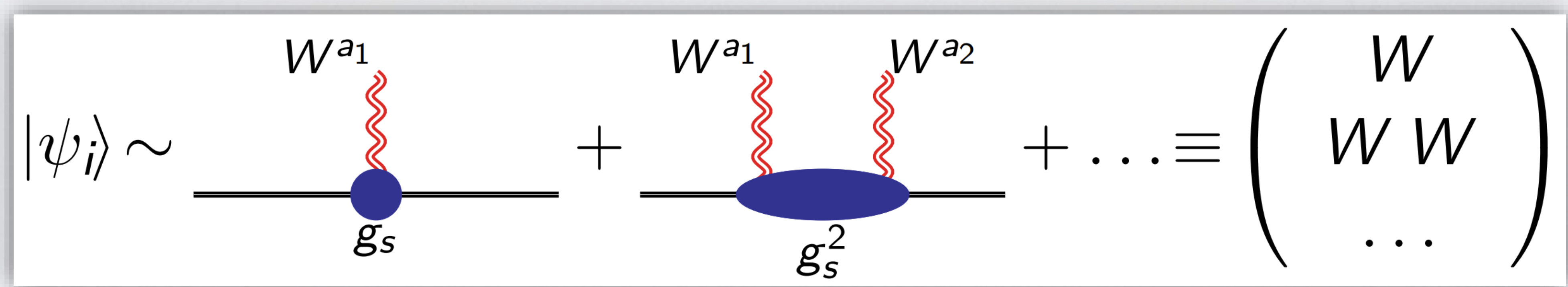
FROM BALITSKY-JIMWLK TO AMPLITUDES

- The Balitsky-JIMWLK equation is non-linear: leads to the phenomenon of saturation.
- For scattering amplitudes, we can consider the dilute regime: expand Wilson lines around unity in an effective degree of freedom dubbed as "Reggeon":

$$U^{\eta}(z_\perp) = \mathcal{P}\exp\left[ig_s\,\mathbf{T}^a \int_{-\infty}^{+\infty} dx^+\, A_+^a(x^+, x^- = 0, z_\perp)\right] \equiv e^{ig_s\,\mathbf{T}^a\,W^a(z_\perp)}\,.$$

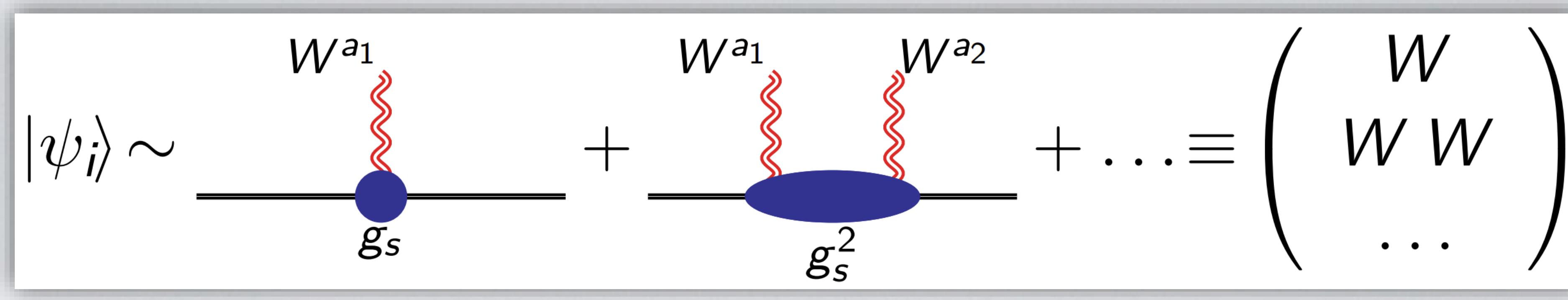
Caron-Huot, 2013

  - T^a group generator in the parton representation
  - $\eta = L$ (implicit) cutoff
- Scattering states (target and projectile) are expanded in Reggeon fields W^a:



FROM BALITSKY-JIMWLK TO AMPLITUDES

- Scattering states (target and projectile) are expanded in Reggeon fields W^a:



- Evolution in rapidity resums the high-energy log:

$$\frac{d}{dL}|\psi_i\rangle = -H|\psi_i\rangle.$$

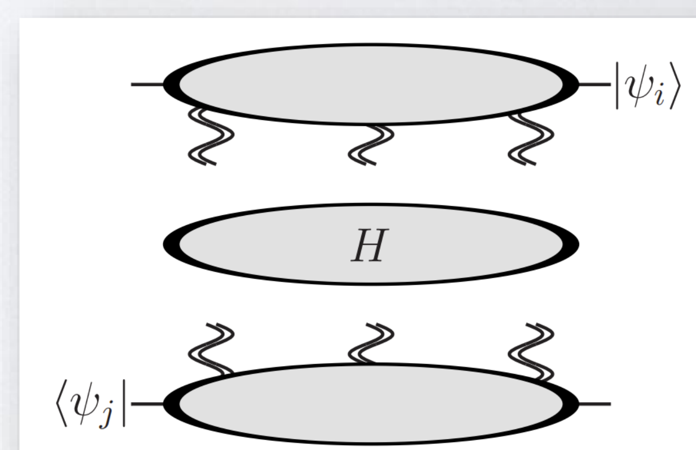
H = Balitsky-JIMWLK Hamiltonian

***NLL:* Balitsky Chirilli, 2013; Kovner, Lublinsky, Mulian, 2013, 2014, 2016; (some) *NNLL:* Caron-Huot, Gardi, Vernazza, 2017.**

- Scattering amplitude: expectation value of Wilson lines evolved to equal rapidity:

$$\frac{i}{2s}\frac{1}{Z_i Z_j}\mathcal{M}_{ij\to ij} = \langle\psi_j|e^{-LH}|\psi_i\rangle.$$

(Z_i = collinear poles)



Caron-Huot, 2013, Caron-Huot, Gardi, LV, 2017

FROM BALITSKY-JIMWLK TO AMPLITUDES

- Structure of the leading-order Balitsky-JIMWLK equation:

$$H\begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \dots \end{pmatrix} = \begin{pmatrix} H_{1\to1} & 0 & H_{3\to1} & 0 & H_{5\to1} & \dots \\ 0 & H_{2\to2} & 0 & H_{4\to2} & 0 & \dots \\ H_{1\to3} & 0 & H_{3\to3} & 0 & H_{5\to3} & \dots \\ 0 & H_{2\to4} & 0 & H_{4\to4} & 0 & \dots \\ H_{1\to5} & 0 & H_{3\to5} & 0 & H_{5\to5} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \dots \end{pmatrix}$$

$$\sim \begin{pmatrix} g_s^2 & 0 & g_s^4 & 0 & g_s^6 & \dots \\ 0 & g_s^2 & 0 & g_s^4 & 0 & \dots \\ g_s^4 & 0 & g_s^2 & 0 & g_s^4 & \dots \\ 0 & g_s^4 & 0 & g_s^2 & 0 & \dots \\ g_s^6 & 0 & g_s^4 & 0 & g_s^2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \dots \end{pmatrix}.$$

LO BFKL kernel

From LO B-JIMWLK

Terms in NNLO B-JIMWLK predicted by symmetry $H = H^T$

Caron-Huot, 2013, Caron-Huot, Gardi, LV, 2017

- At NLL we need *m* → *m* transition only → the LO BFKL kernel.
- At NNLL we need the *m* → *m+2* transition from the LO B-JIMWLK kernel.
- Define the reduced amplitude: subtract single-Reggeon exchange:

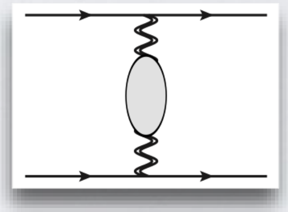
$$\frac{i}{2s}\hat{\mathcal{M}}_{ij\to ij} = \langle\psi_j|e^{-(H-H_{1\to1})L}|\psi_i\rangle \equiv \langle\psi_j|e^{-\hat{H}L}|\psi_i\rangle.$$

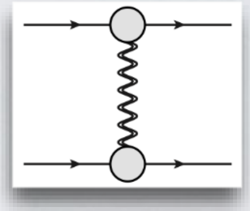
THE ODD AMPLITUDE

- A few examples: decompose the amplitude

$$\frac{i}{2s}\hat{\mathcal{M}}_{ij\to ij} \xrightarrow{\text{Regge}} \frac{i}{2s}\left(\hat{\mathcal{M}}^{(+)}_{ij\to ij} + \hat{\mathcal{M}}^{(-)}_{ij\to ij}\right) \equiv \langle\psi_j^{(+)}|e^{-\hat{H}L}|\psi_i^{(+)}\rangle + \langle\psi_j^{(-)}|e^{-\hat{H}L}|\psi_i^{(-)}\rangle.$$

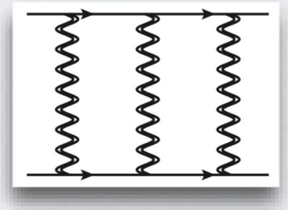
- One has e.g.:





$$\frac{i}{2s}\hat{\mathcal{M}}^{(-)\,\text{1-loop}}_{ij\to ij} = -\,L\,\langle\psi_{j,1}|\,\hat{H}_{1\to 1}\,|\psi_{i,1}\rangle^{\text{LO}} + \langle\psi_{j,1}|\psi_{i,1}\rangle^{\text{NLO}},$$

$$\frac{i}{2s}\hat{\mathcal{M}}^{(-)\,\text{2-loops}}_{ij\to ij} = \frac{L^2}{2}\,\langle\psi_{j,1}|\,(\hat{H}_{1\to 1})^2\,|\psi_{i,1}\rangle^{\text{LO}} - L\,\langle\psi_{j,1}|\,\hat{H}_{1\to 1}\,|\psi_{i,1}\rangle^{\text{NLO}} + \langle\psi_{j,3}|\psi_{i,3}\rangle^{\text{LO}} + \langle\psi_{j,1}|\psi_{i,1}\rangle^{\text{NNLO}}.$$



- Taking into account that

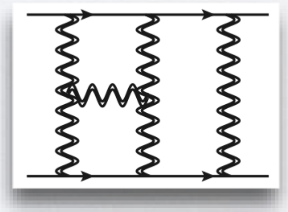
$$\langle\psi_{j,1}|\,\hat{H}_{1\to 1}O\,|\psi_{i,n}\rangle = \langle\psi_{j,n}|\,O\hat{H}_{1\to 1}\,|\psi_{i,1}\rangle = 0,$$

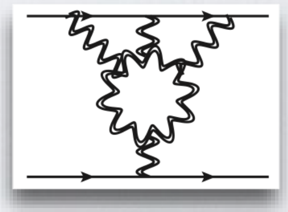
up to three loops the odd amplitude reads

Caron-Huot, Gardi, LV, 2017

$$\frac{i}{2s}\hat{\mathcal{M}}^{(-)\,\text{1-loop}}_{ij\to ij} = \langle\psi_{j,1}|\psi_{i,1}\rangle^{\text{NLO}},$$

$$\frac{i}{2s}\hat{\mathcal{M}}^{(-)\,\text{2-loops}}_{ij\to ij} = \langle\psi_{j,3}|\psi_{i,3}\rangle^{\text{LO}} + \langle\psi_{j,1}|\psi_{i,1}\rangle^{\text{NNLO}},$$





$$\frac{i}{2s}\hat{\mathcal{M}}^{(-)\,\text{3-loops}}_{ij\to ij} = -\,L\Big[\langle\psi_{j,3}|\,\hat{H}_{3\to 3}\,|\psi_{i,3}\rangle + \langle\psi_{j,3}|\,\hat{H}_{1\to 3}\,|\psi_{i,1}\rangle + \langle\psi_{j,1}|\,\hat{H}_{3\to 1}\,|\psi_{i,3}\rangle\Big]^{\text{LO}}$$
$$+\,\langle\psi_{j,3}|\psi_{i,3}\rangle^{\text{NLO}} + \langle\psi_{j,1}|\psi_{i,1}\rangle^{(\text{N}^3\text{LO})}.$$

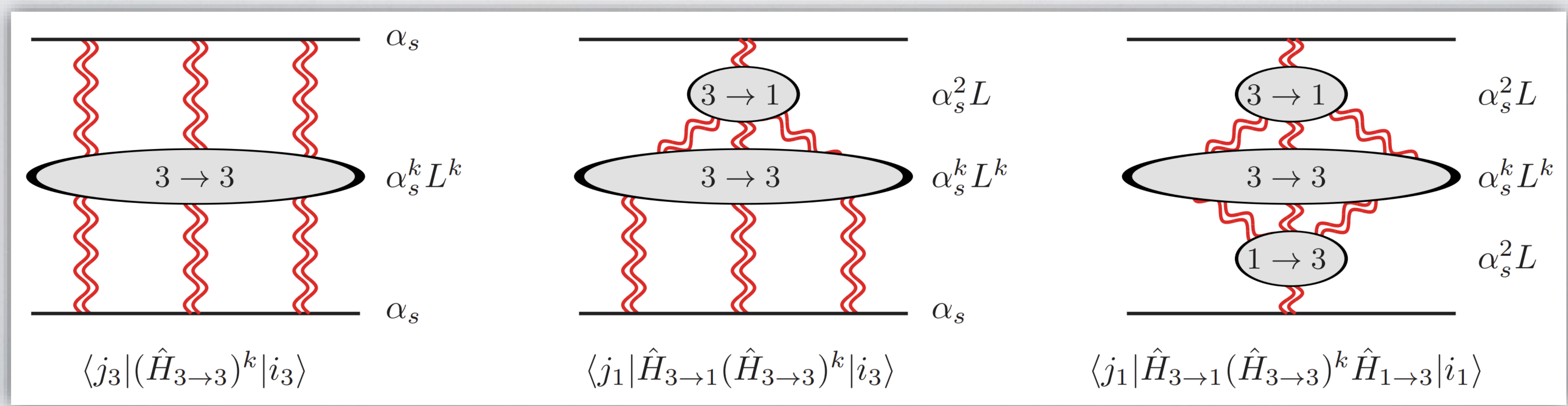
THE ODD AMPLITUDE

- To all orders the amplitude takes the form

Falcioni, Gardi, Milloy, LV, 2020; Falcioni, Gardi, Maher, Milloy, LV, 2021

$$\frac{i}{2s}\hat{\mathcal{M}}^{(-),\mathrm{NNLL}}_{ij\to ij} = \left(\frac{\alpha_s}{\pi}\right)^2 \Bigg\{ r_\Gamma^2 \pi^2 \Bigg[\sum_{k=0}^{\infty} \frac{(-X)^k}{k!} \langle j_3 | \hat{H}^k_{3\to 3} | i_3 \rangle + \sum_{k=1}^{\infty} \frac{(-X)^k}{k!} \Big[\langle j_1 | \hat{H}_{3\to 1} \hat{H}^{k-1}_{3\to 3} | i_3 \rangle + \langle j_3 | \hat{H}^{k-1}_{3\to 3} \hat{H}_{1\to 3} | i_1 \rangle \Big] + \sum_{k=2}^{\infty} \frac{(-X)^k}{k!} \langle j_1 | \hat{H}_{3\to 1} \hat{H}^{k-2}_{3\to 3} \hat{H}_{1\to 3} | i_1 \rangle \Bigg]^{\mathrm{LO}} + \langle j_1 | i_1 \rangle^{\mathrm{NNLO}} \Bigg\} .$$

- In diagrams:



THE ODD AMPLITUDE

- Effective Hamiltonian: one has

Falcioni, Gardi, Milloy, LV, 2020; Falcioni, Gardi, Maher, Milloy, LV, 2021

$$H_{k\to k} = A_{k\to k} + B_{k\to k},$$

with

$$A_{k\to k} = -\int [dp]\, C_A\, \alpha_g(p^2,\mu^2)\, W^a(p) \frac{\delta}{\delta W^a(p)},$$

$$B_{k\to k} = \alpha_s(\mu^2) \int [đq][dp_1][dp_2]\, H_{22}(q;p_1,p_2)\, W^x(p_1+q) W^y(p_2-q)\, (F^x F^y)^{ab} \frac{\delta}{\delta W^a(p_1)} \frac{\delta}{\delta W^b(p_2)},$$

where

$$H_{22}(q;p_1,p_2) = \frac{(p_1+p_2)^2}{p_1^2 p_2^2} - \frac{(p_1+q)^2}{p_1^2 q^2} - \frac{(p_2-q)^2}{q^2 p_2^2}.$$

Furthermore

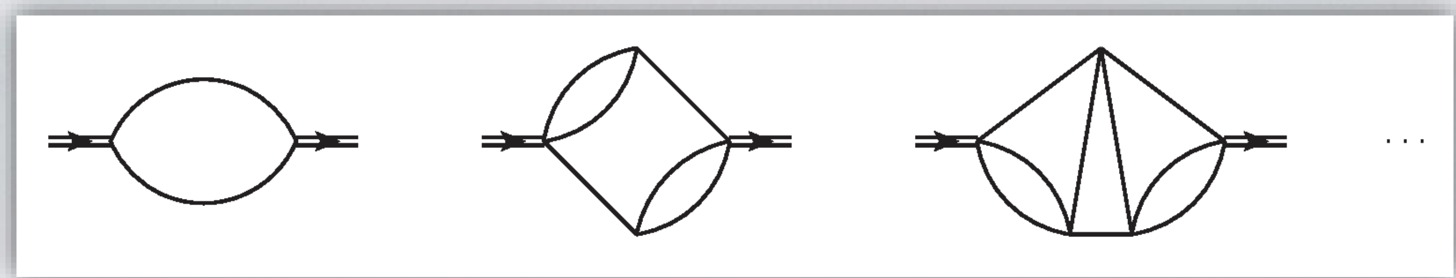
$$H_{1\to 3} = \alpha_s^2(\mu^2) \int [đp_1][đp_2][dp]\, \mathrm{Tr}[F^a F^b F^c F^d]\, W^b(p_1) W^c(p_2) W^d(p_3)\, H_{13}(p_1,p_2,p_3) \frac{\delta}{\delta W^a(p)},$$

with

$$H_{13}(p_1,p_2,p_3) = \frac{r_\Gamma}{3\epsilon} \left[\left(\frac{\mu^2}{(p_1+p_2+p_3)^2}\right)^\epsilon + \left(\frac{\mu^2}{p_2^2}\right)^\epsilon - \left(\frac{\mu^2}{(p_1+p_2)^2}\right)^\epsilon - \left(\frac{\mu^2}{(p_2+p_3)^2}\right)^\epsilon \right].$$

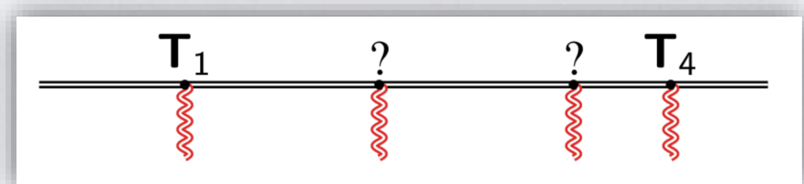
THE ODD AMPLITUDE

- Two tasks: 1) evaluate (Euclidean) integrals in *d = 2-2ε* dimensions:



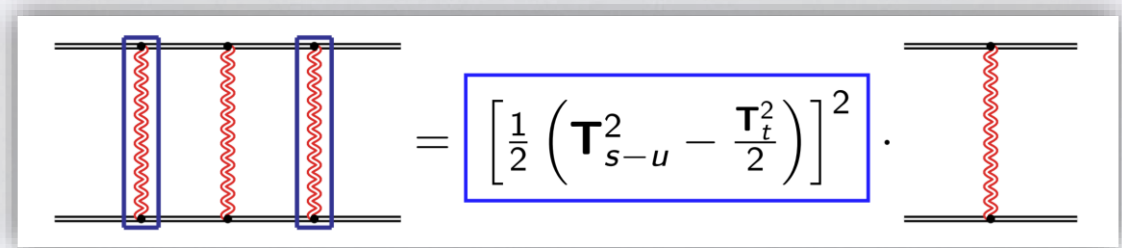
- 2) Express the color factors as operators acting on the tree level amplitude:

Outmost generators clearly associated with external particles

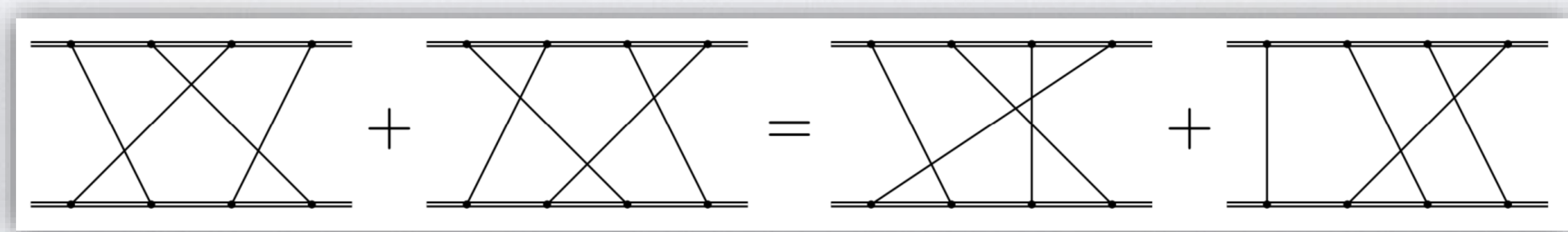


Caron-Huot, Gardi, LV, 2017; Falcioni, Gardi, Milloy, LV, 2020; Falcioni, Gardi, Maher, Milloy, LV, 2021

At lowest order there is no ambiguity



Starting at three loops one has entangled contributions: needs identities such as



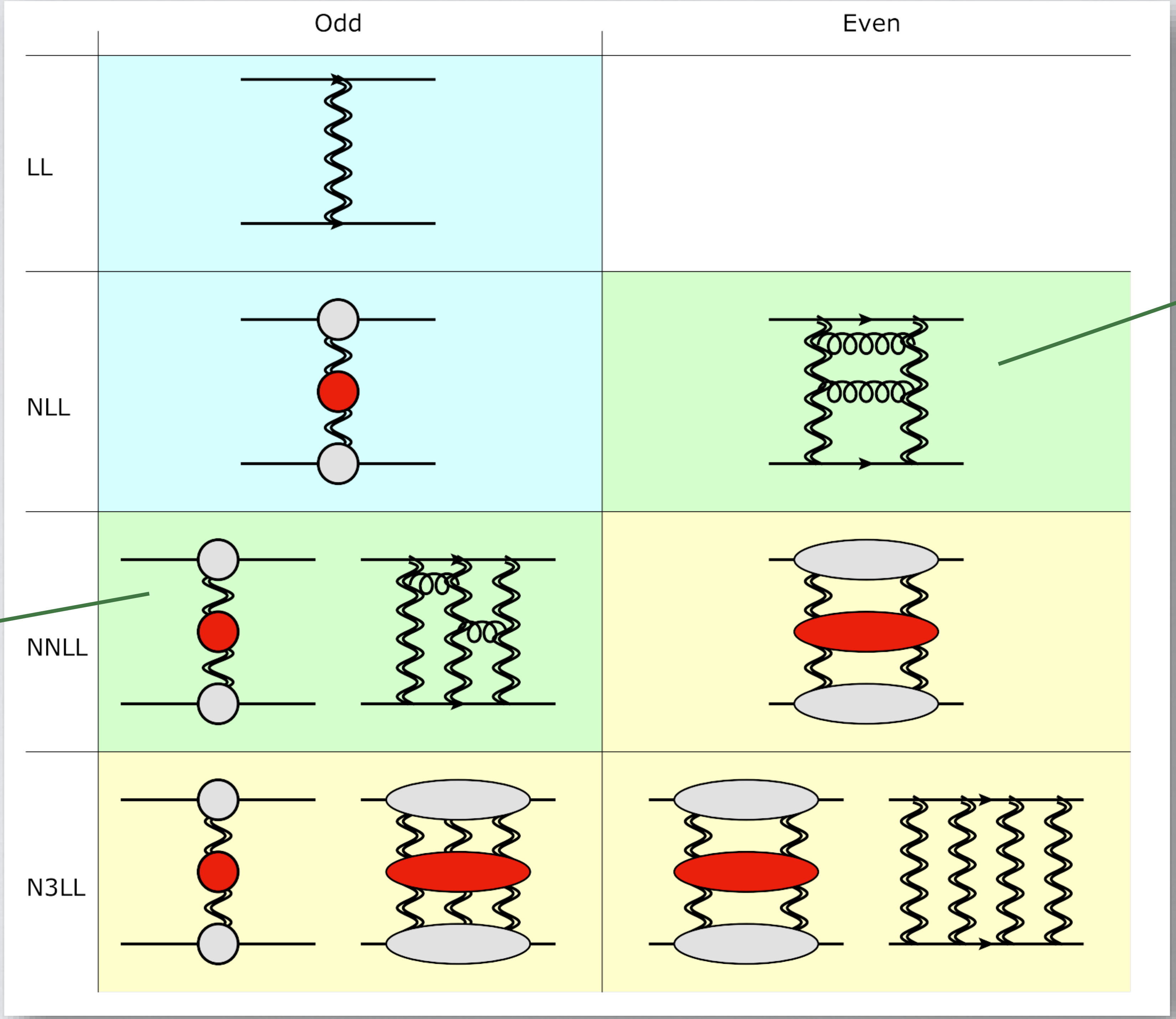
TWO PARTON SCATTERING AMPLITUDES

- We have now a framework for the calculation of amplitudes in the high-energy limit;
- Systematic relation between logarithmic accuracy and number of Reggeons.

Analysed to 2 loops in Del Duca, Falcioni, Magnea, Vernazza 2014;

Calculated to 3 loops in Caron-Huot, Gardi, LV, 2017;

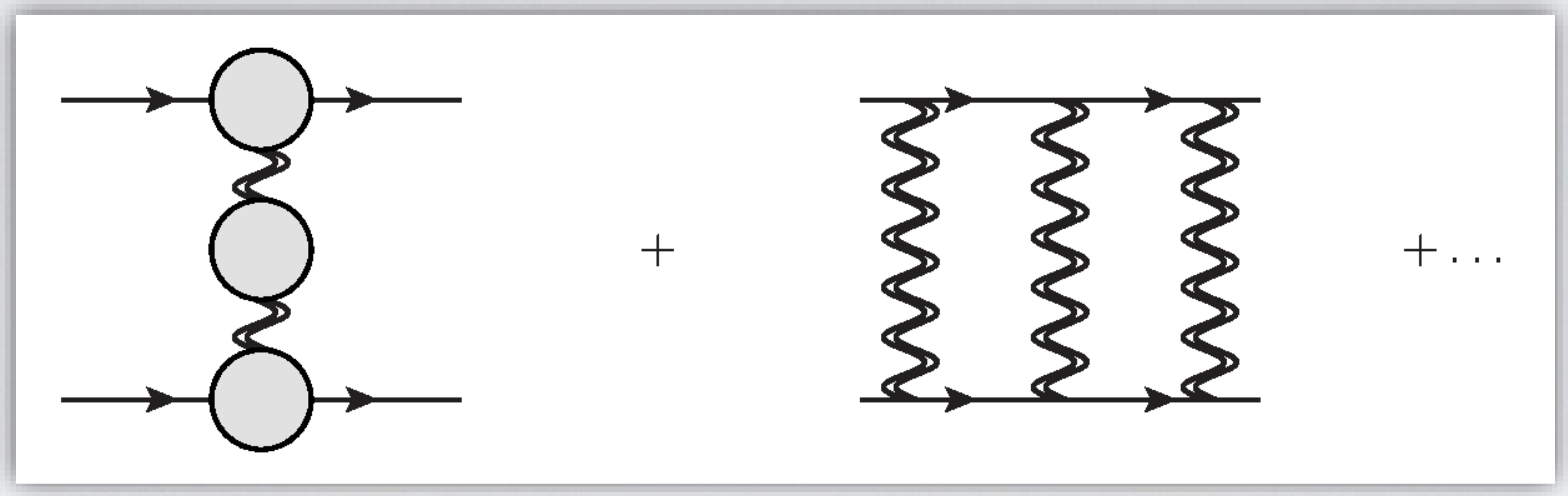
Calculated to 4 loops in Falcioni, Gardi, Milloy, LV, 2020; Falcioni, Gardi, Maher, Milloy, LV, 2021.



IR rivergences calculated to all orders in Caron-Huot, Gardi, Reichel, LV, 2017;

Finite terms calculated to 13 loops in Caron-Huot, Gardi, Reichel, LV, 2020.

TASK 2: REGGE POLE AND CUT



THE ODD AMPLITUDE

- Result at two and three loops:

$$\mathcal{M}^{(-,2,0)} = \Bigg[\underbrace{C_i^{(2)} + C_j^{(2)} + C_i^{(1)} C_j^{(1)}}_{\text{SR}} \underbrace{+\pi^2 r_\Gamma^2 \, S^{(2)}(\epsilon) \left((\mathbf{T}_{s-u}^2)^2 - \frac{1}{12} C_A^2 \right)}_{\text{MR}} \Bigg] \mathcal{M}_{\text{tree}},$$

$$\hat{\mathcal{M}}^{(-,3,1)} = \Bigg\{ \underbrace{C_A \Big[\alpha_g^{(3)} + \alpha_g^{(2)} \Big(C_i^{(1)} + C_j^{(1)} \Big) + \alpha_g^{(1)} \Big(C_i^{(2)} + C_j^{(2)} + C_i^{(1)} C_j^{(1)} \Big) \Big]}_{\text{SR}}$$

$$\underbrace{-\pi^2 r_\Gamma^3 \Big[S_A^{(3)}(\epsilon) \mathbf{T}_{s-u}^2 \left[\mathbf{T}_{s-u}^2, \mathbf{T}_t^2 \right] + S_B^{(3)}(\epsilon) \left[\mathbf{T}_{s-u}^2, \mathbf{T}_t^2 \right] \mathbf{T}_{s-u}^2 + S_C^{(3)}(\epsilon) C_A^3 \Big]}_{\text{MR}} \Bigg\} \mathcal{M}_{\text{tree}},$$

where

$$S^{(2)}(\epsilon) = -\frac{1}{8\epsilon^2} + \frac{3}{4}\epsilon\zeta_3 + \frac{9}{8}\epsilon^2\zeta_4 + \mathcal{O}(\epsilon^3), \qquad S_A^{(3)}(\epsilon) = \frac{1}{48\epsilon^3} + \frac{37\,\hat{\zeta}_3}{24} + \mathcal{O}\left(\epsilon^2\right),$$

$$S_B^{(3)}(\epsilon) = \frac{1}{24\epsilon^3} + \frac{\hat{\zeta}_3}{12} + \mathcal{O}\left(\epsilon^2\right), \qquad S_C^{(3)}(\epsilon) = -\frac{1}{432} \left(\frac{1}{2\epsilon^3} - 35\hat{\zeta}_3 + \mathcal{O}\left(\epsilon^2\right) \right).$$

- Matching to the explicit calculation of the amplitude gives the Regge trajectory and impact factors in the "SR/MR" scheme.

Caola, Chakraborty, Gambuti, von Manteuffel, Tancredi, 2021

REGGE POLE AND CUT

- We have written the amplitude in terms of SR/MR exchange contribution:

$$\mathcal{M}^{(-)} = \mathcal{M}^{(-),\,\mathrm{SR}} + \mathcal{M}^{(-),\,\mathrm{MR}} = e^{C_A \alpha_g(t) L}\, C_i(t)\, C_j(t)\, \mathcal{M}^{\mathrm{tree}}_{ij\to ij} + \mathcal{M}^{(-),\,\mathrm{MR}}\,.$$

- It would be good to express the amplitude in terms of Regge pole and cut contribution:

$$\mathcal{M}^{(-)}_{ij\to ij} = \mathcal{M}^{(-),\,\mathrm{pole}}_{ij\to ij} + \mathcal{M}^{(-),\,\mathrm{cut}}_{ij\to ij} = e^{C_A \tilde{\alpha}_g(t) L}\, \tilde{C}_i(t)\, \tilde{C}_j(t)\, \mathcal{M}^{\mathrm{tree}}_{ij\to ij} + \mathcal{M}^{(-),\,\mathrm{cut}}_{ij\to ij}\,.$$

- This task is non-trivial, because the high-energy analytic properties are only manifest upon resumming the entire perturbative series: it is not at all obvious how to disentangle the Regge pole from the Regge cut in an order-by-order computation.
- However, we have some "guiding principles" which can help with this task:
- The pole contribution is universal:

$$\left(\frac{\mathcal{M}^{(-)}_{qg\to qg}|_{\mathrm{NLL}}}{\mathcal{M}^{\mathrm{tree}}_{qg\to qg}}\right)^2 = \frac{\mathcal{M}^{(-)}_{gg\to gg}|_{\mathrm{NLL}}}{\mathcal{M}^{\mathrm{tree}}_{gg\to gg}} \cdot \frac{\mathcal{M}^{(-)}_{qq\to qq}|_{\mathrm{NLL}}}{\mathcal{M}^{\mathrm{tree}}_{qq\to qq}}\,.$$

- The pole has a "good" infrared behaviour: ***Korchemskaya, Korchemsky, 1994,1996***

$$\tilde{\alpha}_g(t) = K + \mathcal{O}(\epsilon^0)\,, \qquad K(\alpha_s(\mu^2)) \equiv -\frac{1}{4}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\gamma_K(\alpha_s(\lambda^2)) = \frac{1}{2\epsilon}\frac{\alpha_s(\mu^2)}{\pi} + \ldots$$

where K is the integral over the scale of the cusp anomalous dimension.

REGGE POLE AND CUT

- While the Regge cut arises exclusively due to MR contributions to the amplitude, MR exchanges do contribute also to the Regge pole.
- This is evident in the large-Nc limit, where it is known that the amplitude only features a Regge pole, and yet, MR contributions are present.

Eden, Landshoff, Olive, Polkinghorne, 1966; P. D. B. Collins, 2009

- It is also known that Regge cuts only arise due to nonplanar diagrams: the Regge cut should be identified as the nonplanar part of the MR contribution, while the Regge pole corresponds to SR plus the planar MR contributions:

Mandelstam 1963; P. D. B. Collins 2009

- Putting together all these requirements, we make the ansatz:

Gardi, Falcioni, Maher, Milloy, LV, 2021.

$$\begin{aligned}\mathcal{M}^{(-)}_{ij\to ij} &= \underbrace{\mathcal{M}^{(-)\,\mathrm{SR}}_{ij\to ij} + \mathcal{M}^{(-)\,\mathrm{MR}}_{ij\to ij}\Big|_{\mathrm{planar}}} + \underbrace{\mathcal{M}^{(-)\,\mathrm{MR}}_{ij\to ij}\Big|_{\mathrm{nonplanar}}} \\ &= \qquad\quad \mathcal{M}^{(-)\,\mathrm{pole}}_{ij\to ij} \qquad\quad + \qquad \mathcal{M}^{(-)\,\mathrm{cut}}_{ij\to ij}\,.\end{aligned}$$

REGGE POLE AND CUT

- In the end we get

Gardi, Falcioni, Maher, Milloy, LV, 2021.

$$\tilde{C}^{(2)}_{i/j} = C^{(2)}_{i/j} + N_c^2 (r_\Gamma)^2 \frac{\pi^2}{12} S^{(2)}(\epsilon),$$

$$\tilde{\alpha}^{(3)}_g = \alpha^{(3)}_g - (r_\Gamma)^3 N_c^2 \frac{\pi^2}{18} \left(S_A^{(3)}(\epsilon) - S_B^{(3)}(\epsilon) \right).$$

explicitly:

$$\begin{aligned}\tilde{\alpha}^{(3)}_g &= K^{(3)} + C_A^2 \left(\frac{297029}{93312} - \frac{799\zeta_2}{1296} - \frac{833\zeta_3}{216} - \frac{77\zeta_4}{192} + \frac{5}{24}\zeta_2\zeta_3 + \frac{\zeta_5}{4} \right) \\ &+ C_A n_f \left(\frac{103\zeta_2}{1296} + \frac{139\zeta_3}{144} - \frac{5\zeta_4}{96} - \frac{31313}{46656} \right) \\ &+ C_F n_f \left(\frac{19\zeta_3}{72} + \frac{\zeta_4}{8} - \frac{1711}{3456} \right) + n_f^2 \left(\frac{29}{1458} - \frac{2\zeta_3}{27} \right) + \mathcal{O}(\epsilon).\end{aligned}$$

- The Regge-pole contribution is universal among all two-parton scattering processes, but theory dependent (i.e. different in N=4 SYM, QCD, etc);
- The Regge–cut contribution is different for each channel but depends only on the action of color operators in the gauge theory considered.

See also: Gao, Moult, Raman, Ridgway, Stewart 2023; Fadin 2023

REGGE POLE AND CUT

- The scheme dependence does not give rise to infinite freedom: once the impact factors at two loops and the Regge trajectory at three loops have been fixed, there are no more free parameters at NNLL to be adjusted.
- Consequence: from four loops all MR contributions must be entirely nonplanar!
- We have verified this explicitly at four loops:

$$\mathcal{M}^{(-,4,2)} = \frac{\pi^2 r_\Gamma^4}{2}\left[\frac{1}{\epsilon^4}\mathbf{K}^{(4)} + \left(\frac{1}{\epsilon}\zeta_3 + \frac{3}{2}\zeta_4\right)\mathbf{K}^{(1)} + \mathcal{O}(\epsilon)\right]\hat{\mathcal{M}}_{\text{tree}},$$

where the color operators reads

Gardi, Falcioni, Maher, Milloy, LV, 2021

$$\mathbf{K}^{(4)} = \frac{1}{96}\Big[\mathbf{T}^2_{s-u},\Big[\mathbf{T}^2_{s-u},\mathbf{T}^2_t\Big]\Big]\mathbf{T}^2_t + \frac{7}{576}\mathbf{T}^2_t\left[\left(\mathbf{T}^2_{s-u}\right)^2,\mathbf{T}^2_t\right]$$
$$- \frac{1}{192}\left[\mathbf{T}^2_{s-u},\mathbf{T}^2_t\right]\mathbf{T}^2_t\mathbf{T}^2_{s-u} - \frac{5}{192}\mathbf{T}^2_{s-u}\left[\mathbf{T}^2_{s-u},\mathbf{T}^2_t\right]\mathbf{T}^2_t,$$

$$\mathbf{K}^{(1)} = \frac{49}{48}\Big[\mathbf{T}^2_{s-u},\Big[\mathbf{T}^2_{s-u},\mathbf{T}^2_t\Big]\Big]\mathbf{T}^2_t - \frac{47}{288}\mathbf{T}^2_t\left[\left(\mathbf{T}^2_{s-u}\right)^2,\mathbf{T}^2_t\right]$$
$$+ \frac{101}{96}\left[\mathbf{T}^2_{s-u},\mathbf{T}^2_t\right]\mathbf{T}^2_t\mathbf{T}^2_{s-u} - \frac{49}{48}\mathbf{T}^2_{s-u}\left[\mathbf{T}^2_{s-u},\mathbf{T}^2_t\right]\mathbf{T}^2_t + \frac{1}{24}\left(\frac{d_{AA}}{N_A} - \frac{C_A^4}{24}\right).$$

CONCLUSION

- Modern approach to high-energy scattering via Wilson lines:
  - → new theoretical control up to NNLL.
  - → $2 \to 2$ amplitudes obtained by iteration of the Balitsky-JIMWLK Hamiltonian.
- Imaginary part at NLL to all orders in the strong coupling: (not discussed here)
  - → Extracted the soft anomalous dimension to all orders;
  - → Numerical studies on the convergence of the perturbative expansion.
- Real part at NNLL up to four loops:
  - → Extracted the corresponding term of the soft anomalous dimension;
  - → Real part of the $2 \to 2$ amplitude in QCD and N=4 SYM at four loops.
  - → Identified the Regge pole as the planar contribution of single- and multi-Reggeon exchange, and the Regge cut as the non-planar part of the multi-Reggeon exchange.

EXTRA SLIDES

REGGE VS INFRARED FACTORISATION

- Applications: 1) test (and predict) the analytic structure of infrared divergences.
- The infrared divergences of amplitudes are controlled by a renormalization group equation:

$$\mathcal{M}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \mathbf{Z}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right)\mathcal{H}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right),$$

where $\boldsymbol{Z}_n$ is given as a path-ordered exponential of the soft-anomalous dimension:

Becher, Neubert, 2009; Gardi, Magnea, 2009

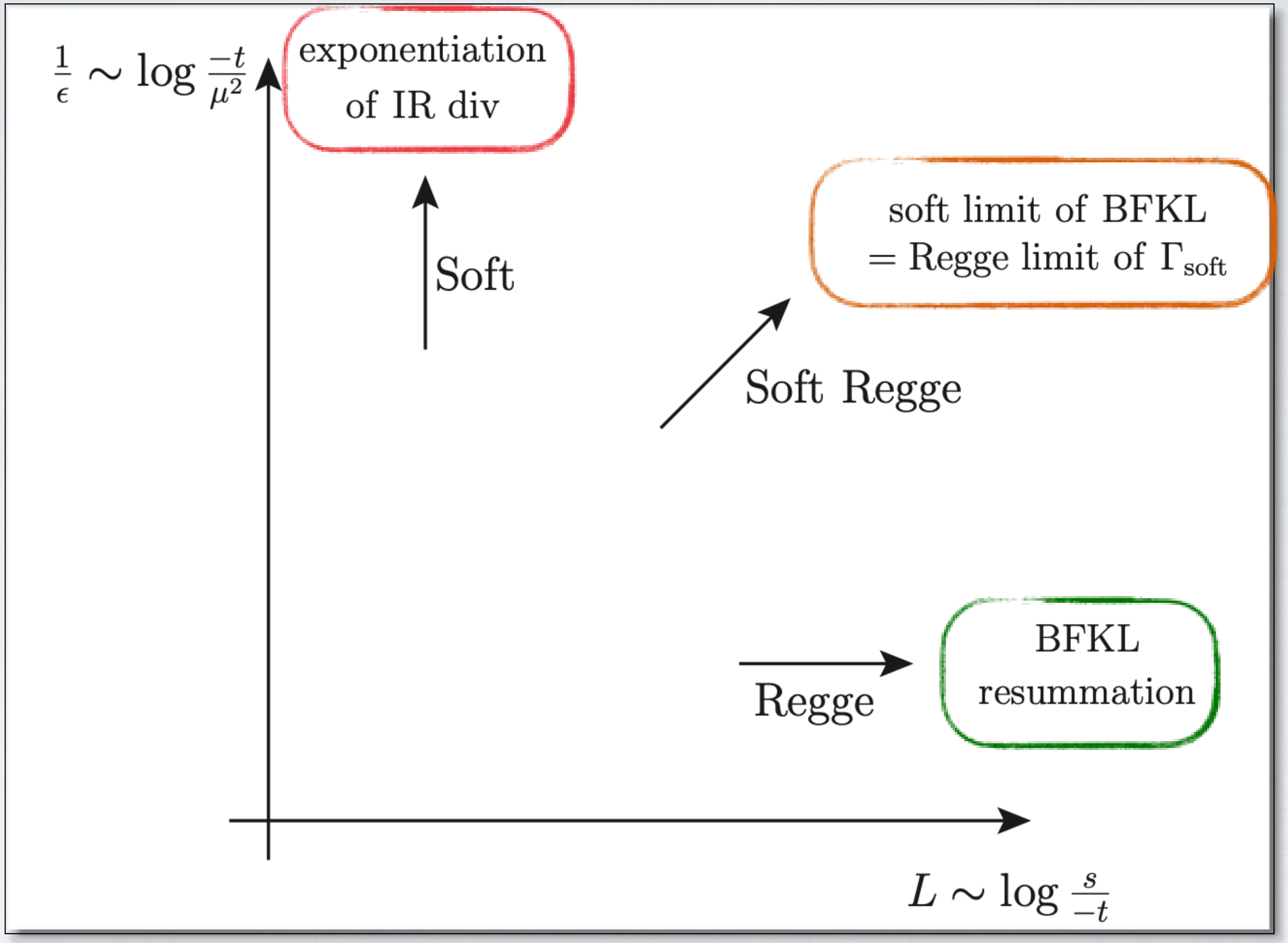
$$\mathbf{Z}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \mathcal{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right)\right\},$$

- The soft anomalous dimension for scattering of massless partons is an operator in color space given by

$$\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \mathbf{\Gamma}_n^{\text{dip.}}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) + \mathbf{\Delta}_n\left(\{\rho_{ijkl}\}\right).$$

- Given M_n as calculated in the high-energy limit, use IR factorisation to extract the soft anomalous dimension.

REGGE VS INFRARED FACTORISATION



- Use amplitudes calculated in the high-energy limit to extract the soft anomalous dimension in that limit;
- Bootstrap the result to constrain the structure of infrared divergences in general kinematic.

REGGE VS INFRARED FACTORISATION

Re	L^0	L^1	L^2	L^3	L^4	L^5	L^6
α_s^1	$\frac{1}{4}\widehat{\gamma}_K^{(1)} \ln\frac{-t}{\lambda^2}\sum_{i=1}^4 C_i + \sum_{i=1}^4 \gamma_i^{(1)}$	$\frac{1}{2}\widehat{\gamma}_K^{(1)}\mathbf{T}_t^2$					
α_s^2	$\frac{1}{4}\widehat{\gamma}_K^{(2)} \ln\frac{-t}{\lambda^2}\sum_{i=1}^4 C_i + \sum_{i=1}^4 \gamma_i^{(2)}$	$\frac{1}{2}\widehat{\gamma}_K^{(2)}\mathbf{T}_t^2$	0				
α_s^3	$\frac{1}{4}\widehat{\gamma}_K^{(3)} \ln\frac{-t}{\lambda^2}\sum_{i=1}^4 C_i + \sum_{i=1}^4 \gamma_i^{(3)} + \Delta^{(+,3,0)}$	$\frac{1}{2}\widehat{\gamma}_K^{(3)}\mathbf{T}_t^2$	0	0			
α_s^4			$\Delta^{(+,4,2)}$	0	0		
α_s^5					0	0	
α_s^6						0	0

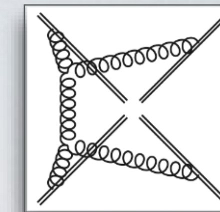
Im	L^0	L^1	L^2	L^3	L^4	L^5	L^6
α_s^1	$\frac{1}{2}\widehat{\gamma}_K^{(1)} i\pi\mathbf{T}_{s-u}^2$	0					
α_s^2	$\frac{1}{2}\widehat{\gamma}_K^{(2)} i\pi\mathbf{T}_{s-u}^2$	0	0				
α_s^3	$\frac{1}{2}\widehat{\gamma}_K^{(3)} i\pi\mathbf{T}_{s-u}^2 + \Delta^{(-,3,0)}$	$\Delta^{(-,3,1)}$	0	0			
α_s^4				$\Delta^{(-,4,3)}$	0		
α_s^5					$\Delta^{(-,5,4)}$	0	
α_s^6						$\Delta^{(-,6,5)}$	0

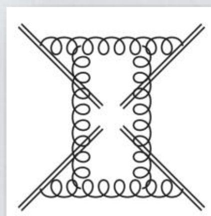
Caron-Huot, Gardi, LV, 2017; Caron-Huot, Gardi, Reichel, LV, 2017; Gardi, Falcioni, Milloy, LV, 2020; Gardi, Falcioni, Maher, Milloy, LV, 2021.

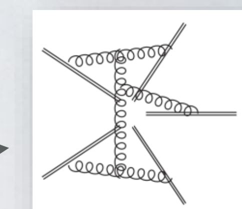
REGGE VS INFRARED FACTORISATION

- Structure of the soft anomalous dimension in general kinematic up to four loops:

$$\begin{aligned}\mathbf{\Gamma}_n\big(\{s_{ij}\},\mu,\alpha_s(\mu^2)\big) =& -\frac{\gamma_K(\alpha_s)}{4}\sum_{(i,j)}\mathbf{T}_i\cdot\mathbf{T}_i\,\log\frac{-s_{ij}}{\mu^2}+\sum_i\gamma_i(\alpha_s)\\ &+f(\alpha_s)\sum_{(i,j,k)}\mathcal{T}_{iikj}+\sum_{(i,j,k,l)}\mathcal{T}_{ijkl}\,\mathcal{F}(\beta_{ijlk},\beta_{iklj};\alpha_s)\\ &-\sum_R\frac{g^R(\alpha_s)}{2}\Bigg[\sum_{(i,j)}\big(\mathcal{D}^R_{iijj}+2\mathcal{D}^R_{iiij}\big)\ln\frac{-s_{ij}}{\mu^2}+\sum_{(i,j,k)}\mathcal{D}^R_{ijkk}\ln\frac{-s_{ij}}{\mu^2}\Bigg]\\ &+\sum_R\sum_{(i,j,k,l)}\mathcal{D}^R_{ijkl}\mathcal{G}^R(\beta_{ijlk},\beta_{iklj};\alpha_s)+\sum_{(i,j,k,l)}\mathcal{T}_{ijkli}\mathcal{H}_1(\beta_{ijlk},\beta_{iklj};\alpha_s)\\ &+\sum_{(i,j,k,l,m)}\mathcal{T}_{ijklm}\mathcal{H}_2(\beta_{ijkl},\beta_{ijmk},\beta_{ikmj},\beta_{jiml},\beta_{jlmi};\alpha_s)+\mathcal{O}(\alpha_s^5).\end{aligned}$$







- From the Regge limit we obtain constrains, useful for a bootstrap approach:

Gardi, Falcioni, Maher, Milloy, LV, 2021.

	Signature even				Signature odd		
	L^3	L^2	L^1 (conj.)		L^3	L^2	L^1
$\mathcal{F}_A^{(+,4)}$	0	$-\frac{C_A}{8}\zeta_2\zeta_3$	0	$\mathcal{F}_A^{(-,4)}$	$i\pi\frac{C_A}{24}\zeta_3$	?	?
$\mathcal{F}_F^{(+,4)}$	0	0	0	$\mathcal{F}_F^{(-,4)}$	0	?	?
$\mathcal{G}_A^{(+,4)}$	0	$\frac{1}{2}\zeta_2\zeta_3$	$\frac{1}{6}g_A^{(4)}$				
$\mathcal{G}_F^{(+,4)}$	0	0	$\frac{1}{6}g_F^{(4)}$				
$\mathcal{H}_1^{(+,4)}$	0	0	0	$\mathcal{H}_1^{(-,4)}$	0	?	?
				$\tilde{\mathcal{H}}_1^{(-,4)}$	0	?	?

See e.g. Almelid, Duhr, Gardi, McLeod, White, 2017