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Exploring type-I seesaw under S_3 modular symmetry

[In collaboration with Dr. Pawin Ittisamai, Dr. Chakrit Pongkitivanichkul, Dr. Patipan Uttayarat]

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Flavor Physics and CP Violation (FPCP)

22nd Conference

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Introduction

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S_3 symmetry

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S_3 symmetry

- The symmetric group S_3, also known as the symmetric group on three elements, is the group of all permutations of a three-element set.
- In other words, it consists of all possible ways to rearrange three distinct objects, i.e., $3! = 6$.

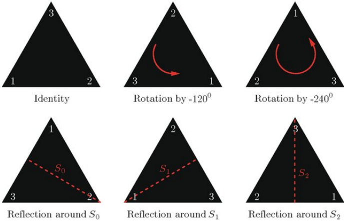
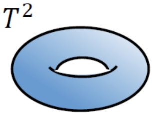


Figure: Symmetries of the equilateral triangle isomorphic to S_3 discrete symmetry

Drawbacks

- Previously, many models have used discrete symmetry to explain neutrino phenomenology.
- But while doing so, there are many flavon fields (scalar multiplets) involved in the play, due to which higher dimensional operators are non-renormalizable in nature.
- These higher dimensional operator comes with unknown coefficients which reduces the predictions of the model.
- In addition to the above, it is also difficult to handle the VEV alignments of these multiple flavon fields.
- Hence **MODULAR SYMMETRY** comes to our rescue discussed below.

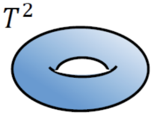
Modular Symmetry



Adapted from Morimitsu Tanimoto ppt

- The modular symmetry is a geometrical symmetry of the two-dimensional torus, T^2.

Modular Symmetry



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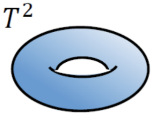
- The modular symmetry is a geometrical symmetry of the two-dimensional torus, T^2.
- The two-dimensional torus is constructed as division of the two-dimensional Euclidean space R^2 by a lattice Λ, $T^2 = \frac{R^2}{\Lambda}$.

Modular Symmetry



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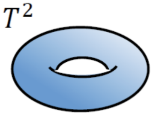
- Instead of R^2, one can use the one-dimensional complex plane.

Modular Symmetry



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- The modular symmetry is a geometrical symmetry of the two-dimensional torus, T^2.
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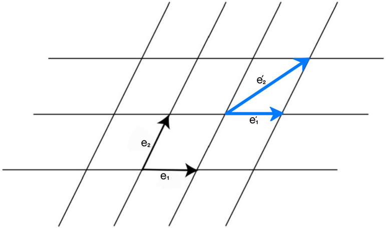
- Instead of R^2, one can use the one-dimensional complex plane.

- The lattice is spanned by two basis vectors, e_1 and e_2 as $m_1 e_1 + m_2 e_2$, where m_1 and m_2 are integer.
- Their ratio is

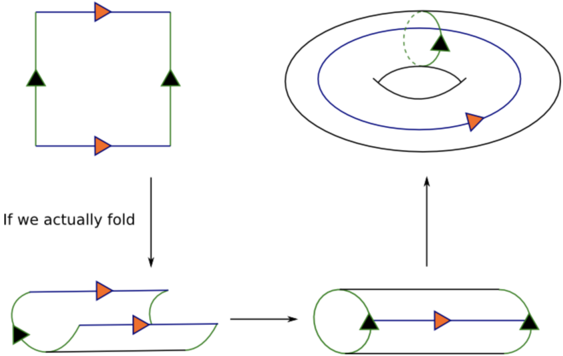
$$\tau = \frac{e_2}{e_1} \qquad (1)$$



- The same lattice can be spanned by other basis vectors, such as

$$\begin{pmatrix} e'_1 \\ e'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \qquad (2)$$

where a, b, c, d are integer satisfying $ad - bc = 1$. That is the $SL(2, Z)$.



- One interesting thing about finite modular **[Feruglio, 1706.08749]** groups are: they are isomorphic to discrete symmetry groups like

$\Gamma_2 \simeq S_3$ Kobayashi et al., 1803.10391	
$\Gamma_3 \simeq A_4$ Feruglio, 1706.08749	$\Gamma'_3 \simeq A'_4$ Liu, Ding, 1907.01488
$\Gamma_4 \simeq S_4$ JP, Petcov, 1806.11040	$\Gamma'_4 \simeq S'_4$ Novichkov, JP, Petcov, 2006.03058
$\Gamma_5 \simeq A_5$ Novichkov et al., 1812.02158	$\Gamma'_5 \simeq A'_5$ Wang, Yu, Zhou, 2010.10159

- Kobayashi et al., 1804.06644; Kobayashi, Tamba, 1811.11384; de Anda et al., 1812.05620; Baur et al., 1901.03251, 1908.00805; Kariyazono et al., 1904.07546; Nilles et al., 2001.01736, 2004.05200, 2006.03059; Kobayashi, Otsuka, 2001.07972, 2004.04518; Abe et al., 2003.03512; Ohki et al., 2003.04174; Kikuchi et al., 2005.12642 **& many more** $\cdots (> 200)$

- The modular group is defined as a group of 2×2 matrices having integer entries and determinant 1.

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z); \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (mod\ N) \right\}$$

- Therefore, the group $\Gamma(N)$ acts on the complex variable τ, varying in the upper-half $\mathcal{H} = \mathrm{Im}(\tau) > 0$, as linear fractional transformation given by

$$\gamma(\tau) = \frac{a\tau + b}{c\tau + d}, \quad \mathcal{H} = \{\tau \in \mathbb{C},\ \mathrm{Im}(\tau) > 0\}, \ \ \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathrm{Z}). \tag{3}$$

- The generators of the modular group being

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} . \tag{4}$$

$$S : \tau \to -\frac{1}{\tau} \qquad T : \tau \to \tau + 1 . \tag{5}$$

- The field transformation :

$$\psi = (c\tau + d)^{-k} \; \rho(\gamma) \; \psi \tag{6}$$

- The modular Yukawa coupling form

$$Y(\gamma\tau) \to (c\tau + d)^{k_Y} \rho_Y(\gamma) Y(\tau) \tag{7}$$

$$= Y\left(\frac{a\tau + b}{c\tau + d}\right) \tag{8}$$

- Superpotential

$$W \sim g(\psi_1 \cdots \psi_n)_1 \tag{9}$$

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$$W \sim g(\psi_1 \cdots \psi_n)_1 \tag{9}$$

$$W \sim g(Y(\tau)\psi_1 \cdots \psi_n)_1 \tag{10}$$

- Hence, they should satisfy the relation

$$\begin{aligned} k_Y &= k_1 + k_2 + k_3 + \cdots k_n, \\ \rho_Y \otimes \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n &\supset 1 \end{aligned} \tag{11}$$

S_3 Modular Symmetry

- The group S_3 exhibits three irreducible representations: the doublet 2, the singlet 1, and the pseudo-singlet $1'$.
- The lowest even-weighted modular forms arise when $k_I = 2$ and the corresponding modular coupling is represented as $Y_{\mathbf{2}}^{(2)} = (Y_1(\tau), Y_2(\tau))$.
- The Dedekind eta function $\eta(\tau)$ form of $Y_{\mathbf{2}}^{(2)}$ is given as

$$\begin{aligned} Y_1(\tau) &= \frac{i}{4\pi}\left(\frac{\eta'(\frac{\tau}{2})}{\eta(\frac{\tau}{2})} + \frac{\eta'(\frac{\tau+1}{2})}{\eta(\frac{\tau+1}{2})} - \frac{8\eta'(2\tau)}{\eta(2\tau)}\right), \\ Y_2(\tau) &= \frac{i\sqrt{3}}{4\pi}\left(\frac{\eta'(\frac{\tau}{2})}{\eta(\frac{\tau}{2})} - \frac{\eta'(\frac{\tau+1}{2})}{\eta(\frac{\tau+1}{2})}\right). \end{aligned} \tag{12}$$

where, $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ with $q = e^{2\pi i\tau}$.

- However, the q-expansion form comes in handy during the numerical analysis of the models as given below

$$\begin{aligned} Y_1(\tau) &= \frac{1}{8} + 3q^2 + 3q^4 + 12q^6 + 3q^8 \cdots , \\ Y_2(\tau) &= \sqrt{3}q(1 + 4q^2 + 6q^4 + 8q^6 \cdots). \end{aligned} \tag{13}$$

- The product rule for the S_3 symmetry is given below when $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are two doublets.
  The S_3 product rule yields

$$\begin{aligned} x \otimes y = (x_2y_2 - x_1y_1, x_1y_2 + x_2y_1)_2 \oplus (x_1y_1 + x_2y_2)_1 \\ \oplus (x_1y_2 - x_2y_1)_{1'}. \end{aligned} \tag{14}$$

where, the subscript denotes the S_3 representation.

Model Framework

Type-I seesaw

- Our models are based on the supersymmetric type-I seesaw model extended by 3 electroweak singlet chiral supermultiplets N_i^c.
- The supermultiplets in the leptonic sector are $L_i(2, -1/2)$, $E_i^c(1, -1)$, $N_i^c(1, 0)$ and $H_{u,d}(1/2, \pm 1/2)$ where the numbers in parenthesis are the electroweak charges.
- The superpotential for the lepton sector can be written as

$$\mathcal{W} \supset (Y_\ell)_{ij} E_i^c L_j H_d + (Y_\nu)_{ij} N_i^c L_j H_u + \frac{1}{2}(M_R)_{ij} N_i^c N_j^c. \tag{15}$$

- The light neutrino mass matrix formula is given by

$$M_\nu = M_D^T M_R^{-1} M_D. \tag{16}$$

	Fields	E_1^c	E_2^c	E_3^c	L_1	L_2	L_3	N_1^c	N_2^c	N_3^c	Free Parameters	NO	IO
MODEL A	S_3	2		1	2		1	2		1	7	✓	✓
	k_I	1		-1	1		1	1		3			
MODEL B	S_3	2		1	2		1	1	1	$1'$	7	✓	✓
	k_I	0		0	2		0	0	2	0			
MODEL C	S_3	1	$1'$	1	2		1	2		1	4	✗	✓
	k_I	1	1	-1	1		1	1		1			
MODEL D	S_3	2		$1'$	1	$1'$	$1'$	2		$1'$	9	✓	✓
	k_I	0		0	2	2	0	0		4			

Table: In this table, we depict the particle content of the different models and their charges under S_3 modular symmetry, where k_I is the modular weight. Also, the number of free real parameters, in addition to the complex modulus τ, and the possible ordering of neutrino masses are provided for each model.

Model A

The invariant superpotential is given as

$$\mathcal{W}_\ell = \alpha_\ell (E^c L)_2 Y_2^{(2)} H_d + \beta_\ell (E^c Y_2^{(2)})_1 L_3 H_d + \gamma_\ell E_3^c L_3 H_d, \quad (17)$$

$$\mathcal{W}_\nu = \alpha_D (N^c L)_2 Y_2^{(2)} H_u + \beta_D N_3^c (L Y_2^{(4)})_1 H_u + \gamma_D (N^c Y_2^{(2)}) L_3 H_u + \omega_D N_3^c L_3 Y_1^{(4)} H_u + \alpha_R M (N^c N^c)_2 Y_2^{(2)} + \beta_R M N_3^c (N^c Y_2^{(4)})_1 + M N_3^c N_3^c Y_1^{(6)}. \quad (18)$$

The charged lepton mass matrices for the corresponding superpotential is given by

$$M_\ell = \frac{v_d}{\sqrt{2}} \begin{pmatrix} -\alpha_\ell Y_1 & \alpha_\ell Y_2 & 0 \\ \alpha_\ell Y_2 & \alpha_\ell Y_1 & 0 \\ \beta_\ell Y_1 & \beta_\ell Y_2 & \gamma_\ell \end{pmatrix}. \quad (19)$$

Similarly, the mass matrices related to the neutrino sector are depicted as below.

$$M_D = \frac{\omega_D v_u}{\sqrt{2}} \begin{pmatrix} -\bar{\alpha}_D Y_1 & \bar{\alpha}_D Y_2 & \bar{\gamma}_D Y_1 \\ \bar{\alpha}_D Y_2 & \bar{\alpha}_D Y_1 & \bar{\gamma}_D Y_2 \\ \bar{\beta}_D (Y_{\mathbf{2}}^{(4)})_1 & \bar{\beta}_D (Y_{\mathbf{2}}^{(4)})_2 & Y_{\mathbf{1}}^{(4)} \end{pmatrix}, \tag{20}$$

$$M_R = M \begin{pmatrix} -2\alpha_R Y_1 & 2\alpha_R Y_2 & \beta_R (Y_2^{(4)})_1 \\ 2\alpha_R Y_2 & 2\alpha_R Y_1 & \beta_R (Y_2^{(4)})_2 \\ \beta_R (Y_{\mathbf{2}}^{(4)})_1 & \beta_R (Y_{\mathbf{2}}^{(4)})_2 & Y_{\mathbf{1}}^{(6)} \end{pmatrix}. \tag{21}$$

Model B

The invariant superpotential is given as

$$\mathcal{W}_\ell = \alpha_\ell (E^c L)_2 Y_{\mathbf{2}}^{(2)} H_d + \beta_\ell E_3^c (Y_{\mathbf{2}}^{(2)} L)_1 H_d + \gamma_\ell E_3^c L_3 H_d. \quad (22)$$

$$\mathcal{W}_\nu = \alpha_D N_1^c (L Y_{\mathbf{2}}^{(2)})_1 H_u + \beta_D N_2^c (L Y_{\mathbf{2}}^{(4)})_1 H_u + \gamma_D N_3^c (L Y_{\mathbf{2}}^{(2)})_{1'} H_u + \omega_D N_1^c L_3 H_u + \alpha_R M N_1^c N_1^c + \beta_R M N_2^c N_2^c Y_{\mathbf{1}}^{(4)} + M N_3^c N_3^c. \quad (23)$$

The charged lepton mass matrices for the corresponding superpotential is given by

$$M_\ell = \frac{v_d}{\sqrt{2}} \begin{pmatrix} -\alpha_\ell Y_1 & \alpha_\ell Y_2 & 0 \\ \alpha_\ell Y_2 & \alpha_\ell Y_1 & 0 \\ \beta_\ell Y_1 & \beta_\ell Y_2 & \gamma_\ell \end{pmatrix}. \quad (24)$$

Similarly, the mass matrices related to the neutrino sector are depicted below.

$$M_D = \frac{\omega_D v_u}{\sqrt{2}} \begin{pmatrix} \bar{\alpha}_D Y_1 & \bar{\alpha}_D Y_2 & 1 \\ \bar{\beta}_D (Y_{\mathbf{2}}^{(4)})_1 & \bar{\beta}_D (Y_{\mathbf{2}}^{(4)})_2 & 0 \\ \bar{\gamma}_D Y_2 & -\bar{\gamma}_D Y_1 & 0 \end{pmatrix}, \tag{25}$$

$$M_R = M \begin{pmatrix} \alpha_R & 0 & 0 \\ 0 & \beta_R Y_{\mathbf{1}}^{(4)} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{26}$$

Model C

The invariant superpotential is given as

$$\mathcal{W}_\ell = \alpha_\ell E_1^c (L Y_{\mathbf{2}}^{(2)})_1 H_d + \beta_\ell E_2^c (L_2 Y_{\mathbf{2}}^{(2)})_{1'} H_d + \gamma_\ell E_3^c L_3 H_d \quad (27)$$

$$\mathcal{W}_\nu = \alpha_D \left[(N^c L)_2 Y_{\mathbf{2}}^{(2)} \right]_1 H_u + \beta_D (N^c Y_{\mathbf{2}}^{(2)})_1 L_3 H_u + \\ \omega_D N_3^c (L Y_{\mathbf{2}}^{(2)})_1 H_u + M \left[\left[(N^c N^c)_2 Y_{\mathbf{2}}^{(2)} \right]_1 + \alpha_R N_3^c (N^c Y_{\mathbf{2}}^{(2)})_1 \right]. \quad (28)$$

The charged lepton mass matrix is given by

$$M_\ell = \frac{v_d}{\sqrt{2}} \begin{pmatrix} \alpha_\ell Y_1 & \alpha_\ell Y_2 & 0 \\ \beta_\ell Y_2 & -\beta_\ell Y_1 & 0 \\ 0 & 0 & \gamma_\ell \end{pmatrix}. \quad (29)$$

Similarly, the matrices for the neutrino sector is given by

$$M_D = \frac{\omega_D v_u}{\sqrt{2}} \begin{pmatrix} -\bar{\alpha}_D Y_1 & \bar{\alpha}_D Y_2 & \bar{\beta}_D Y_1 \\ \bar{\alpha}_D Y_2 & \bar{\alpha}_D Y_1 & \bar{\beta}_D Y_2 \\ Y_1 & Y_2 & 0 \end{pmatrix}, \tag{30}$$

$$M_R = M \begin{pmatrix} -Y_1 & Y_2 & \alpha_R Y_1 \\ Y_2 & Y_1 & \alpha_R Y_2 \\ \alpha_R Y_1 & \alpha_R Y_2 & 0 \end{pmatrix}. \tag{31}$$

Model D

The invariant superpotential in the charged lepton sector can be written as

$$\mathcal{W}_\ell = \alpha_\ell (E^c Y_{\mathbf{2}}^{(2)})_1 L_1 H_d + \beta_\ell (E^c Y_{\mathbf{2}}^{(2)})_{1'} L_2 H_d + \gamma_\ell E_3^c L_3 H_d \quad (32)$$

$$\mathcal{W}_\nu = \omega_D (N^c Y_{\mathbf{2}}^{(2)})_1 L_1 H_u + \alpha_D (N^c Y_{\mathbf{2}}^{(2)})_{1'} L_2 H_u + \beta_D N_3^c L_1 Y_{\mathbf{1'}}^{(6)} H_u + \gamma_D N_3^c L_2 Y_{\mathbf{1}}^{(6)} H_u + \eta_D N_3^c L_3 Y_{\mathbf{1}}^{(4)} H_u + M \left[(N^c N^c)_1 + \alpha_R N_3^c (N^c Y_{\mathbf{2}}^{(4)})_{1'} + \beta_R N_3^c N_3^c Y_{\mathbf{1}}^{(8)} \right] . \quad (33)$$

The charged lepton mass matrix is given by

$$M_\ell = \frac{v_d}{\sqrt{2}} \begin{pmatrix} \alpha_\ell Y_1 & \beta_\ell Y_2 & 0 \\ \alpha_\ell Y_2 & -\beta_\ell Y_1 & 0 \\ 0 & 0 & \gamma_\ell \end{pmatrix} . \quad (34)$$

Similarly, the mass matrices for the neutrino sector are given by

$$M_D = \frac{\omega_D v_u}{\sqrt{2}} \begin{pmatrix} Y_1 & \bar{\alpha}_D Y_2 & 0 \\ Y_2 & -\bar{\alpha}_D Y_1 & 0 \\ \bar{\beta}_D Y^{(6)}_{\mathbf{1}'} & \bar{\gamma}_D Y^{(6)}_{\mathbf{1}} & \bar{\eta}_D Y^{(4)}_{\mathbf{1}} \end{pmatrix}, \quad (35)$$

$$M_R = M \begin{pmatrix} 1 & 0 & \alpha_R (Y^{(4)}_{2})_2 \\ 0 & 1 & -\alpha_R (Y^{(4)}_{2})_1 \\ \alpha_R (Y^{(4)}_{\mathbf{2}})_2 & -\alpha_R (Y^{(4)}_{\mathbf{2}})_1 & \beta_R Y^{(8)}_{\mathbf{1}} \end{pmatrix}. \quad (36)$$

Numerical Analysis

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- We numerically analyze the neutrino mass matrices depicted in the previous section.
- This is achieved by performing a scan on the model parameter space. For each model, we identify the region of parameter space consistent with neutrino oscillation data at the 2-σ level.

Parameter	Best-fit value and 1-σ range	
	NO	**IO**
$\Delta m^2_{\text{sol}}/(10^{-5}\text{ eV}^2)$	$7.41^{+0.21}_{-0.20}$	$7.41^{+0.21}_{-0.20}$
$\Delta m^2_{\text{atm}}/(10^{-3}\text{ eV}^2)$	$2.507^{+0.028}_{-0.027}$	$2.486^{+0.025}_{-0.028}$
$\sin^2\theta_{12}$	$0.303^{+0.012}_{-0.011}$	$0.303^{+0.012}_{-0.012}$
$\sin^2\theta_{13}$	$0.02225^{+0.00056}_{-0.00059}$	$0.02223^{+0.00058}_{-0.00058}$
$\sin^2\theta_{23}$	$0.451^{+0.019}_{-0.016}$	$0.569^{+0.016}_{-0.021}$
J_{CP}	$-0.0254^{+0.0115}_{-0.0080}$	$-0.0330^{+0.0044}_{-0.0011}$

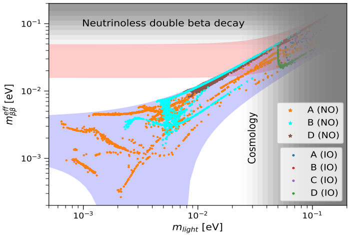
[Esteban et al., 2007.14792]

- In our scan, the parameters $\bar{\alpha}_D$, $\bar{\beta}_D$, $\bar{\gamma}_D$, $\bar{\eta}_D$, α_R and β_R are varied within the range $[10^{-4}, 10^4]$.
- The modulus τ is varied within the fundamental domain defined by

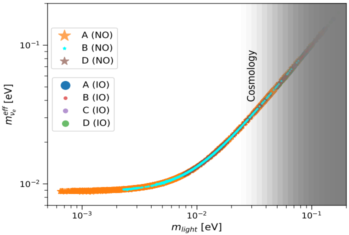
$$\text{Im}(\tau) > 0, \quad \Big|\text{Re}(\tau)\Big| \leq \frac{1}{2}, \text{ and } |\tau| \geq 1. \qquad (37)$$

	Ordering	m_{light} [eV]	$m^{eff}_{\nu_e}$ [eV]	$m^{eff}_{\beta\beta}$ [eV]
MODEL A	NO	$(6.4 \times 10^{-3}, 0.12)$	$(8.8 \times 10^{-3}, 0.12)$	$(3.3 \times 10^{-3}, 5.7 \times 10^{-2})$
	IO	$(5.8 \times 10^{-2}, 0.22)$	$(5.8 \times 10^{-2}, 0.22)$	$(2.6 \times 10^{-2}, 0.11)$
MODEL B	NO	$(2.3 \times 10^{-3}, 7.7 \times 10^{-2})$	$(9.0 \times 10^{-3}, 7.7 \times 10^{-2})$	$(1.6 \times 10^{-3}, 7.7 \times 10^{-2})$
	IO	$(5 \times 10^{-2}, 0.14)$	$(4.9 \times 10^{-2}, 0.13)$	$(4.8 \times 10^{-2}, 0.13)$
MODEL C	IO	$(7.1 \times 10^{-2}, 7.2 \times 10^{-2})$	$(7.1 \times 10^{-2}, 7.2 \times 10^{-2})$	$(5.5 \times 10^{-2}, 6.2 \times 10^{-2})$
MODEL D	NO	$(7.5 \times 10^{-3}, 0.11)$	$(1.2 \times 10^{-2}, 0.11)$	$(8.2 \times 10^{-3}, 0.10)$
	IO	$(5 \times 10^{-2}, 0.15)$	$(5 \times 10^{-2}, 0.15)$	$(2.2 \times 10^{-2}, 0.14)$

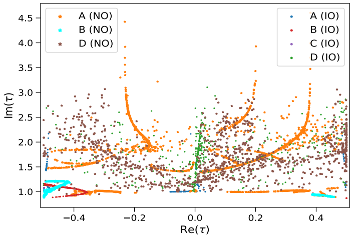
Table: The range of m_{light}, $m^{eff}_{\nu_e}$ and $m^{eff}_{\beta\beta}$ in each scenario.



(a)



(b)



(c)

Figure: Plot (a) and (b) shows the correlation of m_{light} eV with $m_{\beta\beta}^{eff}$ [eV] and $m_{\nu_e}^{eff}$ [eV] respectively, whereas, (c) shows the correlation between $Re(\tau)$ and $Im(\tau)$.

Conclusion

- Despite being the simplest discrete symmetry, S_3 modular symmetry allows us to discuss neutrino phenomenology correctly.
- We were able to keep the main motivation of minimally using the flavon fields in our models.
- In support of the above statement, we were able to identify 4 different realizations of the S_3 modular symmetry.
- All four models are compatible with IO, in addition model A, B and D are compatible with NO too.

Thank you

BACKUP SLIDES

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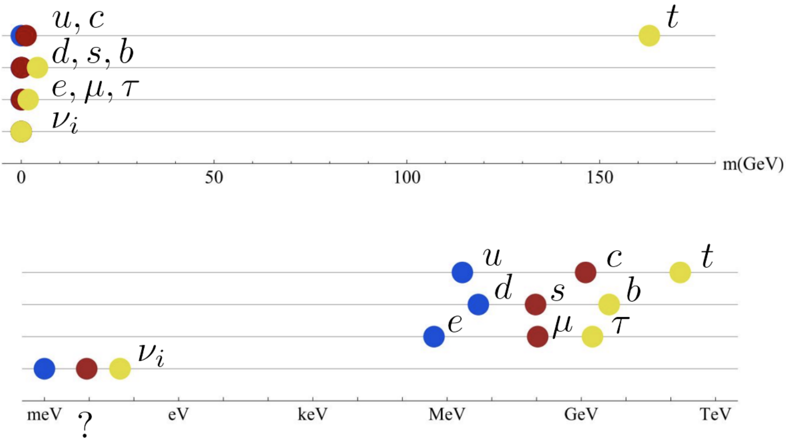


Figure: adapted from R. Toorop's PhD thesis

Higher order Yukawa couplings

$$
\begin{aligned}
Y_1^{(4)} &= Y_1^2 + Y_2^2, \quad Y_1^{(6)} = 3Y_1^2 Y_2 - Y_1^3, \quad Y_{1'}^{(6)} = 3Y_1 Y_2^2 - Y_2^3, \\
Y_2^{(4)} &= \begin{bmatrix} Y_2^2 - Y_1^2 \\ 2Y_1 Y_2 \end{bmatrix} = \begin{bmatrix} (Y_2^{(4)})_1 \\ (Y_2^{(4)})_2 \end{bmatrix}, \\
Y_2^{(6)} &= \begin{bmatrix} Y_1^3 + Y_1 Y_2^2 \\ Y_2^3 + Y_1^2 Y_2 \end{bmatrix} = \begin{bmatrix} (Y_2^{(6)})_1 \\ (Y_2^{(6)})_2 \end{bmatrix}, \\
Y_1^{(8)} &= (Y_1^2 + Y_2^2)^2.
\end{aligned} \tag{38}
$$

The q-expansion form of the Dedekind eta function $\eta(\tau)$ is given by:

$$
\eta(\tau) = q^{1/24} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} \tag{39}
$$

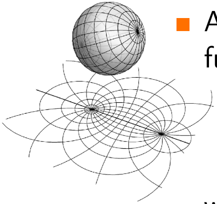
The charged lepton masses through the relations

$$\mathrm{Tr}\left(M_\ell M_\ell^\dagger\right) = m_e^2 + m_\mu^2 + m_\tau^2 \,, \tag{40}$$

$$\mathrm{Det}\left(M_\ell M_\ell^\dagger\right) = m_e^2 m_\mu^2 m_\tau^2 \,, \tag{41}$$

$$\frac{1}{2}\left[\mathrm{Tr}\left(M_\ell M_\ell^\dagger\right)\right]^2 - \frac{1}{2}\mathrm{Tr}\left[(M_\ell M_\ell^\dagger)^2\right] = m_e^2 m_\mu^2 + m_\mu^2 m_\tau^2 + m_\tau^2 m_e^2 \,. \tag{42}$$

Linear Fractional Transformation



- A linear fractional transformation (LFTs) is a function of the form

$$\gamma(\tau) = \frac{a\tau + b}{c\tau + d} \tag{43}$$

where, a, b, c, d are complex constants and the coefficients corresponds to a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{44}$$

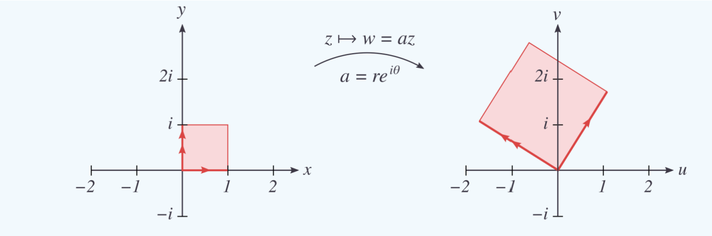
with $ad - bc \neq 0$.

- When a LFT is performed, symmetry is always maintained.
- In the complex plane, an LFT takes lines or circles onto lines or circles.

- If $ad - bc = 0$ then $T(z)$ is a constant function.

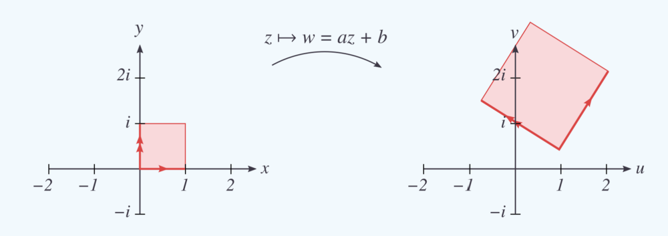
$$T(z) = \frac{(a/c)(cz + d)}{cz + d} = \frac{a}{c} \tag{45}$$

- Let $T(z) = az$. If a = r is real, this scales the plane. If $a = e^{i\theta}$, it rotates the plane. If $a = re^{i\theta}$, it does both at once.



Multiplication by $a = re^{i\theta}$ scales by r and rotates by θ.

- Let $T(z) = az + b$. Adding the b term introduces a translation to the previous example.



The map $w = az + b$ scales, rotates and shifts the square.

- Note that T is the fractional linear transformation with coefficients

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \tag{46}$$

Modular Form

How to find the concrete form of modular form with weight 2 and non-trivial rep. of $\Gamma(N)$

- Suppose functions $f_i(\tau)$ to be modular forms with weight k_i
- Also suppose $\sum_i k_i = 0$

⇒ $\frac{d}{d\tau}\sum_i \log f_i(\tau)$ is a modular form with weight 2

Proof

Modular transformation: $\tau' = \dfrac{a\tau + b}{c\tau + d}, ad - bc = 1$

$$\frac{d}{d\tau'} = \frac{d\tau}{d\tau'}\frac{d}{d\tau} = (c\tau + d)^2 \frac{d}{d\tau}, \qquad f_i(\tau') = (c\tau + d)^{k_i} f_i(\tau)$$

⇓

$$\frac{d}{d\tau'}\sum_i \log f_i(\tau') = (c\tau + d)^2 \frac{d}{d\tau}\sum_i [\log f_i(\tau) + \underbrace{k_i(c\tau + d)}_{=0}]$$

$$= (c\tau + d)^2 \frac{d}{d\tau}\sum_i \log f_i(\tau)$$

Suppose $f_i(\tau)$ with modular weight k_i

$$f_i(\tau) \to (c\tau + d)^{k_i} f_i(\tau)$$

$\frac{d}{d\tau}\sum_i \log f_i(\tau)$ is modular function with weight 2 if $\sum_i k_i = 0$

$$\frac{d}{d\tau}\sum_i \log f_i(\tau) \to (c\tau + d)^2 \frac{d}{d\tau}\sum_i \log f_i(\tau) + c(c\tau + d)\sum_i k_i.$$

$$\begin{aligned}
\eta(3\tau) &\to e^{i\pi/4}\eta(3\tau), \\
\eta(\tau/3) &\to \eta((\tau+1)/3), \\
\eta((\tau+1)/3) &\to \eta((\tau+2)/3), \\
\eta((\tau+2)/3) &\to e^{i\pi/12}\eta(\tau/3),
\end{aligned}$$

$$\begin{aligned}
\eta(3\tau) &\to \sqrt{\frac{-i\tau}{3}}\eta(\tau/3), \\
\eta(\tau/3) &\to \sqrt{-i3\tau}\eta(3\tau), \\
\eta((\tau+1)/3) &\to e^{-i\pi/12}\sqrt{-i\tau}\eta((\tau+2)/3), \\
\eta((\tau+2)/3) &\to e^{i\pi/12}\sqrt{-i\tau}\eta((\tau+1)/3).
\end{aligned}$$

T : $\tau \to \tau + 1$

S : $\tau \to -1/\tau$

NDBD

- The effective Majorana neutrino mass $m_{\beta\beta}$ is a crucial parameter in the study of neutrinoless double-beta decay ($0\nu\beta\beta$). The KamLAND-Zen experiment has set stringent bounds on $m_{\beta\beta}$, with the current best limit being $m_{\beta\beta} \leq 0.036 - 0.156$ eV at 90% confidence level (CL).

- The range in the upper bounds reflects the uncertainties primarily due to nuclear matrix element (NME) calculations.

  The rate of neutrinoless double-beta decay is related to the effective Majorana neutrino mass $m_{\beta\beta}$, given by:

  $(T^{0\nu}_{1/2})^{-1} = G^{0\nu}|M^{0\nu}|^2 \left(\frac{m_{\beta\beta}}{m_e}\right)^2$ where: - $T^{0\nu}_{1/2}$ is the half-life of the neutrinoless double-beta decay. - $G^{0\nu}$ is the phase space factor, which depends on the kinematics of the decay and is relatively well-known. - $M^{0\nu}$ is the nuclear matrix element (NME), which encodes the nuclear physics involved in the decay. - m_e is the electron mass.

Uncertainties in Nuclear Matrix Elements The NME, $M^{0\nu}$, is not directly measurable and must be calculated theoretically. These calculations are complex and involve significant uncertainties due to:

- Nuclear Structure Models: Different theoretical models are used to describe the nuclear structure and the interactions within the nucleus. Common models include the Interacting Shell Model (ISM), the Quasiparticle Random Phase Approximation (QRPA), the Energy Density Functional (EDF) approach, and the interacting boson model (IBM). Each model has different approximations and assumptions, leading to variations in the calculated NMEs.
- Short-Range Correlations: The treatment of short-range correlations between nucleons affects the NME. Different methods for including these correlations (such as the Jastrow method or the Unitary Correlation Operator Method) result in different NME values.

- Nucleon Form Factors: The form factors describing the spatial distribution of nucleons within the nucleus and the finite size of nucleons introduce uncertainties. Variations in the assumed form factors lead to differences in the NMEs.
- Axial Coupling Constant: The value of the axial coupling constant g_A used in the calculations influences the NME. The effective value of g_A in nuclear matter may be quenched compared to its free nucleon value, adding another layer of uncertainty.

Impact on the Effective Mass Bound These uncertainties in the NME directly affect the interpretation of experimental results. Specifically:
- Range of $m_{\beta\beta}$ Values: The spread in the NME calculations translates into a range of possible values for $m_{\beta\beta}$. A higher NME results in a lower bound on $m_{\beta\beta}$, and a lower NME gives a higher bound.
- Upper Bound Variation: The quoted range $0.036 - 0.156$ eV for $m_{\beta\beta}$ reflects the upper bound's dependence on different NME calculations. The lower end of this range corresponds to the scenario with the most favorable (largest) NME, while the upper end corresponds to the least favorable (smallest) NME.