Low-lying hadrons in the matrix model of two-color QCD at extreme strong coupling: Quantum phases and spin-puzzle

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Based on : arXiv:2406.06055

ICHEP,2024

20 July, 2024

Plan of talk

- Motivation: Why Matrix Model?
- Why 2-color 1-falvor QCD?
- Hamiltonian of 2-color 1-flavor QCD and its symmetries
- Results: Quantum Phase Transitions(QPT) and Spin Puzzle
- Summary
- Future Work Plans

Motivation: Why Matrix Model?

- Non abelian gauge theories : Underlying dynamics of subatomic particles.

  Examples:
  - Quantum Chromodynamics (QCD) – Strong interaction of quarks and gluons
  - Electroweak theory – Weak and EM interaction of quarks and leptons

- Many aspects are well understood, especially in the weak coupling regime. Very challenging at strong coupling – hence use of computational methods. For example, QCD in the strong coupling regime is nonperturbative.
- For computations in the strongly coupled regime:
  - Most popular candidate is Lattice QCD: very successful in hadronic sector but computationally very expensive.
- Gauge matrix model proposal:
  - captures certain key features (of course, not all!) of a non-Abelian gauge theory
  - quantum mechanical model $\Longrightarrow$ provides a simplified computational platform.
  - Has been shown to reproduce light hadron masses with surprising accuracy.

Matrix Model

The matrix model[1] can be described as follows:

- Quantum mechanical approximation of SU(N) Yang-Mills theory on $\mathbb{R} \times S^3$
- Building blocks : $3 \times (N^2 - 1)$ real matrices M_{ia}.
- Spatial index $i = 1, 2, 3$ and color index $a = 1, \cdots, (N^2 - 1)$
- Gauge fields are Hermitian matrices $A_i(t) = M_{ia} T^a$,
  where the $T^a =$ generators of $SU(N)$ in the fundamental rep.
- Rotations: $A_i \to R_{ij} A_j, \quad R \in SO(3)_{rot}$
- Gauge transformations : $A_i \to g A_i g^\dagger, \quad g \in SU(N)$
- The configuration space: $\mathcal{M}_N / AdSU(N)$,
  $\mathcal{M}_N =$ space of all $3 \times (N^2 - 1)$ real matrices.
  This bundle is twisted $\Rightarrow$ Gribov Ambiguity
- Field Strength, $F_{ij} = -\epsilon_{ijk} A_k - ig\,[A_i, A_j]$.

[1](Narsimhan-Ramdas 1979, Singer 1978 and Balachandran et.al, 2014)

2-color 1-flavor QCD

Why 2-color 1-flavor QCD?

- Gauge group is $SU(2) \to$ Simplest non-Abelian gauge theory.
- computationally less challenging.
- has many interesting features:
  a) baryons (di-quarks and tetra-quarks) are bosonic states,
  b) there are additional global symmetry (Pauli-Gürsey symmetry):
  Fundamental rep of $SU(2)$ is pseudo-real $\Rightarrow U(1)_V$ extended to $SU(2)_B$.
- The fermionic determinant in the path integral has no sign problem.

Pure SU(2) matrix model:

$$\left.\begin{array}{ll} \text{Chromoelectric field:} & E_i = \partial_t A_i, \\ \text{Chromomagnetic field:} & B_i = \frac{1}{2}\epsilon_{ijk}F_{jk} \\ & \quad = -A_i - \frac{ig}{2}\epsilon_{ijk}[A_j, A_k] \end{array}\right\} \begin{array}{c} \text{Pure YM Hamiltonian:} \\ H_{YM} = \text{Tr}\Big[E_iE_i + B_iB_i\Big] \end{array}$$

• Potential: $V(A) = B_iB_i = \text{Tr}\left(A_iA_i + ig\epsilon_{ijk}[A_i, A_j]A_k - \frac{g^2}{2}[A_i, A_j][A_i, A_j]\right)$

• V(A) has upto quartic terms $\Longrightarrow$ multi-dimensional quartic oscillator.

Fermions in SU(2) Matrix model

- Fermions are Grassmann valued matrix $\psi(t)$ transforming as:

$$\text{Fundamental rep. of color:} \qquad \psi \to u(h)\psi, \qquad h \in SU(2)$$
$$\text{spin-}\frac{1}{2}\ \text{rep. of rotations:} \qquad \psi \to D^{1/2}(R)\psi, \qquad R \in SO(3)_{rot}$$

- Dirac fermion, $\psi = \begin{pmatrix} b_{\alpha A} \\ d^{\dagger}_{\alpha A} \end{pmatrix}$, $\alpha, A = 1, 2$.
- $b^{\dagger}_{\alpha A}$ and $d^{\dagger}_{\alpha A}$ create quarks and anti-quarks respectively.

$$\text{anti-commutation algebra:} \qquad \{b_{\alpha A}, b^{\dagger}_{\beta B}\} = \delta_{\alpha\beta}\delta_{AB} = \{d_{\alpha A}, d^{\dagger}_{\beta B}\}$$

- Fermion Hamiltonian, $H_f = gH_{int} + mH_m + \tilde{c}H_c$, where

$$H_{int} = \bar{\psi}\gamma^i A_a \psi,\ \ H_m = \left(\cos\theta\bar{\psi}\psi + i\sin\theta\bar{\psi}\gamma^5\psi\right),\ H_c = \bar{\psi}\,\gamma^0\gamma^5\psi$$

- Baryon chemical potential term, $\tilde{\mu}H_{\mu} = \frac{1}{2}\tilde{\mu}\bar{\psi}\gamma^0\psi = \tilde{\mu} \times$ Baryon number
- Total Hamiltonian, $H = H_{YM} + gH_{int} + mH_m + \tilde{c}H_c + \tilde{\mu}H_{\mu}$

Symmetries of the Hamiltonian

- Global Symmetries:

  For $m = 0$: $\psi \to e^{i\theta\gamma^5}\psi \Leftarrow U(1)_A$
  Generator of $U(1)_A \to Q_0$

  Pauli-Gürsey Symmetry:
  $U(1)_B \xrightarrow{\text{extended to}} SU(2)_B$
  Generators of $SU(2)_B \to B_1, B_2, B_3$

  $$[B_i, B_j] = \epsilon_{ijk} B_k, \quad [Q_0, B_{1,2,3}] = 0$$

  Three different cases:
  - $m = 0,\ \mu = 0$: $U(1)_A \xrightarrow{\text{Anomaly}} \mathbb{Z}_2$
    Residual Symmetry
    $\Rightarrow SU(2)_B \times \mathbb{Z}_2$
  - $m \neq 0,\ \mu = 0$: $U(1)_A \to \mathbb{Z}_2$
    Residual Symmetry
    $\Rightarrow SU(2)_B \times \mathbb{Z}_2$
  - $\mu \neq 0$: $SU(2)_B \to U(1)_B$
    Residual Symmetry $\Rightarrow U(1)_B \times \mathbb{Z}_2$

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- Spatial Rotation:

Glue and Quark transforms in the spin-1 and spin-1/2 representations.

$$L_i = -2\epsilon_{ijk} Tr\left(\Pi_j A_k\right)$$
$$S_i = \frac{1}{2}\left(b^\dagger_{\alpha A}\sigma^i_{\alpha\beta} b_{\beta A} + d_{\alpha A}\sigma^i_{\alpha\beta} d^\dagger_{\beta A}\right)$$
$$J_i = L_i + S_i$$

- Gauge Symmetry:

Gauss Law Genarators, G_a:

$$G_a = -f_{abc} P_{ib} M_{ic} + \bar{\psi}\gamma^0 T^a \psi$$
$$[G_a, G_b] = i f_{abc} G_c, \quad [H, G_a] = 0$$

Physical states are anihilated by G_a

$$G_a \left|\cdot\right\rangle = 0, \qquad \left|\cdot\right\rangle \in \text{physical states}$$

Strong Coupling Regime

- Recale the Hamiltonian as $\Rightarrow A_i \to g^{-\frac{1}{3}} A_i$ and $P_i \to g^{\frac{1}{3}} P_i$

$$H = e_0 \left[Tr \left(P_i P_i + g^{-\frac{4}{3}} A_i A_i + i g^{-\frac{2}{3}} \epsilon_{ijk} \left[A_i , A_j \right] A_k - \frac{1}{2} \left[A_i , A_j \right]^2 \right) + c H_c + H_{int} + M H_m + \mu H_\mu \right]$$

- $e_0 = g^{2/3}/R$, where R is the radius of S^3.
- Double scaling limit: $g \to \infty$, $R \to \infty$, $e_0 =$finite $\Rightarrow H$ has well-defined spectra.

Numerical Strategy:

- Hilbert space, $\mathcal{H} = \mathcal{H}_{Fermion} \otimes \mathcal{H}_{Boson}$
- $\mathcal{H}_{Boson}$ is infinite dimensional.
- Use Rayleigh-Ritz (truncate $\mathcal{H}_{Boson}$ to a given boson number).

Construction of states

- Eigenstates and eigenvalues labelled as spin(s), $SU(2)_B$ casimir($B_r B_r$) and B_3 charge

$$H \left|\psi_n^{j,B,B_3}\right\rangle = E_n^{j,B,B_3} \left|\psi_n^{j,B,B_3}\right\rangle, \qquad B_3 \left|\psi_n^{j,B,B_3}\right\rangle = B_3 \left|\psi_n^{j,B,B_3}\right\rangle$$
$$B_r B_r \left|\psi_n^{j,B,B_3}\right\rangle = B(B+1) \left|\psi_n^{j,B,B_3}\right\rangle$$

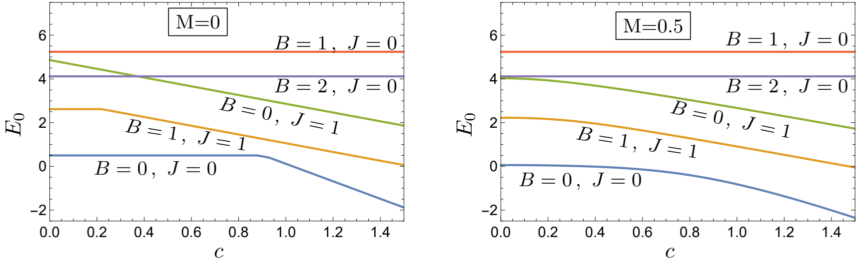
- 5 non-interacting sectors :

Sectors	Mesons	di-(anti-) quark	tetra-(anti-) quark
J=0, B=0	✓	✗	✗
J=0, B=1	✓	✓	✗
J=0, B=2	✓	✓	✓
J=1, B=0	✓	✗	✗
J=1, B=1	✓	✓	✗

- $\mu = 0 \Rightarrow$ mesons, di-(anti-) quarks and tetra-(anti-) quarks are degenerate

Results

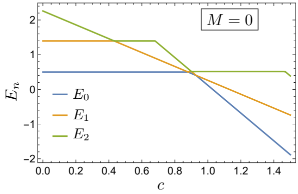
- Low-lying eigenvalues as a function of c from each sector(at $\mu = 0$).

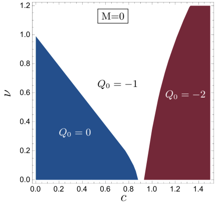


- Ground state is unique and belongs to $B = 0, J = 0$ sector.
- Our findings:
  1. Quantum phase transitions(QPT) in the sectors $(B, J) = (0, 0), (1, 1), (0, 1)$ due to level crossing.
  2. QPTs are first order: $\langle Q_0 \rangle = \frac{\partial E}{\partial c}$ is discontinuous.
  3. Interesting division of spin between the quark and glue. (Spin Puzzle)
  4. Non-trivial Phase structure after adding the Baryon Chemical Potential.

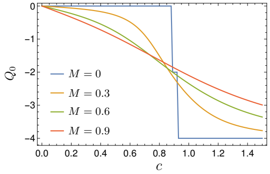
Quantum Phase Transition

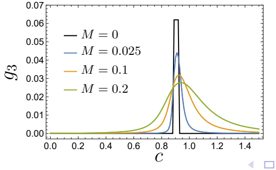
- Level crossing in the (B,J)=(0,0) is rather special $\Rightarrow$ Triple crossing.
- Plot of ν $(= g^{-2/3})$ vs c shows three distinct phases. For $g \to \infty$ or $\nu \to 0$ two transition lines merge at the triple point.

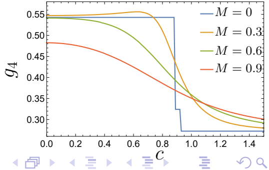




- Critical point $(c, M) \approx (0.928, 0) \Rightarrow Q_0$ is discontinuous; third and fourth Binder cumulants (g_3 and g_4) shows singular behaviour.

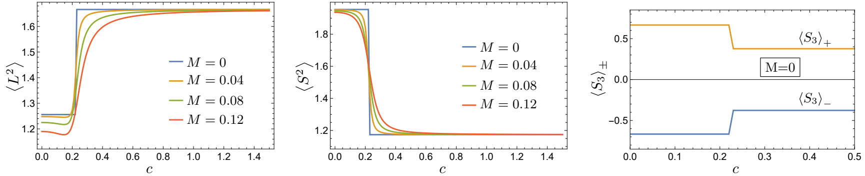






A possible solution to spin puzzle?

- Quarks carry (4 – 24%) of Proton spin: Proton Spin Puzzle! (EMC, 1988)
- For (B,J)=(1,1) the spin division is interesting compared to other sectors.
- For (B,J)=(1,1): QPT occurs at $(c, M) = (c_1^*, M) \approx (0.22, 0)$
- Glue (L) and quark (S) spin contribution of gs:



- When $c < c_1^*$ quark spin contributes significantly and it is opposite for $c > c_1^*$.
- Distribution of spin is further clarified by $\langle S_3 \rangle_\pm$:

$$\text{For } M = 0: \ \langle S_3 \rangle_\pm = \begin{cases} \pm 0.67 & \text{for } c < c_1^* \\ \pm 0.33 & \text{for } c > c_1^* \end{cases}$$

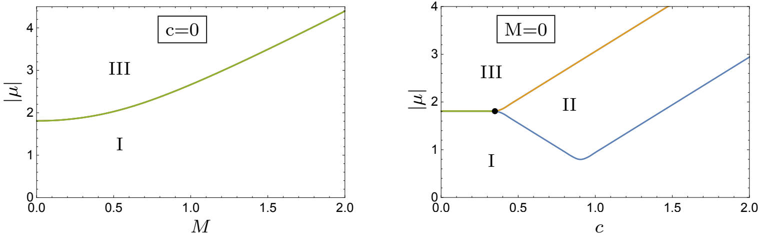
- We observed, at heavy quark limit ($M \gg 1$), $\langle S_3 \rangle_\pm \approx 1$, irrespective of c.

Baryon Chemical Potential

- Adding Baryon chemical potential $g^{-2/3}\mu B_3$, $SU(2)_B \xrightarrow{\text{explicitly broken}} U(1)_B$

$$E(\mu) = E(\mu = 0) + \mu B_3$$

- Degeneracy between mesons, di-quarks and tetra-quarks are lifted.
- New phases emerges:



- **Ground State:**
  1. Phase-I: spin-0 Meson
  2. Phase-II: spin-1 di-quark
  3. Phase-III: spin-0 tetra-quark

- Phase-II: gs is spin-1-di-quark $\implies$ $SO(3)_{rot}$ is spontaneously broken

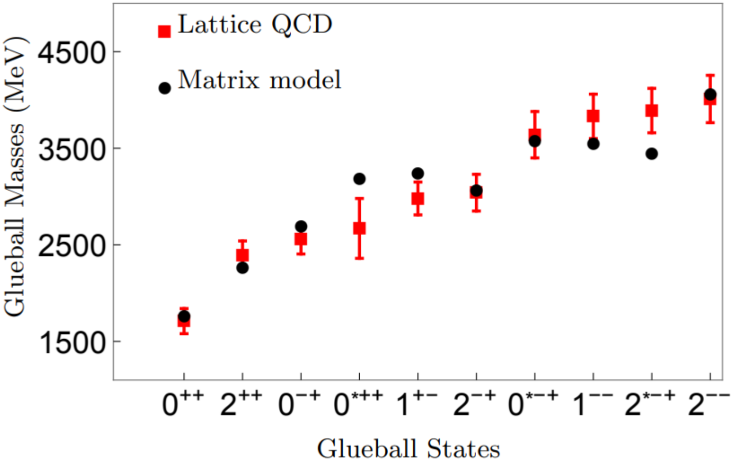
Summary and Future Work

- SU(2) Gauge theory coupled to a fundamental Dirac Fermion.
- Enhanced Global symmetry (Pauli-Gürsey): $U(1)_v \rightarrow SU(2)_B$
- spin-0 and spin-1 Hadrons can be arranged in 5 different sectors. Each sector is labelled by $B(SU(2)_B$ charge) and J (Total Spin).
- QPTs (when we tune c) in different (B,J) sectors – Level crossings in the gs.
- QPTs are 1st order: $\langle Q_0 \rangle = \frac{\partial E_0}{\partial c}$, is discontinuous.
- We studied the **distribution of Spin** among the quark and the glue for different Hadrons belonging to different sectors and different phases.
- Addition of Baryon chemical potential:
  - $SU(2)_B \xrightarrow{\text{Explicitly broken to}} U(1)_B \Rightarrow U(1)_B \times \mathbb{Z}_2$
  - with sufficiently large μ spin-1 di-(anti-) quark can have lower energy than spin-0 meson $\Rightarrow SO(3)_{rot}$ is spontaneously broken.
- Ongoing work: We are currently investigating
  - SU(2) Gauge theory coupled to an adjoint Weyl Fermion. $\Rightarrow \mathcal{N} = 1$ SUSY.
  - 2-color QCD with multiple flavors: for example, 2-flavor case has $SU(4)_B$ symmetry with SSB to Sp(4).
  - SU(3) 3-color 3-flavor QCD

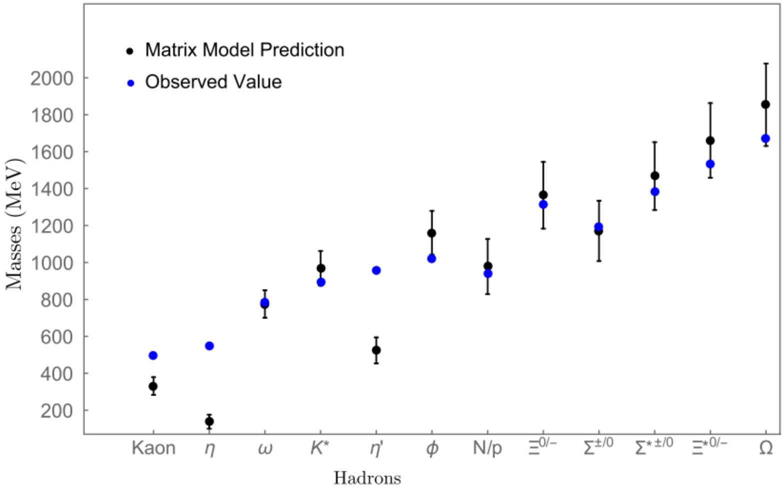
Thank You

Back-up Slides

- Glueball masses obtained from matrix model of SU(3) Yang-Mills theory (N. Acharyya, A. P. Balachandran, M. Pandey, S. Sanyal, S. Vaidya, 2018)



- Matrix model estimation of light Hadron masses (M. Pandey, S. Vaidya, 2020)



Characteristic Polynomial and Configuration space

- It is Interesting to see the Characteristic equation of interaction matrix,

$$\hat{H}_{int} = -\frac{1}{2}\left(\sigma^i \otimes \tau^a\right) M_{ia}$$

- Characteristic Polynomial:

$$\lambda^4 - \frac{\lambda^2}{2} Tr\left(M^T M\right) + \lambda det\, M \frac{1}{16}\left[2\, Tr\left(M^T M\right)^2 - \left(Tr\left(M^T M\right)\right)^2\right] = 0$$

- Re-scale: $x = \frac{\lambda}{\left(\frac{1}{3} Tr(M^T M)\right)^{1/2}} = \frac{\lambda}{g_2}$

$$\Rightarrow x^4 - \frac{3}{2} x^2 - g_3 x + g_4 = 0$$

- where,

$$g_3 = \frac{\det M}{\left(\frac{1}{3} Tr\left(M^T M\right)\right)^{3/2}}, \qquad g_4 = \frac{1}{16}\left[\frac{2\, Tr\left(M^T M\right)^2}{\left(\frac{1}{3} Tr\left(M^T M\right)\right)^2} - 9\right]$$

Characteristic Polynomial and Configuration space

- Discriminant of the qudratic polynomial:

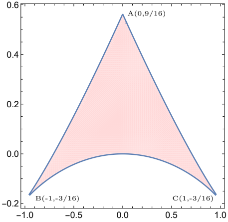
$$\Delta(g_3, g_4) = \frac{1}{2}\left(27g_3^2 - 54g_3^4 + 162g_4 - 432g_3^2 g_4 - 576g_4^2 + 512g_4^3\right)$$

- Four-roots are always real, so $\Delta(g_3, g_4) \geq 0 \Rightarrow$ At Boundary, $\Delta(g_3, g_4) = 0$
- Parity-invariant configurations($M_{ia} \to -M_{ia}$) have, $g_3 = 0$.

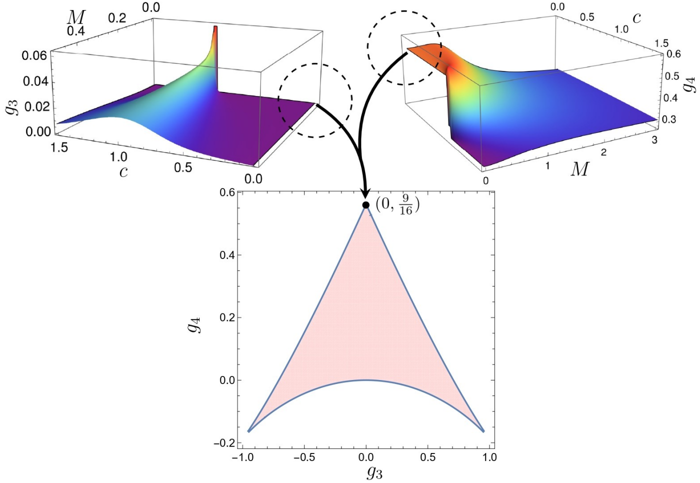
$$\Delta(0, g_4) = g_4\left(16g_4 - 9\right)^2 = 0 \Rightarrow g_4 = 0 \text{ and } 9/16$$

- Allowed values of g_3 and g_4 lie inside the shaded region.

$\Rightarrow$



- For $M = 0$, $c < c_0^*$, the quarks and glue are entangled.
- In this phase, $g_3 \approx 0$ and $g_4 \approx 9/16$
- In $g_3 - g_4$ plane this corresponds to the point $(g_3, g_4) = (0, 9/16)$



Numerical strategy: Diagonalizing the Hamiltonian

- The Hamiltonian is not exactly diagonalizable $\rightarrow$ use variational principle
- Reference Hamiltonian: 9-dim harmononic oscillator $\rightarrow$ Eigenstates $\{|\Phi_n\rangle\}$
- To find the eigenvalues and eigenstates of H (matrix model):
  - $|\Psi_n\rangle \rightarrow$ colorless eigenstates of H:

$$H\,|\Psi_n\rangle = \mathcal{E}_n\,|\Psi_n\rangle$$

  - Expand $|\Psi_n\rangle$ in the basis of $\{|\Phi_n\rangle : n = 0, 1, 2, ...\}$:

$$|\Psi_n\rangle = \sum_{\lambda}\sum_{k=0}^{\infty} C^n_{k,\lambda}\,|\Phi_k\rangle \otimes |F_\lambda\rangle\,, \quad \sum_{\lambda}\sum_{k=0}^{\infty} |C^n_{k,\lambda}|^2 = 1$$

Determine $C^n_{k,\lambda}$ and $\mathcal{E}_n$ from :

$$h^{\lambda\lambda'}_{kk'}\,C^n_{k',\lambda'} = \mathcal{E}_n C^n_{k,\lambda}, \quad h^{\lambda\lambda'}_{kk'} = \Big(\langle F_\lambda| \otimes \langle \Phi_k|\Big) H \Big(|\Phi_{k'}\rangle \otimes |F_{\lambda'}\rangle\Big)$$

Numerical Strategy:

1. Construct $h^{\lambda\lambda'}_{kk'}$ as a matrix $\Leftarrow$ truncate at N_b boson level
2. No truncation in the fermionic Hilbert space
3. Find eigensystem of $h^{\lambda\lambda'}_{kk'}$
4. Increase N_b till the convergence is achieved