Predictions for pion and its excited states spectrum and other observables using holographic QCD plus 't Hooft equation.

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Overview

Light-front QCD and Light-front wavefunction

$$H^{\text{LF}}_{\text{QCD}}|\Psi(P)\rangle = M^2|\Psi(P)\rangle$$

where $H^{\text{LF}}_{\text{QCD}} = P^+P^- - P^2_\perp$ is the LF QCD Hamiltonian and M is the hadron mass. At equal light-front time ($x^+ = 0$) and in the light-front gauge $A^+ = 0$, the hadron state $|\Psi(P)\rangle$ admits a Fock expansion, i.e.

$$|\Psi(P^+, \mathbf{P}_\perp, S_z)\rangle = \sum_{n,h_i} \int [\mathrm{d}x_i][\mathrm{d}^2\mathbf{k}_{\perp i}] \frac{1}{\sqrt{x_i}} \Psi_n(x_i, \mathbf{k}_{\perp i}, h_i)|n : x_i P^+, x_i \mathbf{P}_\perp + \mathbf{k}_{\perp i}, h_i\rangle$$

where $\Psi_n(x_i, \mathbf{k}_{\perp i}, h_i)$ is the LFWF of the Fock state with n constituents and the integration measures are given by

$$[\mathrm{d}x_i] \equiv \prod_i^n \mathrm{d}x_i \delta(1 - \sum_{j=1}^n x_j) \qquad [\mathrm{d}^2\mathbf{k}_{\perp i}] \equiv \prod_{i=1}^n \frac{\mathrm{d}^2\mathbf{k}_{\perp i}}{2(2\pi)^3} 16\pi^3 \delta^2(\sum_{j=1}^n \mathbf{k}_{\perp i}) \, .$$

$(k_i^+, k_i^-, \mathbf{k}_{\perp i})$ and h_i are the momentum and helicity of the i^{th} constituent and $x_i = k_i^+/P^+$.

The valence meson LFWF

For $n = 2$,

$$\mathbf{k}_{\perp 1} = -\mathbf{k}_{\perp 2} = \mathbf{k}_{\perp}$$

$$x_1 = 1 - x_2 = x$$

The position-space conjugate of $\mathbf{k}_{\perp}$, denoted by $\mathbf{b}_{\perp} = b_{\perp} e^{i\varphi}$, is the transverse separation between the quark and the antiquark.

Introduce a new light-front variable $\boldsymbol{\zeta} = \sqrt{x(1-x)}\mathbf{b}_{\perp} = \zeta e^{i\varphi}$ leads to the meson LFWF in the position-space:

$$\Psi(\zeta, x, \varphi) \xlongequal{\text{factorization}} \frac{\phi(\zeta)}{\sqrt{2\pi\zeta}} e^{iL\varphi} X(x)$$

$\phi(\zeta)$ and $X(x) = \sqrt{x(1-x)}\chi(x)$ are referred to as the transverse and longitudinal modes.

The normalization condition: $\int \mathrm{d}x \mathrm{d}^2\mathbf{b}_{\perp} |\Psi(x, \mathbf{b}_{\perp})|^2 = 1$

Holographic Schrödinger equation

Brodsky, de Teramond (PRL, 09)
Brodsky, de Teramond, Dosch, Erlich (Phys. Rep. 15)

In the semi-classical limit, i.e. zero quark mass and no quantum loop, based on AdS/CFT, one can show that the transverse mode of LFWF of the valence ($n = 2$ for mesons) state can be obtained from a 1-dimensional Schrödinger-like wave equation for the:

$$\left(-\frac{d^2}{d\zeta^2} - \frac{1-4L^2}{4\zeta^2} + U_\perp(\zeta)\right)\phi(\zeta) = M_\perp^2\phi(\zeta)$$

the potential is uniquely determined from the conformal symmetry breaking mechanism and correspondence with weakly coupled string modes in AdS_5 space, which results in a light-front harmonic oscillator potential in physical spacetime with confinement scale κ:

$$U_\perp(\zeta, J) = \kappa^4\zeta^2 + \kappa^2(J-1)$$

$J = L + S$ is the total meson angular momemtum where $L \equiv |L_z^{\max}|$.

Solutions to holographic Schrödinger equation

With the confining potential specified, one can solve the holographic Schrödinger equation to obtain the meson mass spectrum,

$$M_{\perp}^2(n_{\perp}, J, L) = 4\kappa^2 \left(n_{\perp} + \frac{J+L}{2}\right)$$

which, as expected, predicts a massless pion. The corresponding normalized eigenfunctions are given by

$$\phi_{n_{\perp} L}(\zeta) = \kappa^{1+L} \sqrt{\frac{2n_{\perp}!}{(n_{\perp}+L)!}} \zeta^{1/2+L} \exp\left(\frac{-\kappa^2\zeta^2}{2}\right) L_{n_{\perp}}^{L}(\kappa^2\zeta^2)$$

To completely specify the holographic meson wavefunction, we need the analytic form of the longitudinal mode $X(x)$. This is obtained by matching the expressions for the pion EM or gravitational form factor in physical spacetime and in AdS space leading to $\chi(x) = 1$ or $X(x) = \sqrt{x(1-x)}$.

π, π' and π'' wavefunction in the chiral limit

In the chiral-limit holographic light-front wavefunction for the pion:

$$\Psi^{\pi}_{m_q=0}(x,\zeta^2) \propto \sqrt{x(1-x)}\,\exp\left(\frac{-\kappa^2\zeta^2}{2}\right) ,$$

which in momentum-space can be written as

$$\Phi^{\pi}_{m_q=0}(x,k_\perp^2) \propto \frac{1}{\sqrt{x(1-x)}}\exp\left(-\frac{k_\perp^2}{2\kappa^2 x(1-x)}\right)$$

Similarly for π' and π'':

$$\Phi^{\pi'}_{m_q=0}(x,k_\perp^2) \propto \frac{\left(k_\perp^2 - x(1-x)\kappa^2\right)}{x^{3/2}(1-x)^{3/2}}\exp\left(-\frac{k_\perp^2}{2\kappa^2 x(1-x)}\right)$$

$$\Phi^{\pi''}_{m_q=0}(x,k_\perp^2) \propto \frac{\left(k_\perp^4 - 4k_\perp^2 x(1-x)\kappa^2 + 2x^2(1-x)^2\kappa^4\right)}{x^{5/2}(1-x)^{5/2}}\exp\left(-\frac{k_\perp^2}{2\kappa^2 x(1-x)}\right)$$

Introducing quark mass

To move away from the chiral limit, Brodsky and de Téramond (BdT) suggested a prescription based on the observation that the chiral-limit of invariant mass of quark-anti-quark pair,

$$\mathcal{M}_{q\bar{q}}^2 = \frac{k_\perp^2 + (1-x)m_q^2 + x m_{\bar{q}}^2}{x(1-x)}$$

appears in $\Phi^\pi_{m_q=0}$, i.e.

$$\Phi^\pi_{m_q=0}(x, k_\perp^2) \propto \frac{1}{\sqrt{x(1-x)}} \exp\left(-\frac{\mathcal{M}_{q\bar{q}}^2\,|_{m_q=0}}{2\kappa^2}\right)$$

Therefore, away from chiral limit where $m_q \neq 0$

$$\Phi^\pi(x, k_\perp^2) \propto \frac{1}{\sqrt{x(1-x)}} \exp\left(-\frac{k_\perp^2 + (1-x)m_q^2 + x m_{\bar{q}}^2}{2\kappa^2 x(1-x)}\right)$$

BdT non-dynamical prescription for the longitudinal mode

Brodsky-de Téramond prescription implies that the longitudinal mode becomes:

$$X(x) = \sqrt{x(1-x)} \exp\left(-\frac{(1-x)m_q^2 + x m_{\bar{q}}^2}{x(1-x)}\right)$$

The bound-state mass eigenvalue consequently receives a first-order correction given by

$$\Delta M^2 = \int \frac{\mathrm{d}x}{x(1-x)} X^2(x) \left(\frac{m_q^2}{x} + \frac{m_{\bar{q}}^2}{1-x}\right)$$

leading to

$$M_\pi^2 = \Delta M^2 \propto 2m_q^2 (\ln\left(\kappa^2/m_q^2\right) - \gamma_E)\ ,$$

where $\gamma_E = 0.577216$ is the Euler's constant. This is on contradiction to GMOR.

Gell-Mann-Oakes-Renner (GMOR) relation

For a pseudoscalar meson of mass M_P, decay constant f_P, made up of quarks with mass m_q, chiral symmetry breaking and confinement are encapsulated in the Gell-Mann-Oakes-Renner (GMOR) relation:

$$M_P^2 f_P^2 = -2\langle q\bar{q}\rangle m_q + \mathcal{O}(m_q^2)\,,$$

where $\langle q\bar{q}\rangle$ is interpreted, in light-front QCD, as the in-meson quark condensate.

While this relation holds for the ground state pion and its excited states, only the ground state pion is a Nambu-Goldstone boson, with nonvanishing decay constant in the chiral limit $\longrightarrow M_\pi^2 \propto m_q$.

For the excited pions, $\pi' \equiv \pi(1300)$ and $\pi'' \equiv \pi(1800)$, GMOR implies that $f^2_{\pi',''} \propto m_q$, provided that their masses do not vanish in the chiral limit.

Longitudinal dynamics using the 't Hooft Equation

G. 't Hooft, A Two-Dimensional Model for Mesons, Nucl. Phys. B 75 (1974) 461470

In an earlier approach, 't Hooft derived a Schrödinger-like equation for the longitudinal mode, starting from the QCD Lagrangian in $(1+1)$-dim in the $N_c \gg 1$ approximation:

$$\left(\frac{m_q^2}{x} + \frac{m_{\bar{q}}^2}{1-x}\right)\chi(x) + U_{\parallel}(x)\chi(x) = M_{\parallel}^2\chi(x)$$

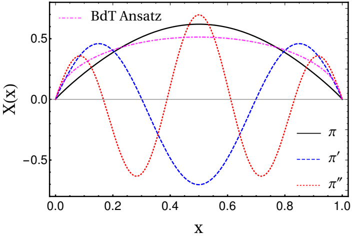
$$U_{\parallel}(x)\chi(x) = \frac{g^2}{\pi}\mathcal{P}\int \mathrm{d}y \frac{\chi(x)-\chi(y)}{(x-y)^2}$$

with g being the longitudinal confinement scale and $\mathcal{P}$ denoting the Cauchy principal value.

- Unlike the holographic light-front Schrödinger Equation, the 't Hooft Equation does not admit analytical solutions and has to be solved numerically.
- Using both the holographic Schrödinger Equation together with the 't Hooft Equation, the meson mass is then given by

$$M^2(n_\perp, n_\parallel, J, L) = 4\kappa^2 \left(n_\perp + \frac{J+L}{2} \right) + M_\parallel^2(n_\parallel, m_q, m_{\bar{q}}, g)$$

Figure shows our numerical results for the dynamical longitudinal mode of the ground-state pion compared to those of the excited states π' and π''.



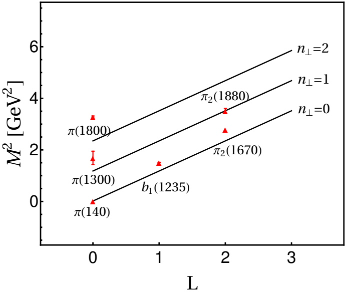
Input parameters

- We now fix the only 3 free parameters: the longitudinal and transverse confinement scales, g and κ, and the light quark mass, $m_{u/d}$, in order to fit the spectroscopic data for the pion family.
- $m_{u/d} = 0.046$ GeV which is the value assumed in hLFQCD together with the Brodsky-de Téramond ansatz.
- We use $\kappa = 0.523$ GeV and $g = 0.109$ GeV.
- The meson's parity and charge conjugation quantum numbers are given by:

$$P = (-1)^{L+1} \, , \quad C = (-1)^{L+S+n_{\parallel}}$$

Mass spectrum

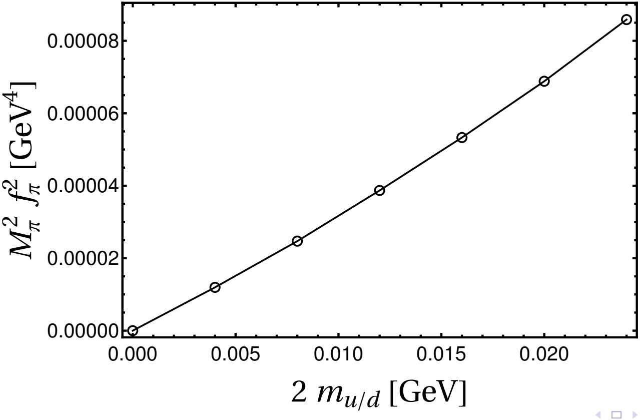
$J^{P(C)}$	Name	$n_\perp$	$n_\parallel$	L	M (MeV)
0^-	$\pi(140)$	0	0	0	134
0^{-+}	$\pi(135)$	0	0	0	134
1^{+-}	$b_1(1235)$	0	2	1	1087
0^{-+}	$\pi(1300)$	1	2	0	1087
2^{-+}	$\pi_2(1670)$	0	4	2	1533
0^{-+}	$\pi(1800)$	2	4	0	1533
2^{-+}	$\pi_2(1880)$	1	6	2	1875



Our computed masses (last column) for $\pi(140)$, $\pi(135)$, and $\pi_2(1880)$ are consistent with experimental data, whereas our predictions for the states $b_1(1235)$, $\pi(1300)$, $\pi_2(1670)$, and $\pi(1800)$ somewhat deviate from experiment. We also notice that our model predicts degeneracies between the states $b_1(1235)$ and $\pi(1300)$, as well as $\pi_2(1670)$ and $\pi(1800)$, which are not realized in Nature.

Quark condensate

It is instructive to check numerically that, as the quark mass goes to zero, our pion mass and decay constant satisfy the GMOR relation. This is indeed the case, as shown in figure. From the slope of the straight line , we can extract a prediction for the in-pion quark condensate: $\langle q\bar{q}\rangle = -(0.143)^3\ \mathrm{GeV}^3$.



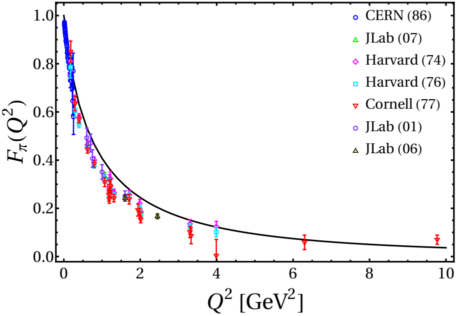
Electromagnetic form factor

The pion's electromagnetic (EM) form factor is defined as

$$\langle \pi(p') | J^{\mu}_{\rm EM}(0) | \pi(p) \rangle = 2(p+p')^{\mu} F_{\pi}(Q^2) \ ,$$

where $p' = p + q$ and $Q^2 = -q^2$. Using the Drell-Yan-West formula, $F_\pi(Q^2)$ can be written as:

$$F_{\pi}(Q^2) = \int \mathrm{d}x \mathrm{d}^2\mathbf{b}_{\perp} J_0[(1-x)b_{\perp}Q] |\Psi^{\pi}(x, \mathbf{b}_{\perp})|^2 \ .$$



Pion charge radius and parton distribution function (PDF)

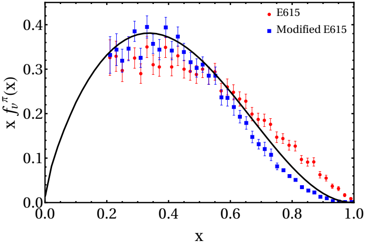
The root-mean-square charge radius of the pion is given by

$$\sqrt{\langle r_\pi^2 \rangle} = \left[\frac{3}{2}\int \mathrm{d}x\mathrm{d}^2\mathbf{b}_\perp [b_\perp(1-x)]^2 |\Psi^\pi(x,\mathbf{b}_\perp)|^2\right]^{1/2}$$

We predict $\sqrt{\langle r_\pi^2 \rangle} = 0.65$ fm, in excellent agreement with the measured value 0.657 ± 0.003 fm.

We also present our prediction for pion PDF, which is defined as:

$$f_v^\pi(x) = \int \mathrm{d}^2\mathbf{b}_\perp |\Psi^\pi(x,\mathbf{b}_\perp)|^2$$



Decay constant

The twist-2 pion DA is defined as:

$$\langle 0|\bar{\Psi}_d(z)\gamma^+\gamma_5\Psi_u(0)|\pi^+\rangle = f_\pi P^+ \int \mathrm{d}x e^{ix(P.z)}\phi_\pi(x,\mu) \; ,$$

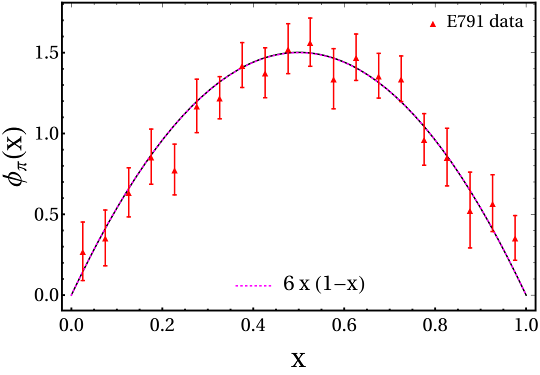
where $z^2 = 0$ and f_π is pion decay constant given as

$$f_\pi = \sqrt{\frac{2N_c}{\pi}} \int \mathrm{d}x \Psi^\pi(x, \mathbf{b}_\perp)\big|_{\mathbf{b}_\perp=0} \, .$$

DC	Our results	Experimental data
f_π	166.46	130.2 ± 1.7
$f_{\pi'}$	1.44	-
$f_{\pi''}$	0.65	-

Pion Distribution Amplitude (DA)

Our predicted DA, after evolution from our model scale, $\mu_0^2 = 0.24\ \mathrm{GeV}^2$, to the experimental scale $\mu^2 = 10\ \mathrm{GeV}^2$ is shown in Fig. 1. While it coincides with the asymptotic form, $6x(1-x)$, and is consistent with the E791 data.



Our prediction (solid-black curve) for pion DA and the asymptotic pion DA (dotted-magenta curve) compared with the E791 data.

Photon-to-pion transition form factor (TFF)

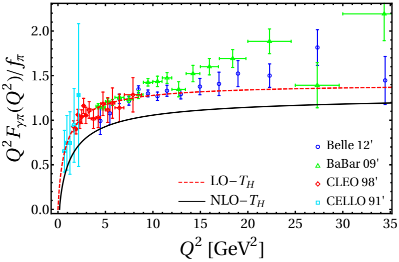
Using the pion DA , we predict the photon-to-pion transition form factor (TFF) given by:

$$Q^2 F_{\gamma\pi}(Q^2) = \frac{\sqrt{2}}{3} f_\pi \int_0^1 \mathrm{d}x\, T_{\rm H}(x, Q^2)\phi_\pi(x, (1-x)Q),$$

where $T_{\rm H}(x, Q^2)$ up to the next-to-leading order (NLO) is given by

$$T_{\rm H}(x, Q^2) = \frac{1}{1-x} + \frac{\alpha_s(Q^2)}{4\pi} C_{\rm F} \frac{1}{1-x}\left[-9 - \frac{1-x}{x}\ln(1-x) + \ln^2(1-x)\right].$$

Here, the color factor is given by $C_{\rm F} = \frac{4}{3}$. We use the QCD scale parameter $\Lambda_{\rm QCD} = 0.204$ GeV in the strong running coupling $\alpha_s(Q^2)$. Our prediction for $Q^2 F_{\gamma\pi}(Q^2)/f_\pi$ using the LO and NLO hard scattering kernel $T_{\rm H}$. As we can see, using the LO kernel, we agree well with the CLEO, CELLO and Belle data but not with the rapidly rising BaBar data.



Conclusion

- We have shown that the 't Hooft Equation is a suitable candidate to complement the holographic Schrödinger Equation of hLFQCD.
- The two equations together provide a more realistic description of the three-dimensional, non-perturbative internal dynamics of the pion and its excited states.
- The result is a qualitative but simultaneous description of pion spectroscopy and dynamics while also being consistent with the chiral-limit constraints implied by the GMOR relation.

Light-front coordinates

Lorentz transformation mixes the components of the space-time 4-vector. $x^\mu \equiv (x^0, x^1, x^2, x^3)$.

However, one can define combinations of the 4-vector components which are the eigenstates of the Lorentz Transformation and so get scaled under Lorentz boost:

$$x^+ = x^0 + x^3\,,\;\; x^- = x^0 - x^3\,,\;\; x^\perp = x^1,\, x^2$$

$$x^2 = x^\mu x_\mu = x^+ x^- - {x^\perp}^2$$

$x^+ \rightarrow$ Light-front time $\qquad x^- \rightarrow$ Light-front distance

For energy momentum 4-vector $p^\mu \equiv (p^0, p^1, p^2, p^3)$
Light-front energy $\rightarrow p^- = p^0 - p^3$
Light-front momentum $\rightarrow p^+ = p^0 + p^3$

Meson holographic LFWF

The meson holographic LFWFs for massless quarks can thus be written in closed form:

$$\Psi_{nL}(\zeta, x, \phi) = e^{iL\phi}\sqrt{x(1-x)}(2\pi)^{-1/2}\kappa^{1+L}\sqrt{\frac{2n!}{(n+L)!}}\zeta^L \exp\left\{\left(\frac{-\kappa^2\zeta^2}{2}\right)\right\} L_n^L(x^2\zeta^2)$$

For non-zero quark mass, Brodsky and de Teramond prescription is to shift the longitudinal mode:

$$X(x) = \sqrt{x(1-x)} \longrightarrow X_{\mathrm{BdT}}(x) = \sqrt{x(1-x)} \exp\left(-\frac{(1-x)m_q^2 + x m_{\bar{q}}^2}{2\kappa^2 x(1-x)}\right)$$

Example: pion LFWF ($m_q = m_{\bar{q}}$)

$$\Psi^{\pi}(x, \zeta^2) = \mathcal{N}\sqrt{x(1-x)} \exp\left\{\left[-\frac{\kappa^2\zeta^2}{2}\right]\right\} \exp\left\{\left[-\frac{m_q^2}{2\kappa^2 x(1-x)}\right]\right\}$$

Meson spectrum

The shift in meson mass when moving away from chiral limit:

$$\Delta M^2_{\mathrm{BdT}} = \int \frac{\mathrm{d}x}{x(1-x)} \times X^2_{\mathrm{BdT}}(x) \left(\frac{m_q^2}{x} + \frac{m_{\bar{q}}^2}{1-x} \right)$$

$$M^2 = M_\perp^2 + \Delta M^2_{\mathrm{BdT}}$$

For pion $M_\perp = 0$ and the above prescription leads to $M_\pi^2 = \Delta M^2_{\mathrm{BdT}} \propto m_q^2$ which is not in agreement with Gell-Mann-Oakes-Renner (GMOR) relation $M_\pi^2 \propto m_q$.

Another problem with this prescription is that it is the same for ground state and excited states mesons.

Numerical solutions

Expand the longitudinal mode onto a Jacobi polynomial basis:

$$\chi(x) = \sum_n c_n f_n(x) \tag{1}$$

with

$$f_n(x) = N_n x^{\beta_1}(1-x)^{\beta_2} P_n^{(2\beta_2,2\beta_1)}(2x-1), \tag{2}$$

where $P_n^{(2\beta_2,2\beta_1)}$ are the Jacobi polynomials and

$$N_n = \sqrt{(2n+\tilde{\beta}_1+\tilde{\beta}_2)} \times \sqrt{\frac{n!\Gamma(n+\tilde{\beta}_1+\tilde{\beta}_2)}{\Gamma(n+\tilde{\beta}_1+1)\Gamma(n+\tilde{\beta}_2)}} \tag{3}$$

Solve the eigenvalue problem for eigenvalues M_L^2 and eigenvectors $\{c_n\}$.

Pion parton distribution function (PDF)

We also present our prediction for pion PDF, which is defined as:

$$f_v^\pi(x) = \int \mathrm{d}^2\mathbf{b}_\perp |\Psi^\pi(x, \mathbf{b}_\perp)|^2 \tag{4}$$

Figure shows our predicted pion valence quark PDF, which we have evolved from an initial scale $\mu_0^2 = 0.240~\mathrm{GeV}^2$ to $\mu^2 = 16~\mathrm{GeV}^2$ relevant to the E615 data using the NNLO DGLAP equations