On the Implications of $|\mathrm{U}_{\mu i}| = |\mathrm{U}_{\tau i}|$ in the Canonical Seesaw Mechanism

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Background

Although neutrino has been proposed by Pauli in 1930[1] and experimentally discovered first by Cowan and Reines in 1956[2], our understanding of neutrinos has never stopped evolving. We now know that neutrino can change from one flavour to another after propagation in spacetime. The most promising reason for this phenomenon is non-degenerate neutrino masses with the mismatch between flavour eigenstates and mass eigenstates. This immediately leads to a natural question: *Where do neutrino masses come from?* At the present stage, the most widely welcome class of mass generation mechanisms are the so-called seesaw mechanisms, in which one or more types of unobserved heavy neutrinos are introduced to the original Standard Model to account for the tiny but non-degenerate masses of the observed neutrinos in a way similar to what a seesaw looks like — one end up and one end down.

[1] W. Pauli, in Proceedings of the Gauverein Conference, Tübingen, Germany, 4 December 1930.

[2] C. L. Cowan Jr. et al., Science **124**, 103–104 (1956).

Background

A large number of models have been proposed for the generation of neutrino masses. As remarked by Edward Witten in the 19th International Conference on Neutrino Physics and Astrophysics (Neutrino 2000) in 2000[3]:

For neutrino masses, the considerations have always been qualitative, and, despite some interesting attempts, there has never been a convincing quantitative model of the neutrino masses.

Witten's opinion is still essentially true after 24 years.

To improve the quantitative predictability of models, one can introduce some constraints, such as flavour symmetries, to reduce those degrees of freedom.

[3] E. Witten, Nucl. Phys. B Proc. Suppl. **91**, 3–8 (2001).

Canonical seesaw mechanism

In **canonical seesaw mechanism**[4], three right-handed neutrino fields $N_{\alpha\mathrm{R}}$ with $\alpha = e, \mu, \tau$ are appended to the Standard Model of particle physics. These new neutrino fields are $\mathrm{SU}(2)_\mathrm{L}$ singlet. The corresponding neutrino mass term with both gauge invariance and Lorentz invariance is as follows:

$$-\mathcal{L}_\nu = \overline{l_\mathrm{L}}\, Y_\nu \tilde{H} N_\mathrm{R} + \frac{1}{2}\overline{(N_\mathrm{R})^c} M_\mathrm{R} N_\mathrm{R} + \mathrm{h.c.}.$$

- l_L: $\mathrm{SU}(2)_\mathrm{L}$ doublet of left-handed lepton fields.
- Y_ν: 3×3 Yukawa coupling matrix.
- $\tilde{H}$: defined with Higgs doublet H and the second Pauli matrix σ_2 as $\tilde{H} := i\sigma_2 H^*$.
- N_R: column vector $(N_{e\mathrm{R}}, N_{\mu\mathrm{R}}, N_{\tau\mathrm{R}})^T$.
- M_R: 3×3 Majorana mass matrix.
- h.c.: "**H**ermitian **c**onjugate".

[4] E.g., P. Minkowski, Phys. Lett. B **67**, 421 (1977). More in the references.

Canonical seesaw mechanism

After spontaneous electroweak symmetry breaking, the part of Lagrangian density responsible for neutrino masses is

$$-\mathcal{L}'_\nu = \frac{1}{2}\begin{pmatrix}\overline{\nu_{\mathrm{L}}} & \overline{(N_{\mathrm{R}})^c}\end{pmatrix}\begin{pmatrix}\mathbf{0} & M_D \\ M_D^T & M_{\mathrm{R}}\end{pmatrix}\begin{pmatrix}(\nu_{\mathrm{L}})^c \\ N_{\mathrm{R}}\end{pmatrix} + \mathrm{h.c.}.$$

- ν_{L}: column vector $(\nu_{e\mathrm{L}}, \nu_{\mu\mathrm{L}}, \nu_{\tau\mathrm{L}})^T$.
- M_D: 3×3 Dirac mass matrix defined by $Y_\nu \langle H \rangle$.
- $\langle H \rangle$: vacuum expectation value of the Higgs field.

Canonical seesaw mechanism

The whole 6×6 mass matrix can be diagonalised by the unitary matrix $\begin{pmatrix} \mathrm{U} & \mathrm{R} \\ \mathrm{S} & \mathrm{Q} \end{pmatrix}$ as follows:

$$\begin{pmatrix} \mathrm{U} & \mathrm{R} \\ \mathrm{S} & \mathrm{Q} \end{pmatrix}^{\dagger} \begin{pmatrix} \mathbf{0} & M_D \\ M_D^T & M_R \end{pmatrix} \begin{pmatrix} \mathrm{U} & \mathrm{R} \\ \mathrm{S} & \mathrm{Q} \end{pmatrix}^{*} = \begin{pmatrix} D_\nu & \mathbf{0} \\ \mathbf{0} & D_N \end{pmatrix}.$$

- $\dagger$: conjugate transpose.
- $*$: complex conjugate.
- D_ν: diagonal matrix with eigenvalues m_1, m_2, m_3.
- D_N: diagonal matrix with eigenvalues M_1, M_2, M_3.

Canonical seesaw mechanism

- Unitarity conditions:

$$\mathrm{U}\mathrm{U}^{\dagger} + \mathrm{R}\mathrm{R}^{\dagger} = \mathrm{S}\mathrm{S}^{\dagger} + \mathrm{Q}\mathrm{Q}^{\dagger} = I,$$
$$\mathrm{U}^{\dagger}\mathrm{U} + \mathrm{S}^{\dagger}\mathrm{S} = \mathrm{R}^{\dagger}\mathrm{R} + \mathrm{Q}^{\dagger}\mathrm{Q} = I,$$
$$\mathrm{U}\mathrm{S}^{\dagger} + \mathrm{R}\mathrm{Q}^{\dagger} = \mathrm{U}^{\dagger}\mathrm{R} + \mathrm{S}^{\dagger}\mathrm{Q} = \mathbf{0}.$$

- Exact seesaw formula:

$$\mathrm{U} D_{\nu} \mathrm{U}^{T} + \mathrm{R} D_{N} \mathrm{R}^{T} = \mathbf{0}.$$

Previous works

In JHEP 06 (2022) 034 (presented in ICHEP2022)[5], it is claimed that the experimentally favoured relation $|\mathrm{U}_{\mu i}| = |\mathrm{U}_{\tau i}|$ (for $i = 1, 2, 3$) necessarily implies $|\mathrm{R}_{\mu i}| = |\mathrm{R}_{\tau i}|$ (for $i = 1, 2, 3$) in the context of canonical seesaw mechanism, from which it is further claimed that in the scenario $\mathrm{U} = \mathcal{P}\mathrm{U}^*$ with $\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ the relation $\mathrm{R} = \mathcal{P}\mathrm{R}^*$ is a necessary consequence and there is a corresponding minimal flavour symmetry in the neutrino mass term under the transformation $\nu_{e\mathrm{L}} \to (\nu_{e\mathrm{L}})^c$, $\nu_{\mu\mathrm{L}} \to (\nu_{\tau\mathrm{L}})^c$, $\nu_{\tau\mathrm{L}} \to (\nu_{\mu\mathrm{L}})^c$ on the left-handed neutrino fields and arbitrary unitary CP transformation on the right-handed neutrino fields.

[5] Z.-z. Xing, JHEP **06**, 034 (2022)

Some results

After carefully re-examining the argument in JHEP 06 (2022) 034, we find that the relation $\mathrm{R} = \mathcal{P}\mathrm{R}^*$ is no more than one of many possibilities that can accommodate $\mathrm{U} = \mathcal{P}\mathrm{U}^*$ in the context of canonical seesaw mechanism. Therefore, the minimal symmetry mentioned earlier is a good guess but does not necessarily exist in the scenario $\mathrm{U} = \mathcal{P}\mathrm{U}^*$

Some results

By substituting $\mathrm{U} = \mathcal{P}\mathrm{U}^*$ into the exact seesaw formula, we have

$$(\mathcal{P}\mathrm{U}^*)D_\nu(\mathcal{P}\mathrm{U}^*)^T + \mathrm{R}D_N\mathrm{R}^T = \mathbf{0}.$$

By simultaneously left- and right-multiplying $\mathcal{P}$ on the above equation, and then taking its complex conjugate, one obtains

$$\mathrm{U}D_\nu\mathrm{U}^T + (\mathcal{P}\mathrm{R}^*)D_N(\mathcal{P}\mathrm{R}^*)^T = \mathbf{0}.$$

Note that we have made use of the properties that D_ν and D_N are both diagonal and real. Comparing the above equation with the previously mentioned exact seesaw formula, one immediately obtain:

$$\mathrm{R}D_N\mathrm{R}^T = (\mathcal{P}\mathrm{R}^*)D_N(\mathcal{P}\mathrm{R}^*)^T.$$

It is claimed in the previous works that the above equation necessarily implies $\mathrm{R} = \mathcal{P}\mathrm{R}^*$. However, this is mathematically not correct, since $\mathrm{R}D_N\mathrm{R}^T = (\mathcal{P}\mathrm{R}^*)D_N(\mathcal{P}\mathrm{R}^*)^T$, as a matrix equation, is not a sufficient condition for $\mathrm{R} = \mathcal{P}\mathrm{R}^*$.

Some results

We show that[6], there exist at least 6 distinct nontrivial classes of 3×3 matrices F, such that for any of these choices the relation $\mathrm{R} D_N \mathrm{R}^T = (\mathrm{R}F) D_N (\mathrm{R}F)^T$ is always true. A more general condition to be satisfied is thus $\mathrm{R}F = \mathcal{P}\mathrm{R}^*$.

[6] J. Lu, A. H. Chan and C. H. Oh, Universe **10**(1), 50 (2024).

Some results

The first class of F has the texture $\begin{pmatrix} 0 & \times & 0 \\ \times & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$:

$$F_1 = \begin{pmatrix} 0 & +\frac{\sqrt{M_1}}{\sqrt{M_2}} & 0 \\ +\frac{\sqrt{M_2}}{\sqrt{M_1}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad F_2 = \begin{pmatrix} 0 & +\frac{\sqrt{M_1}}{\sqrt{M_2}} & 0 \\ -\frac{\sqrt{M_2}}{\sqrt{M_1}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$F_3 = \begin{pmatrix} 0 & -\frac{\sqrt{M_1}}{\sqrt{M_2}} & 0 \\ +\frac{\sqrt{M_2}}{\sqrt{M_1}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad F_4 = \begin{pmatrix} 0 & -\frac{\sqrt{M_1}}{\sqrt{M_2}} & 0 \\ -\frac{\sqrt{M_2}}{\sqrt{M_1}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Some results

The second class of F has the texture $\begin{pmatrix} 0 & 0 & \times \\ 0 & 1 & 0 \\ \times & 0 & 0 \end{pmatrix}$:

$$F_5 = \begin{pmatrix} 0 & 0 & +\frac{\sqrt{M_1}}{\sqrt{M_3}} \\ 0 & 1 & 0 \\ +\frac{\sqrt{M_3}}{\sqrt{M_1}} & 0 & 0 \end{pmatrix}, \qquad F_6 = \begin{pmatrix} 0 & 0 & +\frac{\sqrt{M_1}}{\sqrt{M_3}} \\ 0 & 1 & 0 \\ -\frac{\sqrt{M_3}}{\sqrt{M_1}} & 0 & 0 \end{pmatrix},$$

$$F_7 = \begin{pmatrix} 0 & 0 & -\frac{\sqrt{M_1}}{\sqrt{M_3}} \\ 0 & 1 & 0 \\ +\frac{\sqrt{M_3}}{\sqrt{M_1}} & 0 & 0 \end{pmatrix}, \qquad F_8 = \begin{pmatrix} 0 & 0 & -\frac{\sqrt{M_1}}{\sqrt{M_3}} \\ 0 & 1 & 0 \\ -\frac{\sqrt{M_3}}{\sqrt{M_1}} & 0 & 0 \end{pmatrix}.$$

Some results

The third class of F has the texture $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & 0 \end{pmatrix}$:

$$F_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & +\frac{\sqrt{M_2}}{\sqrt{M_3}} \\ 0 & +\frac{\sqrt{M_3}}{\sqrt{M_2}} & 0 \end{pmatrix}, \qquad F_{10} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & +\frac{\sqrt{M_2}}{\sqrt{M_3}} \\ 0 & -\frac{\sqrt{M_3}}{\sqrt{M_2}} & 0 \end{pmatrix},$$

$$F_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{M_2}}{\sqrt{M_3}} \\ 0 & +\frac{\sqrt{M_3}}{\sqrt{M_2}} & 0 \end{pmatrix}, \qquad F_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{M_2}}{\sqrt{M_3}} \\ 0 & -\frac{\sqrt{M_3}}{\sqrt{M_2}} & 0 \end{pmatrix}.$$

Some results

The fourth class of F has the texture $\begin{pmatrix} \times & 0 & \times \\ 0 & 1 & 0 \\ \times & 0 & \times \end{pmatrix}$: $(\lambda \in \mathbb{R})$

$$F_{13} = \begin{pmatrix} \frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} & 0 & \lambda \\ 0 & 1 & 0 \\ -\frac{\lambda M_3}{M_1} & 0 & \frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}, \quad F_{14} = \begin{pmatrix} \frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} & 0 & \lambda \\ 0 & 1 & 0 \\ \frac{\lambda M_3}{M_1} & 0 & -\frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix},$$

$$F_{15} = \begin{pmatrix} -\frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} & 0 & \lambda \\ 0 & 1 & 0 \\ \frac{\lambda M_3}{M_1} & 0 & \frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}, \quad F_{16} = \begin{pmatrix} -\frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} & 0 & \lambda \\ 0 & 1 & 0 \\ -\frac{\lambda M_3}{M_1} & 0 & -\frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}.$$

Some results

The fifth class of F has the texture $\begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & 1 \end{pmatrix}$: $(\alpha \in \mathbb{R})$

$$F_{17} = \begin{pmatrix} -\frac{\sqrt{M_1 - \alpha^2 M_2}}{\sqrt{M_1}} & \alpha & 0 \\ -\frac{\alpha M_2}{M_1} & -\frac{\sqrt{M_1 - \alpha^2 M_2}}{\sqrt{M_1}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F_{18} = \begin{pmatrix} -\frac{\sqrt{M_1 - \alpha^2 M_2}}{\sqrt{M_1}} & \alpha & 0 \\ \frac{\alpha M_2}{M_1} & \frac{\sqrt{M_1 - \alpha^2 M_2}}{\sqrt{M_1}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$F_{19} = \begin{pmatrix} \frac{\sqrt{M_1 - \alpha^2 M_2}}{\sqrt{M_1}} & \alpha & 0 \\ -\frac{\alpha M_2}{M_1} & \frac{\sqrt{M_1 - \alpha^2 M_2}}{\sqrt{M_1}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F_{20} = \begin{pmatrix} \frac{\sqrt{M_1 - \alpha^2 M_2}}{\sqrt{M_1}} & \alpha & 0 \\ \frac{\alpha M_2}{M_1} & -\frac{\sqrt{M_1 - \alpha^2 M_2}}{\sqrt{M_1}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Some results

The sixth class of F has the texture $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix}$: $(\beta \in \mathbb{R})$

$$F_{21} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} & \beta \\ 0 & -\frac{\beta M_3}{M_2} & -\frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}, \quad F_{22} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} & \beta \\ 0 & \frac{\beta M_3}{M_2} & \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix},$$

$$F_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} & \beta \\ 0 & -\frac{\beta M_3}{M_2} & \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}, \quad F_{24} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} & \beta \\ 0 & \frac{\beta M_3}{M_2} & -\frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}.$$

Some results

Detailed analysis on all these possibilities shows that the $\mathrm{R} = \mathcal{P}\mathrm{R}^*$ is generally not a necessary implication of $\mathrm{U} = \mathcal{P}\mathrm{U}^*$. And the minimal symmetry mentioned earlier is not guaranteed even if $\mathrm{U} = \mathcal{P}\mathrm{U}^*$ is experimentally supported. To reach the genuine flavour symmetry (if it really exists), more constraints are needed.

More details and discussions can be found in our paper **J. Lu, A. H. Chan and C. H. Oh, Universe 10 (2024) 1, 50**.

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Thank you!