The kinematic Hopf algebra for Yang-Mills amplitudes

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Color-kinematic duality

- Color-kinematic duality $\mathscr{A} = \sum \frac{c_i n_i}{d_i}$
- Color factors satisfy various identities (Jacobi, antisymmetry, *or others*) $L(c_i) = 0$
- If there exist kinematic (BCJ) numerators that also satisfy $L(n_i) = 0$
- Then we declare the amplitude satisfies color-kinematics duality, which enables the double copy
- $\mathscr{A}(c_i, n_i) = \sum_i \frac{c_i n_i}{d_i} \to \sum_i \frac{n_i n_i}{d_i} = \mathscr{M}(n_i, n_i)$
- GR = YM x YM, sGAL = NLSM x NLSM, bosonic string = (YM+DF^2) x Z-theory, etc.

Challenges

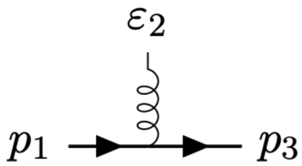
- Computational: find BCJ numerators
- Conceptual: what is the kinematic algebra $L(n_i) = 0$, and where does it come from?
- This talk will be on a new perspective based on "Hopf algebras" [Brandhuber, Chen, Johansson, Travaglini, Wen '22]

HEFT amplitudes

(see talks by Brandhuber, Chen, Travaglini)

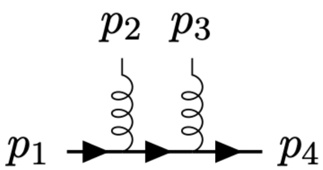
- Heavy mass effective field theory (HEFT) $\qquad p^\mu = m v^\mu$
- Theory coupled to two heavy scalars

$$A_3^{\mathrm{YM-M}}(123) =$$

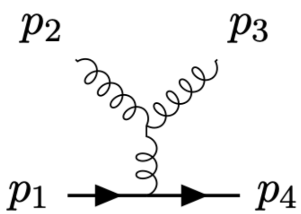


$$= m\varepsilon_2\cdot v$$

$$A_4^{\mathrm{YM-M}}(1234) =$$



$$+$$



$$= 2m\left(-\frac{\varepsilon_2\cdot p_3 v\cdot\varepsilon_3}{s_{23}} - \frac{\varepsilon_2\cdot\varepsilon_3 v\cdot p_2}{s_{23}} + \frac{\varepsilon_3\cdot p_2 v\cdot\varepsilon_2}{s_{23}} + \frac{v\cdot\varepsilon_2 v\cdot\varepsilon_3}{2v\cdot p_2}\right)$$

- Greatly simplifies amplitude but keeps lots of non-trivial structure
- HEFT amps are O(m). [side note: there exists an infinite family of BCJ theories. These extra ones are O(m^3) or higher. see Carrasco's talk]

Hopf algebras

- Hopf algebra= (algebra+coalgebra)+counit+antipode
- Algebra: $V \times V \to V$
- Coalegbra: $V \to V \times V$ (dual to an algebra)
- For us: coproduct of numerators is factorisation $(N \to N_L \times N_R)$ [Brandhuber,Brown,Chen,Gowdy,Travaglini,Wen '23]
- Counit and antipode are relevant for non-abelian Hopf algebra

Hopf algebras, fusion products and maps

- Prenumerator given by a fusion product

$$\widehat{\mathcal{N}}(12\ldots n{-}2, v) = T_{(1)} \star T_{(2)} \cdots \star T_{(n-2)}$$

- Based on a quasi-shuffle product

$$\widehat{\mathcal{N}}(12, v) = T_{(1)} \star T_{(2)} = -T_{(12)}\ ,$$

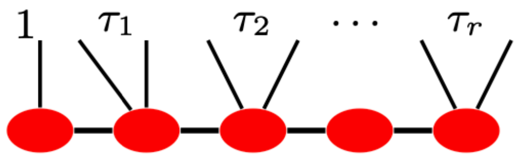
$$\widehat{\mathcal{N}}(123, v) = T_{(1)} \star T_{(2)} \star T_{(3)} = T_{(123)} - T_{(12),(3)} - T_{(13),(2)}$$

- This is algebra is universal

- Actual numerator is given by a map

$$\mathcal{N}([12\ldots n-2], v) = \langle T_{(1)} \star T_{(2)} \star \cdots \star T_{(n-2)} \rangle$$

$$\langle T_{(1\tau_1),(\tau_2),\ldots,(\tau_r)} \rangle :=$$



$$= \frac{1}{n-2} \frac{G_{1\tau_1}(v)}{v{\cdot}p_1} \frac{G_{\tau_2}(p_{\Theta(\tau_2)})}{v{\cdot}p_{1\tau_1}} \cdots \frac{G_{\tau_r}(p_{\Theta(\tau_r)})}{v{\cdot}p_{1\tau_1\ldots\tau_{r-1}}}$$

$$G_\tau(x) = x \cdot F_\tau \cdot v$$

Pure YM from HEFT

- Finally pure YM BCJ numerators can be obtained via a factorization limit
- Not completely satisfactory: not a closed form, has spurious poles
- Question: Are Hopf algebras truly universal? Do they work without HEFT?
- Answer: Yes, and higher derivative corrections in YM lead to a direct pure YM formulation

DF^2

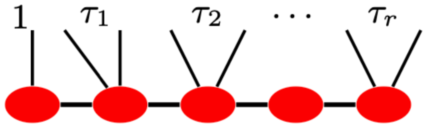
- DF^2 is a theory with massless and massive gluons and tachyons, building block of bosonic strings [Johansson, Nohle '17]

$$A(12,v) = m\frac{\alpha'}{1-2\alpha' p_1.p_2}\frac{\text{Tr}(F_{12})p_1.v}{p_1.p_2}$$

$$\mathcal{N}(12,v) = 2m\frac{\alpha'}{1-2\alpha' p_1.p_2}\text{Tr}(F_{12})p_1.v \longrightarrow \mathcal{N}_{\text{BCJ}}(12,v) = \cdot\ \alpha'\text{tr}(F_{12})p_1\cdot v + \alpha'^2 p_{12}^2 \text{tr}(F_{12})p_1\cdot v$$

Map for DF^2

Same algebra: $\langle T_{(1\tau_1),(\tau_2),\ldots,(\tau_r)}\rangle :=$



$$= \frac{1}{n-2}\frac{G_{1\tau_1}(v)}{v\cdot p_1}\frac{G_{\tau_2}(p_{\Theta(\tau_2)})}{v\cdot p_{1\tau_1}}\cdots\frac{G_{\tau_r}(p_{\Theta(\tau_r)})}{v\cdot p_{1\tau_1\ldots\tau_{r-1}}}$$

Same G function:

$$G_\tau(x) = x\cdot F_\tau\cdot v + \sum_{\sigma_1 i\sigma_2 j\sigma_3=\tau} x\cdot F_{\sigma_1}\cdot p_i W(i\sigma_2 j)p_j\cdot F_{\sigma_3}\cdot v\,,$$

Introduce W':

$$W(\sigma) = \sum_{r=1}^{\lfloor|\sigma|/2\rfloor}\sum_{i_1\rho_1 j_1\sigma_2 i_2\rho_2 j_2\ldots i_r\sigma_r j_r\rho_r=\sigma}\left(\prod_{k=1}^{r} W'(i_k\rho_k j_k)\right)\left(\prod_{k=2}^{r} p_{j_{k-1}}\cdot F_{\sigma_k}\cdot p_{i_k}\right)$$

all α' dependence

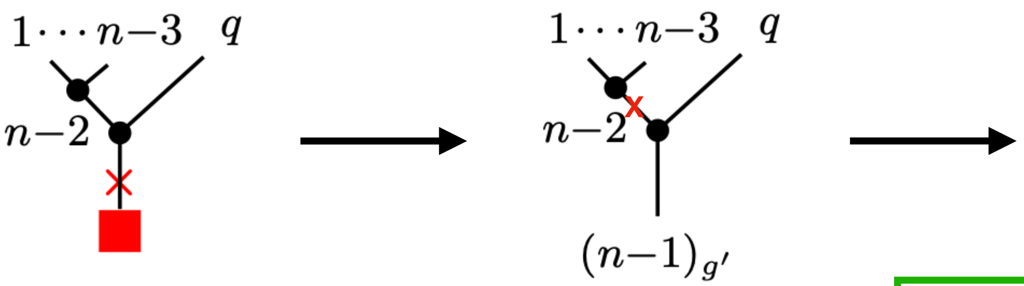
$$W'(12) \sim \frac{\alpha'}{1-\alpha' p_{12}^2}$$

$$\begin{aligned} G_{12}(x) &= x\cdot F_{12}\cdot v + x\cdot p_1 W'(12)p_2\cdot v\,,\\ G_{123}(x) &= x\cdot F_{123}\cdot v + x\cdot F_1\cdot p_2 W'(23)p_3\cdot v\\ &\quad + x\cdot p_1 W'(12)p_2\cdot F_3\cdot v + x\cdot p_1 W'(123)p_3\cdot v \end{aligned}$$

Factorisation

- Two ways to obtain BCJ numerators of pure gluons

1:



gluon BCJ numerator

$$(1-\alpha' p^2_{q1\ldots n-3})(-1)^{n-3} W'(1\ldots n{-}3, q)\, p_q{\cdot}v \quad \longrightarrow \quad \mathcal{N}(12\ldots n{-}3, n{-}2) = (-1)^{n-3} \times \Big((1-\alpha' p^2_{1\ldots(n-3)q}) W'(1\ldots(n{-}3)q)\Big)\Big|_{\text{cuts}}$$

2: $$A(1\ldots n{-}3, n{-}2) = A(1\ldots n{-}3, v)\Big|^{p^2_{1\ldots n-3}=0}_{v\to\varepsilon_{n-2}}$$

Map for DF^2 and limit to YM

- Examples:

$$W'(12) = \frac{\alpha'}{1-\alpha' p_{12}^2} W_0(12)\,,$$

$$W'(123) = \frac{\alpha'}{1-\alpha' p_{123}^2}\Big(W_0(123) - 2W'(23)p_2{\cdot}F_1{\cdot}p_3$$

$$- 2W'(12)p_1{\cdot}F_3{\cdot}p_2 + 2W'(13)p_1{\cdot}F_2{\cdot}p_3\Big)$$

$$W_0(i_1 \ldots i_r) \equiv \mathrm{tr}(F_{[i_1 \ldots i_{r-1}]}F_{i_r})$$

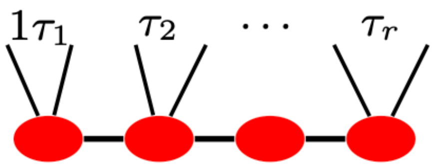
- Take $\alpha' \to 0$?

Map for pure YM

- We can find a map directly for YM

- Modified map $$\mathcal{N}^{\mathrm{YM}}(12\ldots n{-}2) = \langle T_{(1)} \star T_{(2)} \cdots \star T_{(n-3)} \rangle_{\mathrm{YM}}$$

$$\langle T_{(1\tau_1),(\tau_2),\ldots,(\tau_r)} \rangle_{\mathrm{YM}} :=$$



$$= \quad \overline{G}_{1\tau_1} \frac{\overline{G}_{\tau_2}(p_{\Theta(\tau_2)})}{n{-}3{-}|\tau_1|} \cdots \frac{\overline{G}_{\tau_r}(p_{\Theta(\tau_r)})}{n{-}3{-}|\tau_1 \ldots \tau_{r-1}|}$$

$$\overline{G}_{1\tau_1} = \varepsilon_1 \cdot F_{\tau_1} \cdot \varepsilon_{n-2}\,,$$

$$\overline{G}_{\tau_i j}(p_{\Theta(\tau_i j)}) = p_{\Theta(\tau_i j)} \cdot F_{\tau_i} \cdot \varepsilon_j$$

Summary and outlook

- We have shown that the Hopf algebra construction extends beyond YM
- Only the mapping rule requires modification to accommodate DF^2, and the α' corrections are neatly contained in propagators
- Exploiting factorisation on massive gluon poles we can obtain a map directly for pure YM, bypassing the need for HEFT
- Physical implications of Hopf algebras? In scalar/graviton amplitudes?
- Loop level extension?