Superstring loop amplitudes from BCJ numerators at one loop

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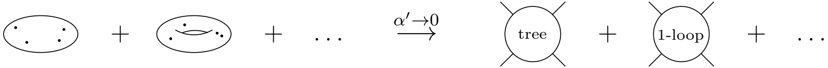
with Yvonne Geyer and Jiachen Guo
arXiv:24xx.xxxxx

From string theory to field theory

Focus on scattering of massless external states.

Field theory limit is $\alpha' \to 0$.



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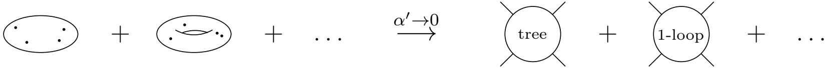
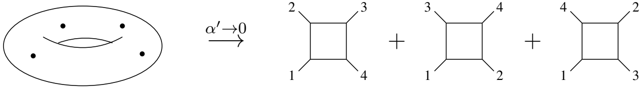


Powerful tool, e.g. first 1-loop results in $\mathcal{N} = 4$ sYM, $\mathcal{N} = 8$ sugra [Green, Schwarz, +Brink 82].



Many lessons, including **double copy**.

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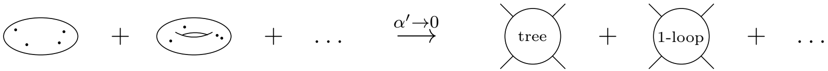

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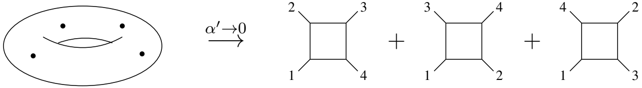

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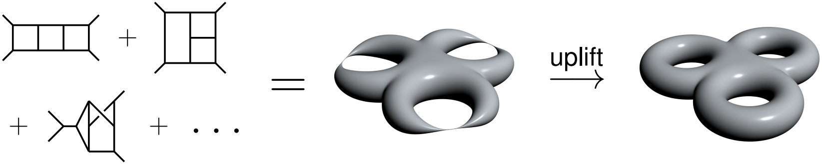
Converse: from field theory to string theory?

Previous success

[Geyer, RM, Stark-Muchão 21]

Complete conjecture for 3-loop 4-pt type II amplitude.



(i) Take SUGRA loop integrand, obtained from SYM. [Bern, Carrasco, Johansson 10]

(ii) Translate it into nodal Riemann sphere.

(iii) Uplift from nodal sphere to higher genus, guided by modular invariance.

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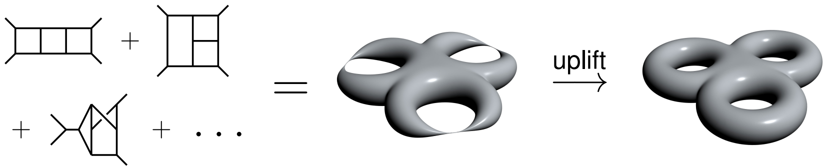

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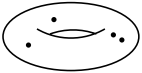
This talk: down to 1 loop, but higher multiplicity ($n = 4, 5, 6, 7, \cdots$)

1-loop type II superstring amplitude

Chiral splitting [D'Hoker, Phong, . . .]



τ

$$\mathcal{A}_n^{(1)} = \int d^{10}\ell \int_{\mathcal{F}} d^2\tau \int_{T^{n-1}} d^2 z_2 \cdots d^2 z_n \left|\mathcal{I}_n\right|^2 \left|\mathrm{KN}_n\right|^2$$

$$\mathrm{KN}_n = \exp \frac{\alpha'}{2}\left(\sum_{1\leq i<j\leq n} k_i \cdot k_j \ln \theta_1(z_{ij},\tau) + 2i\pi\,\ell \cdot \sum_{j=1}^{n} z_j\, k_j \, + i\pi\tau\,\ell^2 \right)$$

1-loop type II superstring amplitude

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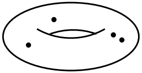

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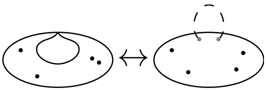
Goal: construct chiral integrand $\mathcal{I}_n$

[builds on: Green, Schwarz, Brink; **Mafra, Schlotterer**, Stieberger]
[Cachazo, He, Yuan; Geyer, Mason, RM, Tourkine; Broedel]
[Adamo, Casali, Skinner; Mizera; Stark-Muchão; ...]

- modular weight
- monodromy invariant: $\mathcal{I}_n\big|^{\ell \to \ell - k_i}_{z_i \to z_i + \tau} = \mathcal{I}_n$
- matches field theory ($\alpha' \to 0$): $\tau \to i\infty$



Construct chiral integrand $\mathcal{I}_n$

- modular invariance: ansatz of weight $n-4$

object	weight	
ℓ_μ	1	
$g_{ij}^{(m)}$	m	coefficients $g^{(m)}(z_i - z_j, \tau)$ of Kronecker-Eisenstein series
G_{2K}	$2K$	absent for $n < 8$

- monodromy invariance: $\mathcal{I}_n\big|^{\ell\to\ell-k_i}_{z_i\to z_i+\tau} = \mathcal{I}_n$ ($g_{ij}^{(m)}$ transform)

- matches field theory: $\tau \to i\infty$

$$(-1)^n(2\pi i)^3 \boxed{\mathcal{I}_n} \prod_{i=2}^{n} dz_i \;\to\; \frac{d^n\sigma}{\text{vol SL}(2,\mathbb{C})} \sum_{\rho\in S_n} \frac{\boxed{N(\rho(1),\rho(2),\cdots,\rho(n);\ell)}}{\sigma_{+\rho(1)}\sigma_{\rho(2)\rho(3)}\cdots\sigma_{\rho(n)-}\sigma_{-+}}$$

$N(\cdots)$ are n-gon BCJ numerators for 10D SYM.

Examples: 4 pt, 5 pt

$\underline{n = 4}$

ST: $\mathcal{I}_4 = t_8(f_1, f_2, f_3, f_4)$ $\qquad f_i^{\mu\nu} = k_i^{[\mu}\epsilon_i^{\nu]}$ $\qquad$ (weight 0)

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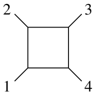

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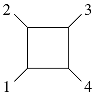

$\underline{n=5}$

ST: $\mathcal{I}_5 = 2\pi i\, C_5^{\mu}\,\ell_\mu + \sum_{\substack{i,j\\ i<j}} C_{5,ij}\, g_{ij}^{(1)}$ $\qquad$ (weight 1)

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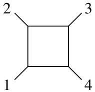

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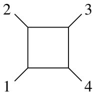

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ST from FT: $C_5^\mu = N(\cdots)|_{\ell_\mu}$ $\quad C_{5,ij} = -N(\cdots[i,j]\cdots)$

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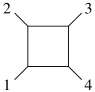

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Can get $\mathcal{I}_5$ from BCJ numerators !

Examples: 6 pt

$\underline{n=6}$

ST: weight 2 ansatz $\left(\ell_\mu\ell_\nu,\ \ell_\mu g^{(1)}_{ij},\ g^{(1)}_{ij} g^{(1)}_{kl},\ g^{(2)}_{kl}\right)$ mod Fay relations

$$\mathcal{I}_6 = (2\pi i)^2\, C_6^{\mu\nu}\,\ell_\mu\ell_\nu + 2\pi i \sum_{\substack{i,j\\ i<j}} C^{\mu}_{6,ij}\,\ell_\mu\, g^{(1)}_{ij} + \sum_{\substack{i,j,k,l\,\mathrm{dist}\\ i<j,k<l,i<k}} C_{6,ij,kl}\, g^{(1)}_{ij}\, g^{(1)}_{kl} + \sum_{\substack{i,j,k\,\mathrm{dist}\\ i<k}} C_{6,ijk}\, g^{(1)}_{ij}\, g^{(1)}_{kj} + \sum_{\substack{i,j\\ i<j}} \tilde{C}_{6,ij}\, g^{(2)}_{ij}$$

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ST from FT: 6-pt BCJ numerators known [Edison, He, Johansson, Schlotterer, Teng, Zhang 22]

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Problem in fixing $\tilde{C}_{6,ij}$: $g^{(2)}_{ij} \to -\pi^2/3$.

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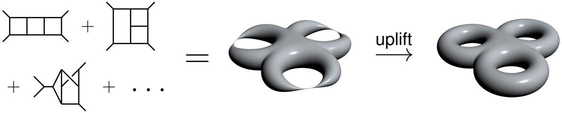
Problem in fixing $\tilde{C}_{6,ij}$: $g^{(2)}_{ij} \to -\pi^2/3$.

Solution: $\tilde{C}_{6,ij}$ uniquely fixed by **monodromy invariance**! $\mathcal{I}_n\big|^{\ell \to \ell - k_i}_{z_i \to z_i + \tau} = \mathcal{I}_n$

Conclusion

Superstring integrands bootstrappable (partly?) by

- modular/monodromy invariance
- field theory limit



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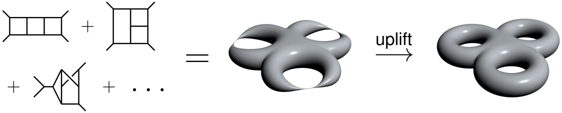

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- higher loops: fate of BCJ?

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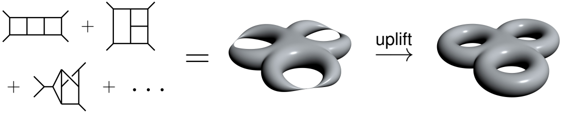

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Other 'string theories' from BCJ-satisfying field theories?