

# Superstring loop amplitudes from BCJ numerators at one loop

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with Yvonne Geyer and Jiachen Guo

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# From string theory to field theory

Focus on scattering of massless external states.

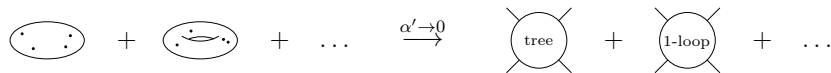
Field theory limit is  $\alpha' \rightarrow 0$ .

The diagrammatic equation shows the field theory limit of string theory amplitudes. On the left, a sum of string theory diagrams is shown: a sphere with four external legs (representing a tree-level amplitude), a sphere with a handle and four external legs (representing a one-loop amplitude), and an ellipsis. An arrow labeled  $\alpha' \rightarrow 0$  points to the right. On the right, a sum of field theory diagrams is shown: a circle with four external legs labeled "tree", a circle with a loop and four external legs labeled "1-loop", and an ellipsis.

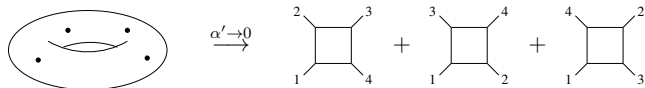
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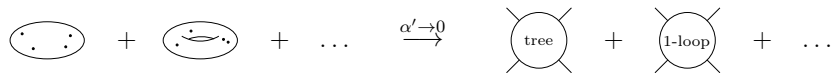


Many lessons, including **double copy**.

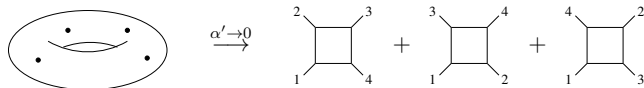
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Converse: **from field theory to string theory?**

Complete conjecture for 3-loop 4-pt type II amplitude.

- (i) Take SUGRA loop integrand, obtained from SYM. [Bern, Carrasco, Johansson 10]
- (ii) Translate it into nodal Riemann sphere.
- (iii) Uplift from nodal sphere to higher genus, guided by modular invariance.

## Previous success

[Geyer, RM, Stark-Muchão 21]

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[This talk](#): down to 1 loop, but higher multiplicity ( $n = 4, 5, 6, 7, \dots$ )

# 1-loop type II superstring amplitude

Chiral splitting [D'Hoker, Phong, ...]



$$\mathcal{A}_n^{(1)} = \int d^{10}\ell \int_{\mathcal{F}} d^2\tau \int_{T^{n-1}} d^2z_2 \cdots d^2z_n |\mathcal{I}_n|^2 |\text{KN}_n|^2$$

$$\text{KN}_n = \exp \frac{\alpha'}{2} \left( \sum_{1 \leq i < j \leq n} k_i \cdot k_j \ln \theta_1(z_{ij}, \tau) + 2i\pi \ell \cdot \sum_{j=1}^n z_j k_j + i\pi\tau \ell^2 \right)$$

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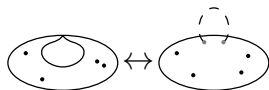
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**Goal:** construct chiral integrand  $\mathcal{I}_n$

[builds on: Green, Schwarz, Brink; **Mafra, Schlotterer**, Stieberger]  
 [Cachazo, He, Yuan; Geyer, Mason, Tourkine; Broedel]  
 [Adamo, Casali, Skinner; Mizera; Stark-Muchão; ...]

- modular weight
- monodromy invariant:  $\mathcal{I}_n|_{z_i \rightarrow z_i + \tau}^{\ell \rightarrow \ell - k_i} = \mathcal{I}_n$
- matches field theory ( $\alpha' \rightarrow 0$ ):  $\tau \rightarrow i\infty$





# Construct chiral integrand $\mathcal{I}_n$

- modular invariance: ansatz of weight  $n - 4$

object	weight	
$\ell_\mu$	1	
$g_{ij}^{(m)}$	$m$	coefficients $g^{(m)}(z_i - z_j, \tau)$ of Kronecker-Eisenstein series
$G_{2K}$	$2K$	absent for $n < 8$

- monodromy invariance:  $\mathcal{I}_n|_{z_i \rightarrow z_i + \tau}^{\ell \rightarrow \ell - k_i} = \mathcal{I}_n$  ( $g_{ij}^{(m)}$  transform)
- matches field theory:  $\tau \rightarrow i\infty$

$$(-1)^n (2\pi i)^3 \boxed{\mathcal{I}_n} \prod_{i=2}^n dz_i \rightarrow \frac{d^n \sigma}{\text{vol SL}(2, \mathbb{C})} \sum_{\rho \in S_n} \boxed{N(\rho(1), \rho(2), \dots, \rho(n); \ell)}$$

$$\sigma_{+\rho(1)} \sigma_{\rho(2)\rho(3)} \cdots \sigma_{\rho(n) - \sigma_{-+}}$$

$N(\dots)$  are  $n$ -gon BCJ numerators for 10D SYM.

## Examples: 4 pt, 5 pt

$$\underline{n = 4}$$

$$\text{ST: } \mathcal{I}_4 = t_8(f_1, f_2, f_3, f_4) \quad f_i^{\mu\nu} = k_i^{[\mu} \epsilon_i^{\nu]} \quad (\text{weight } 0)$$

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ST:  $\mathcal{I}_5 = 2\pi i C_5^\mu \ell_\mu + \sum_{\substack{i,j \\ i < j}} C_{5,ij} g_{ij}^{(1)}$       (weight 1)

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Can get  $\mathcal{I}_5$  from **BCJ numerators** !

## Examples: 6 pt

$$\underline{n = 6}$$

ST: weight 2 ansatz  $(\ell_\mu \ell_\nu, \ell_\mu g_{ij}^{(1)}, g_{ij}^{(1)} g_{kl}^{(1)}, g_{kl}^{(2)})$  mod Fay relations

$$\begin{aligned} \mathcal{I}_6 = & (2\pi i)^2 C_6^{\mu\nu} \ell_\mu \ell_\nu + 2\pi i \sum_{\substack{i,j \\ i < j}} C_{6,ij}^\mu \ell_\mu g_{ij}^{(1)} + \sum_{\substack{i,j,k,l \text{ dist} \\ i < j, k < l, i < k}} C_{6,ij,kl} g_{ij}^{(1)} g_{kl}^{(1)} \\ & + \sum_{\substack{i,j,k \text{ dist} \\ i < k}} C_{6,ijk} g_{ij}^{(1)} g_{kj}^{(1)} + \sum_{\substack{i,j \\ i < j}} \tilde{C}_{6,ij} g_{ij}^{(2)} \end{aligned}$$



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Solution:  $\tilde{C}_{6,ij}$  uniquely fixed by **monodromy invariance!**  $\mathcal{I}_n|_{z_i \rightarrow z_i + \tau}^{\ell \rightarrow \ell - k_i} = \mathcal{I}_n$

# Conclusion

Superstring integrands bootstrappable (partly?) by

- modular/monodromy invariance
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The diagram shows a sum of Feynman diagrams on the left, followed by an equals sign, a 3D rendering of a genus-3 surface (a torus with three holes), and an arrow labeled "uplift" pointing to another 3D rendering of the same genus-3 surface. The Feynman diagrams include a chain of four boxes, a square with two internal lines, a triangle with a line, and an ellipsis.

Lessons for field theory

- superstring ansatz  $\leftrightarrow$  BCJ double copy
- higher loops: fate of BCJ?

# Conclusion

Superstring integrands bootstrappable (partly?) by

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The diagram shows a sum of Feynman diagrams on the left, followed by an equals sign, a 3D rendering of a genus-2 surface (a torus with two holes), and an arrow labeled 'uplift' pointing to another 3D rendering of a genus-2 surface. The Feynman diagrams include a box with three internal lines, a box with two internal lines, and a more complex diagram with multiple internal lines and vertices, followed by an ellipsis.

Lessons for field theory

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Other 'string theories' from BCJ-satisfying field theories?