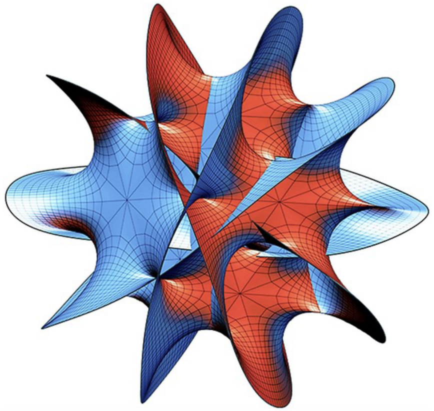

ICHEP 2024

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Constructing polylogarithms on higher-genus Riemann surfaces

Oliver Schlotterer (Uppsala University & Centre for Geometry and Physics)

based on: 2407.11476 with Eric D'Hoker

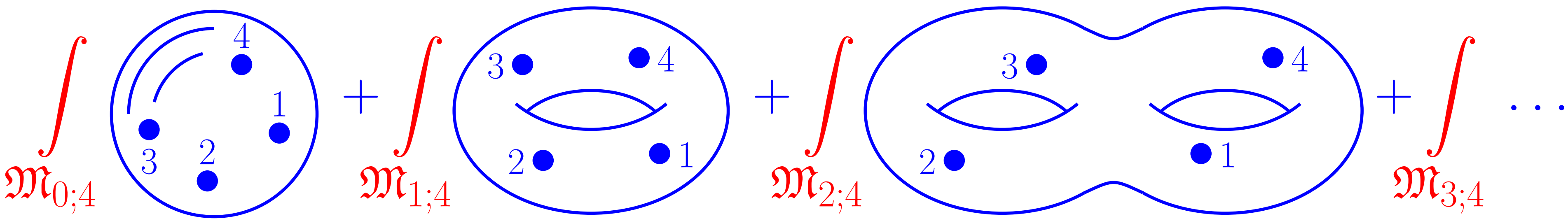
and 2306.08644 with Eric D'Hoker & Martijn Hidding

July 20th 2024

Motivation & Context

Scattering amplitudes are full of challenging integrals

- quantum field theory $\longleftrightarrow$ Feynman integrals (mom. space or duals)
- string theory $\longleftrightarrow$ mod.-space integrals $\mathfrak{M}_{h;n}$ (genus h with n pts.)



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For each geometry, polylogarithms needed as well-adapted function space

- large enough to close under integration over points $\Rightarrow$ primitives for free
- small enough to control properties (functional relations, numerics, ...)

This talk:

- explicit construction of polylogs on Riemann surfaces of arbitrary genus h
- Fay identities = driving force of funct. rel's & closure under integration

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This talk: [sorry, no K3 or higher-dim. Calabi-Yau's this time]

- explicit construction of polylogs on Riemann surfaces of arbitrary genus h
- Fay identities = driving force of funct. rel's & closure under integration

Polylogarithms at genus $h \geq 1$

Polylogs on sphere and torus (genus $h = 0, 1$) have long success story

in both Feynman-integral and string-amplitude computations

- $h = 0$ polylogs = completion of rational fcts. to close under integration

$$G(a_1, \ldots, a_r; z) = \int_0^z \frac{\mathrm{d}x}{x - a_1} G(a_2, \ldots, a_r; x) \, , \qquad G(\emptyset; z) = 1$$

- elliptic polylogs ($h = 1$): Kronecker-Eisenstein kernels $g^{(k \in \mathbb{N}_0)}(x - a|\tau)$

$$\tilde{\Gamma}\left(\begin{smallmatrix} k_1 & \ldots & k_r \\ a_1 & \ldots & a_r \end{smallmatrix} ; z|\tau \right) = \int_0^z \mathrm{d}x \, g^{(k_1)}(x - a_1|\tau) \, \tilde{\Gamma}\left(\begin{smallmatrix} k_2 & \ldots & k_r \\ a_2 & \ldots & a_r \end{smallmatrix} ; x|\tau \right) , \quad \tilde{\Gamma}\left(\begin{smallmatrix} \emptyset \\ \emptyset \end{smallmatrix} ; z|\tau \right) = 1$$

Both close under $\int \mathrm{d}z$ and any $\int \mathrm{d}a_i$ (and $\exists$ algorithms to find primitive)

Amplitudeology was blessed with game-changing help from number theory

& algebraic geometry! **[Bloch, Brown, Dupont, Enriquez, Gangl, Goncharov, Levin, Matthes, Panzer, Racinet, Schnetz, Zagier, Zerbini, . . .]**

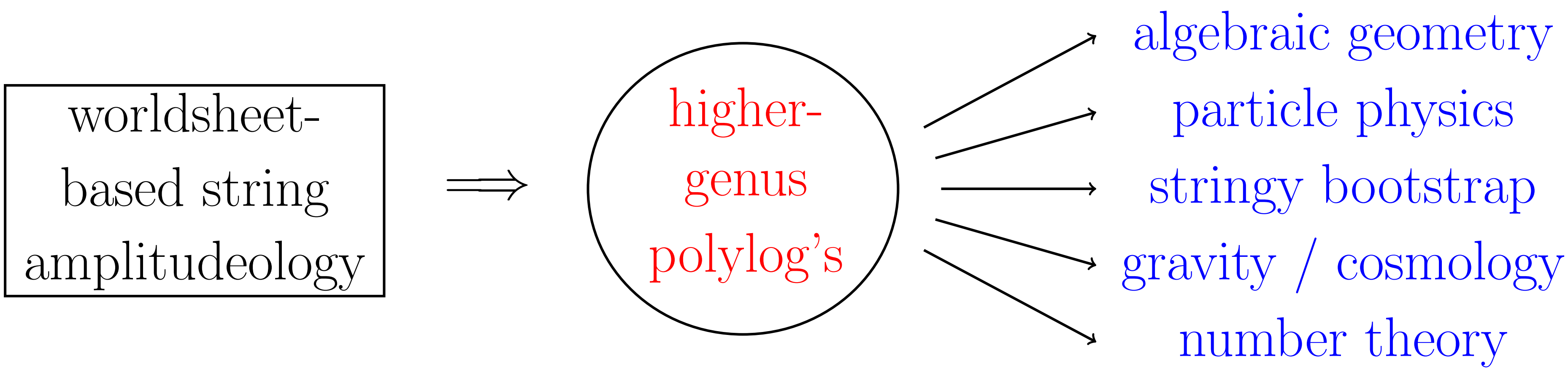
String-theory input for higher-genus polylogs

Strong symbiosis between two questions:

Q1: What is a good set of integration kernels for polylogs on Riemann surfaces of arbitrary genus compatible with closure under integration?

Q2: What is an integration-friendly function space for integrands of multiloop string amplitudes, universal to type I & II / het / bos theories?

Example that string-theoretic objectives / techniques are useful for other fields



[e.g. talks of Giuseppe, Paul, Yoann]

String-theory input for higher-genus polylogs

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$\longrightarrow$ genus zero: $\mathrm{d}\log(z-a)$ kernels of multiple polylogarithms resonate with Parke-Taylor basis for string tree amplitudes in arbitrary theories

$\longrightarrow$ genus one: Kronecker-Eisenstein $g^{(k)}(z|\tau)$ or doubly-periodic $f^{(k)}(z|\tau)$ = unified language for elliptic polylogs & 1-loop string amp's **[Ricardo's talk]**

$\longrightarrow$ now: higher-genus generalization of $f^{(k)}$ **[D'Hoker, Hidding, OS 2306.08644]**

Double-life of Kronecker-Eisenstein kernels at genus one

Instead of mero' / multivalued $g^{(k)}(z|\tau)$ introduce 2×periodic but non-mero' Kronecker-Eisenstein kernels $f^{(k)}(z|\tau) = f^{(k)}(z+1|\tau) = f^{(k)}(z+\tau|\tau)$ by

$$\exp\left(2\pi i\eta \frac{\text{Im } z}{\text{Im } \tau}\right)\frac{\theta_1'(0|\tau)\theta_1(z+\eta|\tau)}{\theta_1(z|\tau)\theta_1(\eta|\tau)} = \frac{1}{\eta} + \sum_{k=1}^{\infty} \eta^{k-1} f^{(k)}(z|\tau)$$

- backbone of elliptic polylogs in formulation of [Brown, Levin 1110.6917]
- function space for 1-loop string integrands (or $g^{(k)}(z|\tau)$ before $\int d^D\ell$)
  [Broedel, Mafra, OS 1412.5535; Gerken, Kleinschmidt, OS 1811.02548]
- $f^{(k)}(z|\tau)$ at rational pt's $z \in \mathbb{Q} + \tau\mathbb{Q}$ $\Rightarrow$ modular forms of congruence subgroups $\Gamma(N)$ $\Rightarrow$ symbol alphabet for elliptic polylogs at rational pt's
  [Broedel, Duhr, Dulat, Penante, Tancredi 1803.10256]
- convolutions of $f^{(k)}$'s $\Rightarrow$ modular graph forms & sv elliptic polylog's
  [Gerken, Kleinschmidt, OS 1911.03476; D'Hoker, Kleinschmidt, OS 2012.09198]

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$$\exp\left(2\pi i\eta \frac{\operatorname{Im} z}{\operatorname{Im} \tau}\right)\frac{\theta_1'(0|\tau)\theta_1(z+\eta|\tau)}{\theta_1(z|\tau)\theta_1(\eta|\tau)} = \frac{1}{\eta} + \sum_{k=1}^{\infty} \eta^{k-1} f^{(k)}(z|\tau)$$

Alternatively construction via bosonic (Arakelov) Green function on $T^2(\tau)$

$$\mathcal{G}(z|\tau) = -\log\left|\frac{\theta_1(z|\tau)}{\eta(\tau)}\right|^2 + 2\pi \frac{(\operatorname{Im} z)^2}{\operatorname{Im} \tau}$$

- base case is derivative: $f^{(1)}(z|\tau) = -\partial_z \mathcal{G}(z|\tau) = \partial_z \log \theta_1(z|\tau) + 2\pi i \frac{\operatorname{Im} z}{\operatorname{Im} \tau}$
- higher $k \geq 2$ kernels recursively obtained from convolutions with $\mathcal{G}$

$$f^{(k)}(x|\tau) = \int_{T^2(\tau)} \frac{\mathrm{d}^2 z}{\operatorname{Im} \tau}\, \partial_x \mathcal{G}(x-z|\tau) f^{(k-1)}(z|\tau)$$

Higher-genus generalization of $f^{(k)}$

Instead of θ_1-representation of $f^{(k)}$, generalize their construction from $\mathcal{G}$:

Arakelov Green function $\mathcal{G}(x, y)$ on higher-genus surface Σ depending on

2 pt's $x, y \in \Sigma$ is uniquely defined by symmetry $\mathcal{G}(x, y) = \mathcal{G}(y, x)$ and

- Laplace eq: $\partial_x \partial_{\bar{x}} \mathcal{G}(x, y) \; = \; \pi\kappa(x) - \pi\delta^2(x, y)$ "locally behaves like log"
- absence of zero mode $\int_\Sigma \mathrm{d}^2x \, \kappa(x)\mathcal{G}(x, y) \; = \; 0$ "integrates to zero"

with $\kappa(x)$ the Kähler form on Σ with unit normalization $\int_\Sigma \mathrm{d}^2x \, \kappa(x) = 1$.

[Faltings '84; Alvarez-Gaumé, Moore, Nelson, Vafa, Bost '86]

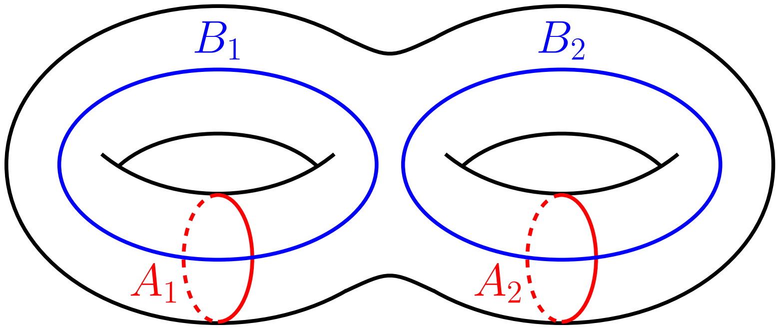
- also $\exists$ representation in terms of the "prime form" (higher-genus θ-fct's)
- separating and non-separating degenerations of $\mathcal{G}(x, y)$ well studied

[D'Hoker, Green, Pioline 1712.06135]

Higher-genus generalization of $f^{(k)}$

Instead of θ_1-representation of $f^{(k)}$, generalize their construction from $\mathcal{G}$:

Convolute with Abelian differentials $\omega_{I=1,2,\ldots,h}(z)$ on genus-h surface Σ



normalization $\displaystyle\oint_{A_I} \omega_J(z)\mathrm{d}z = \delta_{IJ}$

period matrix $\displaystyle\oint_{B_I} \omega_J(z)\mathrm{d}z = \Omega_{IJ}$

with cplx. conjugates $\bar{\omega}^I(z) = [(\mathrm{Im}\,\Omega)^{-1}]^{IJ}\bar{\omega}_J(z)$ @ $I, J = 1, 2, \ldots, h$

Even though $\mathcal{G}(x, z)$ integrates to zero against $\kappa(z) = \frac{1}{h}\bar{\omega}^I(z)\omega_I(z)$ obtain tensorial $f^{(1)}$ kernel from remaining "traceless" h^2-1 vol. forms $\bar{\omega}^J(z)\omega_I(z)$

$$f^I{}_J(x, y) = \int_\Sigma \mathrm{d}^2 z\, \partial_x \mathcal{G}(x, z)\bar{\omega}^J(z)\omega_I(z) - \delta^I_J \partial_x \mathcal{G}(x, y)$$

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Instead of θ_1-representation of $f^{(k)}$, generalize their construction from $\mathcal{G}$:

tensorial $f^{(1)}$ kernel from remaining "traceless" h^2-1 vol. forms $\bar\omega^J(z)\omega_I(z)$

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Higher kernels $f^{(k\geq 2)}$ with $k+1$ free indices mimic recursion of $h=1$ case

$$f^{I_1\ldots I_k}{}_J(x,y) \;=\; \int_\Sigma \mathrm{d}^2 z\, \partial_x \mathcal{G}(x,z)\, \bar\omega^{I_1}(z) f^{I_2\ldots I_k}{}_J(z,y)$$

- kernels $f^{I_1\ldots I_k}{}_J(x,y)$ at rank $k\geq 2$ are regular throughout $\Sigma\times\Sigma$,

  only $k=1$ case has simple pole $\;f^I{}_J(x,y) \;=\; \dfrac{\delta^I{}_J}{x-y} + \mathcal{O}\big((x-y)^0\big)$

- by construction, recover $f^{(k)}(x-y) \;=\; f^{I_1\ldots I_k}{}_J(x,y)\big|_{h=1}$ on torus

[D'Hoker, Hidding, OS 2306.08644]

Higher-genus generalization of elliptic polylog's

Assembly line for higher-genus polylogarithms **[D'Hoker, Hidding, OS 2306.08644]**

- combine f's to flat connection $\mathcal{J}(z,p) = -\pi \mathrm{d}\bar z\, \bar\omega^I(z) b_I + \mathrm{d}z\, \Psi_J(z,p)\, a^J$

  where $\Psi_J(z,p) \ = \ \omega_J(z) + \mathrm{ad}_{b_I} f^I{}_J(z,p) + \mathrm{ad}_{b_{I_1}} \mathrm{ad}_{b_{I_2}} f^{I_1 I_2}{}_J(z,p) + \ldots$

- expand homotopy-inv. path-ordered exp. in words in non-comm. a^J, b_I

$$\mathrm{Pexp}\bigg(\int_y^x \mathcal{J}(z,p) \bigg) = 1 + \int_y^x \mathcal{J}(z,p) + \int_y^x \mathcal{J}(z_1,p) \int_y^{z_1} \mathcal{J}(z_2,p) + \ldots$$

$$= \ 1 + a^J\, \Gamma_J(x,y) + b_I\, \Gamma^I(x,y) + a^J a^K\, \Gamma_{JK}(x,y) + b_I a^J\, \Gamma^I{}_J(x,y;p) + \cdots$$

- polylogarithm $\Gamma_{\ldots J \ldots}{}^{\ldots I \ldots}(x,y;p)$ = coeff. of word $\ldots a^J \ldots b_I \ldots$, e.g.

$$\Gamma^I{}_J(x,y;p) = \int_y^x \mathrm{d}z\, f^I{}_J(z,p) - \pi \int_y^x \mathrm{d}\bar z_1\, \bar\omega^I(z_1) \int_y^{z_1} \mathrm{d}z_2\, \omega_J(z_2)$$

- setting $h=1$ & renaming $a \mapsto a + \frac{\pi b}{\mathrm{Im}\,\tau}$ reproduces elliptic polylogs of

[Brown, Levin 1110.6917]

Fay identities

Closure of polylogs under $\int \mathrm{d}z$ requires bilinear identities among kernels

- genus zero: partial fraction $\frac{1}{(y-z)(z-x)} + \mathrm{cycl}(x,y,z) = 0$

$$\int_0^u \mathrm{d}z \, \frac{G(a_1,\ldots,a_n;z)}{(y-z)(z-x)} = \frac{1}{x-y} \int_0^u \mathrm{d}z \left[\frac{1}{z-x} - \frac{1}{z-y} \right] G(a_1,\ldots,a_n;z)$$
$$= \frac{1}{x-y} \left[G(x,a_1,\ldots,a_n;u) - G(y,a_1,\ldots,a_n;u) \right]$$

- genus one: Fay identities among Kronecker-Eisenstein kernels

$$f^{(s)}(x-z) f^{(r)}(y-z) = -(-1)^s f^{(r+s)}(y-x) + \sum_{\ell=0}^{s} \binom{\ell+r-1}{\ell} f^{(s-\ell)}(x-y) f^{(r+\ell)}(y-z) + \sum_{\ell=0}^{r} \binom{\ell+s-1}{\ell} f^{(r-\ell)}(y-x) f^{(s+\ell)}(x-z)$$

no repeated appearance of z on right-hand side! $\Rightarrow$ friendly to $\int \mathrm{d}z$

[Brown, Levin 1110.6917; Broedel, Mafra, Matthes, OS 1412.5535]

Fay identities

Closure of polylogs under $\int \mathrm{d}z$ requires bilinear identities among kernels

Higher-genus kernels $f^{I_1 \ldots I_k}{}_J(x,y)$ obey tensorial Fay identities such as

$$f^I{}_J(x,y) f^J{}_K(y,z) + f^I{}_J(y,x) f^J{}_K(x,z) - f^I{}_J(x,z) f^J{}_K(y,z)$$
$$+ \omega_J(x) f^{IJ}{}_K(y,x) + \omega_J(y) f^{JI}{}_K(x,z) + \omega_J(x) f^{JI}{}_K(y,z) \ = \ 0$$

- trace w.r.t. I, K yields higher-genus uplift of partial-fraction identity

$$\underbrace{\partial_x \mathcal{G}(x,y) \partial_y \mathcal{G}(y,z) + \partial_y \mathcal{G}(y,x) \partial_x \mathcal{G}(x,z) - \partial_x \mathcal{G}(x,z) \partial_y \mathcal{G}(y,z)}_{\frac{1}{(x-y)(y-z)} + \frac{1}{(z-x)(x-y)} + \frac{1}{(y-z)(z-x)} \, + \, \text{non-singular}} + \text{non-singular} \ = \ 0$$

- at genus one, translation invariance yields cyclic form

$$f^{(1)}(x-y) f^{(1)}(y-z) + f^{(2)}(x-z) + \text{cycl}(x,y,z) = 0$$

Fay identities

Closure of polylogs under $\int \mathrm{d}z$ requires bilinear identities among kernels

Higher-genus kernels $f^{I_1\ldots I_k}{}_J(x,y)$ obey tensorial Fay identities

$$
\begin{aligned}
f^{I_1\ldots I_r}{}_J(z,x) f^{P_1\ldots P_s J}{}_K(y,z) \;=\; & f^{I_1\ldots I_r}{}_J(z,x) f^{P_1\ldots P_s J}{}_K(y,x) \\
& + \sum_{m=0}^{s} (-1)^{m-s-1} \sum_{\ell=0}^{r} f^{(P_s\cdots P_{m+1} \sqcup I_1 \cdots I_\ell)}{}_J(z,y) f^{P_1\cdots P_m J I_{\ell+1}\cdots I_r}{}_K(y,x) \\
& + \sum_{m=0}^{s} (-1)^{m-s-1} f^{P_1\cdots P_m}{}_J(y,x) \big[f^{(P_s\cdots P_{m+1} J \sqcup I_1\cdots I_{r-1}) I_r}{}_K(z,x) \\
& \qquad + f^{(P_s\cdots P_{m+1} \sqcup I_1\ldots I_r) J}{}_K(z,y) \big]
\end{aligned}
$$

no repeated z on RHS!

with shuffles such as $f^{\cdots(P\sqcup I)\cdots}{}_J(x,y) \;=\; f^{\ldots PI\ldots}{}_J(x,y) + f^{\ldots IP\ldots}{}_J(x,y)$

[D'Hoker, OS 2407.11476]

Conclusion & Outlook

- QFT and string amplitudes call for function spaces that close under integration over points – separate construction for different geometries
- explicitly constructed polylog's on Riemann surfaces of arbitrary genus: homotopy-invariant iterated integrals from flat connection $\ni$ f-kernels
- Fay identities generalizing partial fraction are needed for closure under integration and derivation of functional relations

Thank you for your attention !

Applications to string-amplitude computations

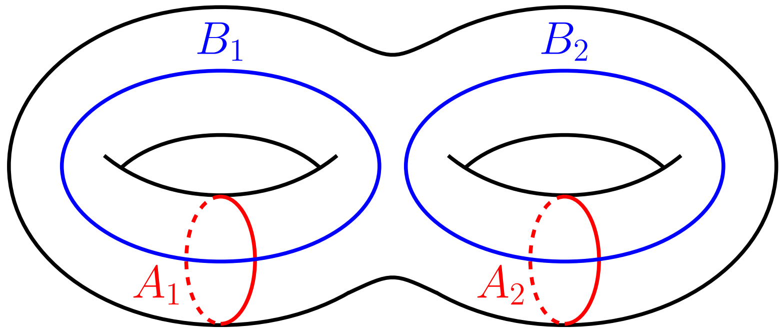
Bottleneck in $h \geq 2$ loop amplitudes of RNS superstring: simplify $\prod$ of

$$S_\delta(x,y) = \frac{\theta[\delta]\big(\int_y^x \omega_I\big)}{\theta[\delta](0)E(x,y)} \quad \text{fermion Green fct's or "Szegö kernel"}$$

and their summation over "spin structures δ"

$\longrightarrow$ 2^{2h} configurations of $\pm$ that 2dim

fermions pick up under A_I, B_J shifts



Higher-genus $f^{I_1\ldots I_k}{}_J(x,y)$-kernels completely disentangle z_i-dependence

from δ-dependence in cyclic products $S_\delta(z_1,z_2)S_\delta(z_2,z_3)\ldots S_\delta(z_n,z_1)$

- $\sum_\delta(S_\delta$-cycles) are essential parts of chiral amplitudes at $h=1,2$ loops

[D'Hoker, Phong 0501197; D'Hoker, OS 2108.01104]

- part of recent proposal for 4pt chiral amplitude at $h=3$ loops

[Geyer, Monteiro, Stark-Muchão 2106.03968]

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$$S_\delta(z_1,z_2)S_\delta(z_2,z_3)S_\delta(z_3,z_1) \ = \ F^{(3)}_{IJK}(\vec z)\, C_\delta^{IJK} + F^{(2)}_{JK}(\vec z)\, C_\delta^{JK} + F^{(0)}(\vec z)$$

z_i-independent, govern SUSY decomposition

with $F^{(3)}_{IJK}(\vec z) = \omega_I(1)\omega_J(2)\omega_K(3)$ and

$$F^{(2)}_{JK}(\vec z) \ = \ \omega_I(1) f^I{}_J(2,3)\omega_K(3) + \text{cycl}(1,2,3)$$

$$F^{(0)}(\vec z) \ = \ \big(\partial_1 \mathcal{G}(1,3) - \partial_1 \mathcal{G}(1,2)\big)\partial_2\partial_3 \mathcal{G}(2,3) - \tfrac{1}{h}\omega_I(1)\partial_3 f^{IK}{}_K(2,3)$$

[D'Hoker, Hidding, OS 2308.05044]

Applications to string-amplitude computations

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from δ-dependence in cyclic products $S_\delta(z_1,z_2)S_\delta(z_2,z_3)\ldots S_\delta(z_n,z_1)$

$$S_\delta(z_1,z_2)S_\delta(z_2,z_3)\ldots S_\delta(z_n,z_1) = F^{(0)}(\vec{z}) + \sum_{r=2}^{n} F^{(r)}_{I_1\cdots I_r}(\vec{z})\, C_\delta^{I_1\cdots I_r}$$

with $F^{(r)}_{I_1\cdots I_r}(\vec{z})$ indep. on δ & modular tensors $C_\delta^{I_1\cdots I_r}$ indep. on z_i

[D'Hoker, Hidding, OS 2308.05044]

Next steps:

- simplify integral representations of $C_\delta^{I_1\cdots I_r}$ & rewrite via θ-fct's
- extend to open chains $S_\delta(x,z_1)S_\delta(z_1,z_2)\ldots S_\delta(z_n,y)$ at $x \neq y$

Meromorphic kernels

How do mero' Kronecker-Eisenstein kernels $g^{(k)}$ generalize beyond genus 1?

$\rightarrow$ Enriquez implicitly defined meromorphic but multi-valued connection ...

... with mero' coefficients $g^{I_1\ldots I_k}{}_J(x,y)$ multiplying $\mathrm{ad}_{b_{I_1}}\ldots\mathrm{ad}_{b_{I_k}}a^J$

... monodromies $g^{I_1\ldots I_k}{}_J(x{+}B_L,y) = \sum_{\ell=0}^{k}\frac{(-2\pi i)^\ell}{\ell!}\delta_L^{I_1}\ldots\delta_L^{I_\ell}g^{I_{\ell+1}\cdots I_k}{}_J(x,y)$

generalizing $g^{(k)}(x{+}\tau) = \sum_{\ell=0}^{k}\frac{1}{\ell!}(-2\pi i)^\ell g^{(k-\ell)}(x)$ to arbitrary genus

... including $\omega_J(x) = g^{\emptyset}{}_J(x,y)$ as $k=0$ instance **[Enriquez 1112.0864]**

- in chiral splitting / before $\prod_{J=1}^{h}\int \mathrm{d}^D\ell_J$, expect $g^{I_1\ldots I_k}{}_J(x,y)$ to be suitable function space for chiral string amplitudes

- expressing $g^{I_1\ldots I_k}{}_J(x,y)$ in terms of $f^{I_1\ldots I_k}{}_J(x,y)$: under investigation

[D'Hoker, Enriquez, OS, Zerbini: work in progress]

Meromorphic kernels

Conjecture: Fay id's of $f^{I_1 \ldots I_k}{}_J(x,y)$ hold in identical form for $g^{I_1 \ldots I_k}{}_J(x,y)$

$$f^I{}_J(x,y) f^J{}_K(y,z) + f^I{}_J(y,x) f^J{}_K(x,z) - f^I{}_J(x,z) f^J{}_K(y,z)$$

$$+\omega_J(x) f^{IJ}{}_K(y,x) + \omega_J(y) f^{JI}{}_K(x,z) + \omega_J(x) f^{JI}{}_K(y,z) \, = \, 0$$

$$g^I{}_J(x,y) g^J{}_K(y,z) + g^I{}_J(y,x) g^J{}_K(x,z) - g^I{}_J(x,z) g^J{}_K(y,z)$$

$$+\omega_J(x) g^{IJ}{}_K(y,x) + \omega_J(y) g^{JI}{}_K(x,z) + \omega_J(x) g^{JI}{}_K(y,z) \, = \, 0$$

[D'Hoker, OS 2407.11476; proof under discussion with Enriquez, Zerbini]

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Conjecture: Fay id's of $f^{I_1\ldots I_k}{}_J(x,y)$ hold in identical form for $g^{I_1\ldots I_k}{}_J(x,y)$

$$g^{I_1\ldots I_r}{}_J(z,x)g^{P_1\ldots P_s J}{}_K(y,z) \;=\; g^{I_1\ldots I_r}{}_J(z,x)g^{P_1\ldots P_s J}{}_K(y,x)$$
$$+\sum_{m=0}^{s}(-1)^{m-s-1}\sum_{\ell=0}^{r} g^{(P_s\cdots P_{m+1}\sqcup I_1\cdots I_\ell)}{}_J(z,y)g^{P_1\cdots P_m J I_{\ell+1}\cdots I_r}{}_K(y,x)$$
$$+\sum_{m=0}^{s}(-1)^{m-s-1}g^{P_1\cdots P_m}{}_J(y,x)\Big[g^{(P_s\cdots P_{m+1}J\sqcup I_1\cdots I_{r-1})I_r}{}_K(z,x)$$
$$+\,g^{(P_s\cdots P_{m+1}\sqcup I_1\ldots I_r)J}{}_K(z,y)\Big]$$

no repeated z on RHS!

[D'Hoker, OS 2406.abcde; proof under discussion with Enriquez, Zerbini]

Alternative to meromorphic & multivalued connection of **[Enriquez 1112.0864]**:

meromorphic and single-valued connection with higher poles $(x-y)^{\leq -2}$

[Enriquez, Zerbini 2110.09341, 2212.03119]