(In)stability of the Higgs vacuum from the $O(N)$ model at large N

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Overview

- Techniques
  - Thermal field theory
  - Large-N expansion
- $\mathrm{O}(N)$ model at large N in $3+1\mathrm{D}$
  - History
  - Renormalization
  - Equation of state and phase structure
  - Non-hermiticity
  - Stability of phases
- Coupling to $\mathrm{U}(1)$ gauge field
  - Mass generation without Higgs mechanism

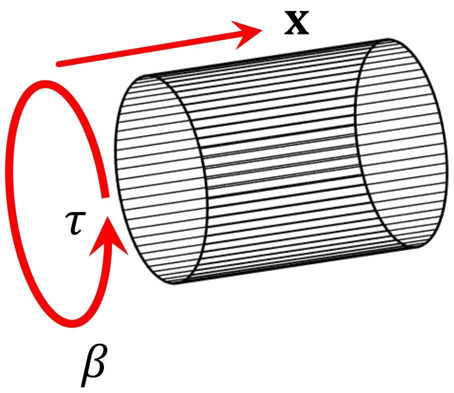
Thermal quantum field theory

- Work in d Euclidean spacetime dimensions
- Coordinates $x = (\tau, \mathbf{x})$; $\tau \in [0, \beta)$ is Euclidean time; $\mathbf{x} \in \mathbb{R}^{d-1}$
- $\beta = 1/T$ is inverse temperature; thermal partition function Z is (with field content Ψ)

$$Z = \mathrm{tr}\left(e^{-\beta H}\right) = \int \mathcal{D}\Psi \; e^{-S_{\mathrm{E}}[\Psi]} \qquad S_{\mathrm{E}}[\Psi] = \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^{d-1}\mathbf{x}\, \mathcal{L}_{\mathrm{E}}(\Psi, \partial_\mu \Psi, \dots)$$

- $\omega_n = 2\pi n/\beta$ are Matsubara frequencies
- $\mathbb{N}$-spin fields satisfy periodic boundary conditions

$$\phi(\tau + \beta, \mathbf{x}) = \phi(\tau, \mathbf{x}),$$



for example, see Vuorinen and Laine, *Basics of Thermal Field Theory* (2016)

Large-N expansion

- Large N gives a useful alternative to perturbative expansion
  - Gives access to non-perturbative physics
  - Valid at all coupling strengths

Romatschke, arXiv:2310.00048 [hep-th]; Moshe and Zinn-Justin, *Phys. Rept.* 385 (2003) 69; Fradkin, *Quantum Field Theory: an Integrated Approach* (2021), Ch. 17

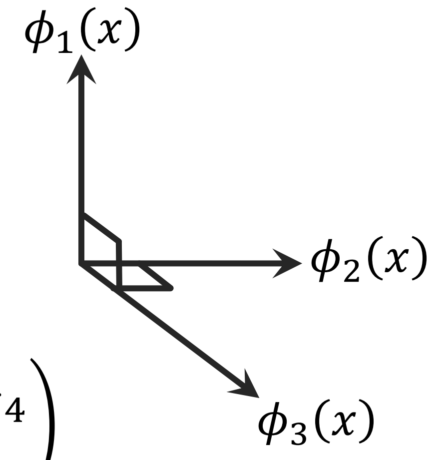
- Scalar $\mathrm{O}(N)$ models, marginal $\mathrm{U}(N)$ fermion models, $\mathrm{U}(N)$ unitary Fermi gases, for example

Romatschke, *Int. J. Mod. Phys. A* 38 (2023) 28, 2350157; Grable and Weiner, *JHEP* 09 (2023) 017

- Renormalization is simpler Romatschke, arXiv:2401.06847 [hep-th]
- Possible to calculate
  - equations of state, phase structures
  - bound states
  - transport coefficients e.g. shear viscosity η/s, curvature coupling κ

Aarts and Martinez Resco, *JHEP* 02 (2004) 061; Romatschke, *PRD* 100 (2019) 5, 054029; Romatschke, *PRL* 127 (2021) 11, 111603; Lawrence and Romatschke, *PRA* 107 (2023) 3, 033327; Weiner and Romatschke, *JHEP* 01 (2023) 046

$O(N)$ model at large N in 3+1D



- N scalar fields $\vec{\phi}(x) \in \mathbb{R}^N$; $\vec{\phi}$ is a Lorentz scalar, $O(N)$ vector
- Thermal partition function in $d = 4$

$$Z \propto \int \mathcal{D}^N\vec{\phi}\ e^{-S[\vec{\phi}]} \qquad S[\vec{\phi}] = \int_{\beta,V} \mathrm{d}^4x \left(\frac{1}{2}\left(\partial_\mu\vec{\phi}\right)^2 + \frac{1}{2}M^2\vec{\phi}^2 + \frac{\lambda}{N}\vec{\phi}^4\right)$$

$$\vec{\phi}^4 \equiv \left(\vec{\phi}\cdot\vec{\phi}\right)^2$$

- Theory of an independent Higgs field for $N = 4$
- Can be solved exactly at large N
  - Can study spontaneous symmetry breaking (SSB)
  - Relevant for Higgs physics

History of the model

- Kobayashi and Kugo, 1975:

> To the present authors' knowledge, there are no rigorous reason to reject the possibility of negative λ. It should be noted that, in the present formalism, we are naturally led to the theory with negative λ and possibly asymptotically free.

Kobayashi and Kugo, *Prog. Theor. Phys.* 54 (1975) 1537

- Coleman, Jackiw, and Politzer, 1974:

> been able to definitely resolve the matter. In any case, the theory in the leading $1/N$ approximation is sick, and furthermore sick in a uniform way;

Coleman, Jackiw, and Politzer, *PRD* 10 (1974) 2491

History of the model

- Abbott, Kang, and Schnitzer, 1976:

The most important global conclusion to be drawn from our calculations is that the $1/N$ expansion appears to be a consistent approximation scheme. The tachyons characteristic of bubble sums can be removed by finding the correct vacuum state for the construction of Green's functions. However, the manner in which this comes about is rather surprising in that spontaneous symmetry breakdown cannot occur in the large-N limit, Goldstone phenomena are not possible, and the ground state of the theory is $O(N)$-symmetric.

Abbott, Kang, and Schnitzer, *PRD* 13 (1976) 2212

History of the model

- Bardeen and Moshe, 1980s:

bosons. This phase reflects the intrinsic instability of this theory, and the large-ϕ_c structure indicates that the renormalized $\lambda\Phi^4$ theory is inconsistent. The only acceptable version of this theory is its regularized form which becomes a free-field theory as the regularization is removed.

from below as seen in Sec. III. We conclude that the renormalized $O(N)\lambda\Phi^4$ vector theory is futile for $T\to\infty$

Bardeen and Moshe, *PRD* 28 (1983) 1372

This strange behavior of the renormalized theory can be traced directly to the negative coupling constant. Since

Bardeen and Moshe, *PRD* 34 (1986) 1229

- Claimed high-temperature instability in theory
- Results were dependent on UV cutoff?

see also **RW**, arXiv:2310.02516 [hep-th]

Rescuing the theory

- We demonstrate the stability and consistency of the model
  - $O(N)$-symmetric phase at **low** temperatures
  - Spontaneous breaking of $O(N)$ symmetry (SSB) at **high** temps
  - Tachyon-free

Su, **RW**, and Romatschke, in preparation

- Results are independent of UV cutoff
  - Negative coupling constant in UV… what could this mean?
  - Can still give a consistent, albeit Non-Hermitian, theory

RW, arXiv:2310.02516 [hep-th]

Solving the model

- $\lambda\vec{\phi}^4$ interaction term dealt with via Hubbard-Stratonovich transformation
  - Introduce auxiliary field ζ:

$$Z \propto \int \mathcal{D}^N\vec{\phi}\,\mathcal{D}\zeta\; e^{-S[\vec{\phi},\zeta]} \qquad S[\vec{\phi},\zeta] = \int_{\beta,V} \mathrm{d}^4x \left(\frac{1}{2}\left(\partial_\mu\vec{\phi}\right)^2 + \frac{1}{2}\left(M^2+\mathrm{i}\zeta\right)\vec{\phi}^2 + \frac{N}{16\lambda}\zeta^2\right)$$

- Split $\zeta, \vec{\phi}$ into zero modes $\zeta_0, \vec{\phi}_0$ plus non-zero modes $\zeta', \vec{\phi}'$
  - At leading order in large N only ζ_0 contributes to partition function
  - Splitting $\vec{\phi} = \vec{\phi}_0 + \vec{\phi}'$ allows to study SSB

$$Z \propto \int \mathcal{D}^N\vec{\phi}'\,\mathrm{d}\zeta_0\,\mathrm{d}\vec{\phi}_0\; e^{-S[\vec{\phi}',\zeta_0,\vec{\phi}_0]}$$

$$S[\vec{\phi}',\zeta_0,\vec{\phi}_0] = \frac{N\beta V}{16\lambda}\zeta_0^2 + \frac{\beta V}{2}\left(M^2+\mathrm{i}\zeta_0\right)\vec{\phi}_0^2 + \int_{\beta,V} \mathrm{d}^4x \left(\frac{1}{2}\left(\partial_\mu\vec{\phi}'\right)^2 + \frac{1}{2}\left(M^2+\mathrm{i}\zeta_0\right)\vec{\phi}'^2\right)$$

Renormalization

- Integrate out non-zero modes $\vec{\phi}'$ (action is quadratic in $\vec{\phi}'$)

$$Z \propto \int \mathrm{d}\zeta_0 \, \mathrm{d}\vec{\phi}_0 \, e^{N\beta V p(\zeta_0, \vec{\phi}_0)}$$

  - $p(\zeta_0, \vec{\phi}_0)$ is pressure per component

$$m^2 = M^2 + \mathrm{i}\zeta_0$$

$$p(\zeta_0, \vec{\phi}_0) = -\frac{\zeta_0^2}{16\lambda} - \frac{1}{2N} m^2 \vec{\phi}_0^2 - \frac{T}{2} \sum_n \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \ln(\omega_n^2 + \mathbf{k}^2 + m^2)$$

- Thermal sum-integral can be done in dim reg in $4 - 2\epsilon$ dimensions

Vuorinen and Laine, *Basics of Thermal Field Theory* (2016)

modified Bessel functions of second kind

$$p(m^2, \vec{\phi}_0) = \frac{(m^2 - M^2)^2}{16\lambda} - \frac{1}{2N} m^2 \vec{\phi}_0^2 + \frac{m^4}{64\pi^2} \left(\frac{1}{\epsilon} + \ln\left(\frac{\bar{\mu}^2}{m^2}\right) + \frac{3}{2} \right) + \frac{m^2}{2\pi^2 \beta^2} \sum_{n=1}^{\infty} \frac{K_2(n\beta m)}{n^2}$$

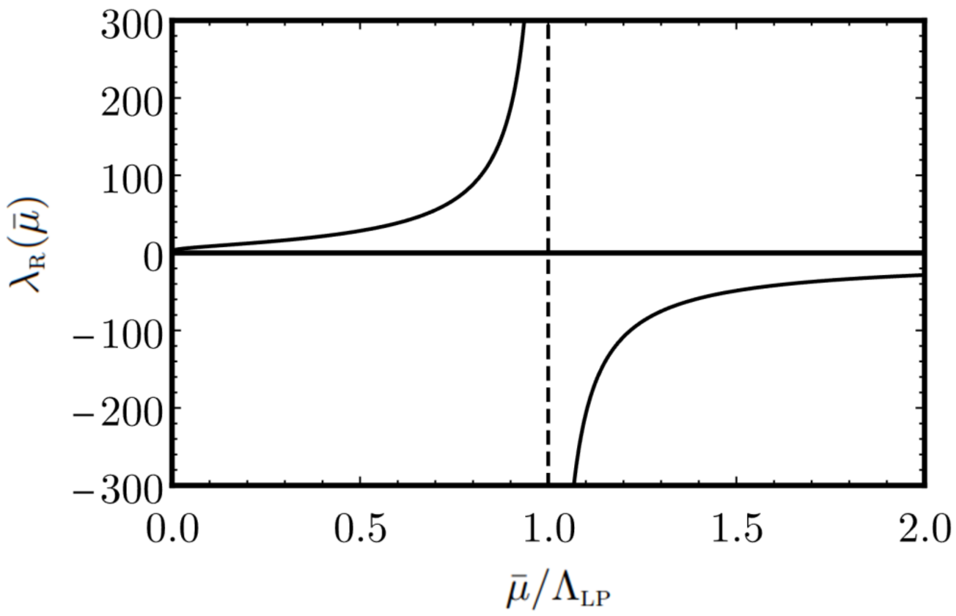
Renormalization

- $\overline{\text{MS}}$ renormalization of $\mathcal{O}(m^4)$ term fixes running coupling
  - Λ_C is dimensionful scale of the theory produced by integration of beta function

$$\frac{1}{\lambda_\text{R}} = \frac{1}{\lambda} + \frac{1}{4\pi^2\epsilon}$$

$$\lambda_\text{R}(\bar{\mu}) = \frac{4\pi^2}{\ln\left(\frac{\Lambda_\text{C}^2}{\bar{\mu}^2}\right)}$$

- $\lambda(\bar{\mu}) < 0$ for $\bar{\mu} > e^{-1/2\epsilon}\Lambda_\text{C}$
  - Negative UV coupling constant!



RW, arXiv:2310.02516 [hep-th]

Renormalization

- Renormalization of $\mathcal{O}(m^2)$ term fixes running of mass
  - α is dimensionless parameter of the theory

$$\frac{M_{\mathrm{R}}^2}{\lambda_{\mathrm{R}}} = \frac{M^2}{\lambda} = \frac{\alpha\Lambda_{\mathrm{C}}^2}{4\pi^2}$$

- Renormalized pressure per component is now cutoff-independent

$$p\left(m^2, \vec{\phi}_0\right) = \frac{m^4}{64\pi^2}\left(\ln\left(\frac{\Lambda_{\mathrm{C}}^2}{m^2}\right) + \frac{3}{2}\right) - \frac{m^2}{8}\left(\frac{\alpha\Lambda_{\mathrm{C}}^2}{4\pi^2} + \frac{4}{N}\vec{\phi}_0^2\right) + \frac{m^2}{2\pi^2\beta^2}\sum_{n=1}^{\infty}\frac{K_2(n\beta m)}{n^2}$$

Equation of state

- At leading order in large N only saddles of $p(\zeta_0, \vec{\phi}_0)$ contribute
- Saddle conditions (gap equation):

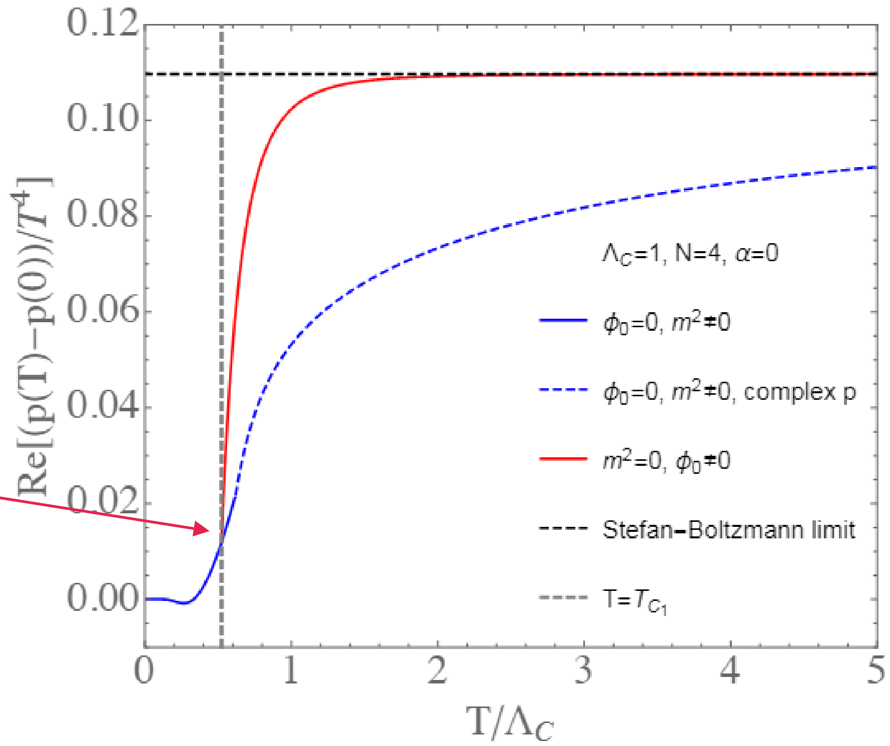
$$\frac{\partial p}{\partial m^2} = 0, \qquad \frac{\partial p}{\partial \vec{\phi}_0} = 0$$

- Solution with dominant pressure gives equation of state (EoS) $p(T)$ at all temperatures
- Three potential phases appear (three different solutions to saddle conditions)
  - Two phases with no SSB, $\vec{\phi}_0 = 0$
  - One phase (degeneracy of phases) with SSB, $\vec{\phi}_0 \neq 0$

Equation of state

- Pressure (vacuum pressure subtracted) divided by T^4 for $\alpha = 0$
  - Approaches Stefan–Boltzmann limit $p(T) = \pi^2 T^4/90$ for non-interacting bosons at high T

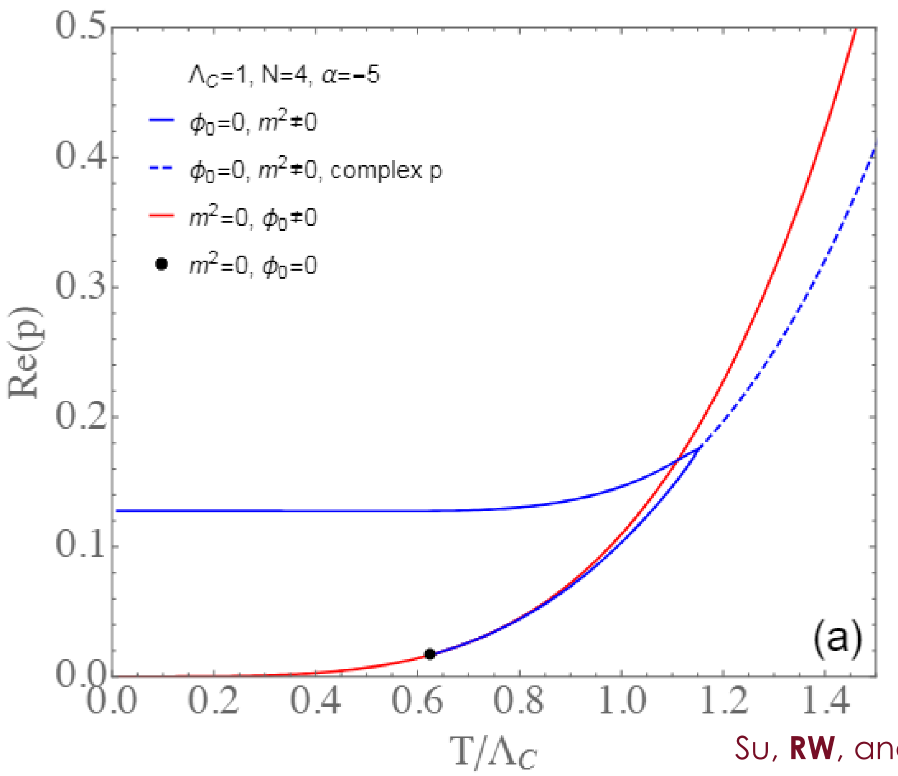
first-order phase transition

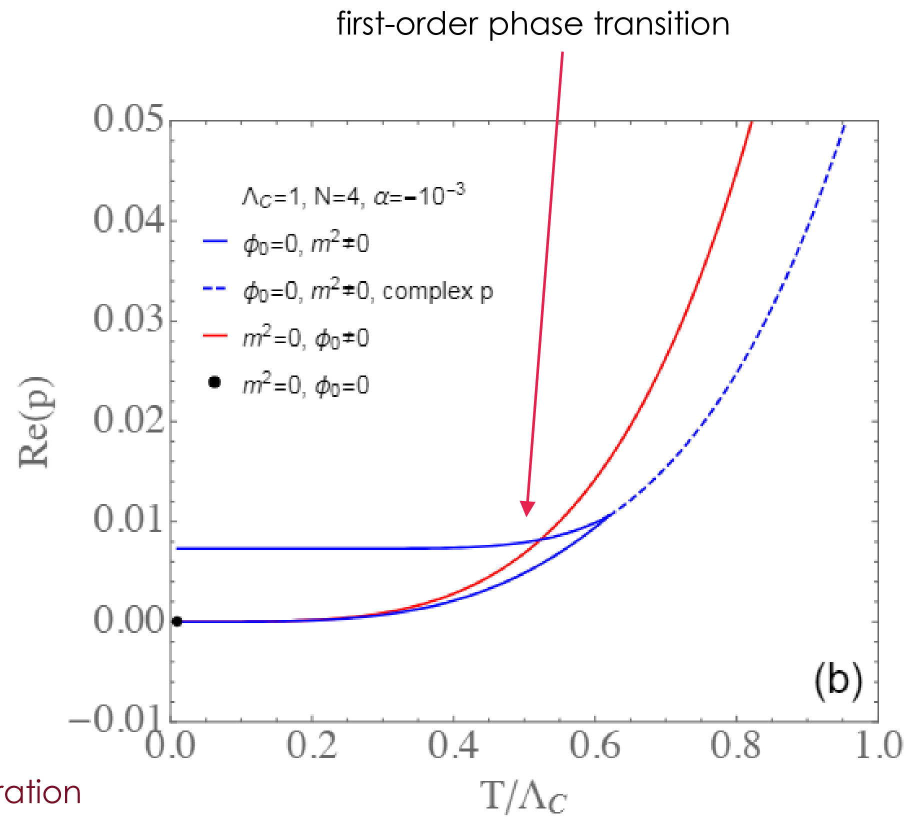


Su, **RW**, and Romatschke, in preparation

Phase structure

- For $\alpha \lesssim 0.824$ there is no SSB at low T; there is SSB at high T
  - Opposite of what is usually assumed

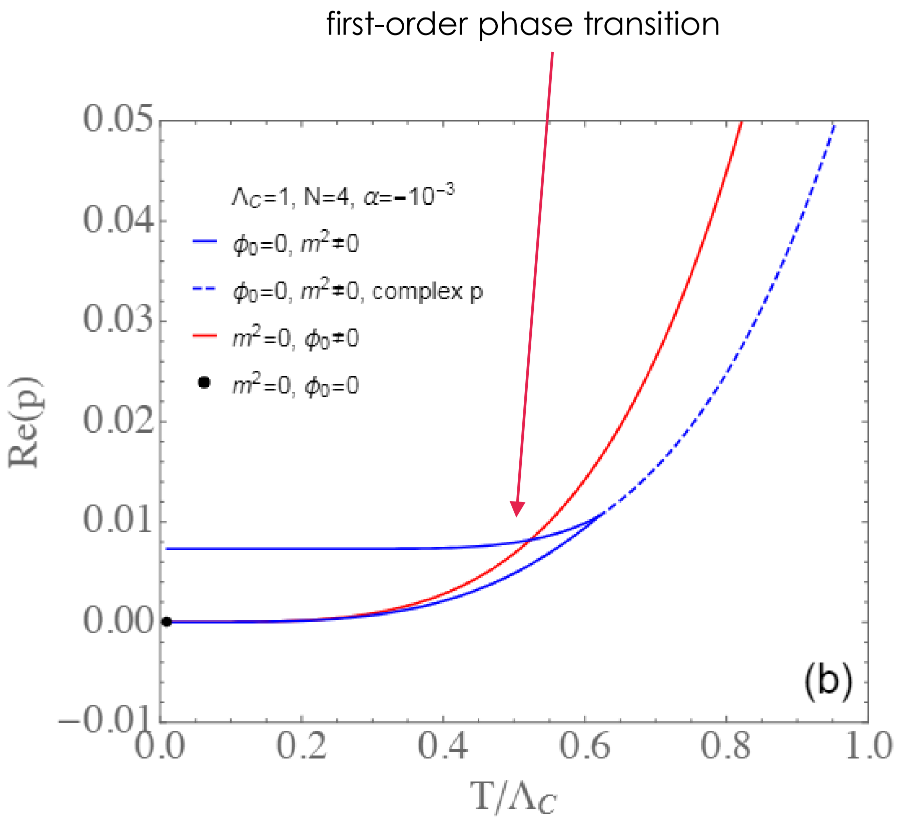




Su, **RW**, and Romatschke, in preparation

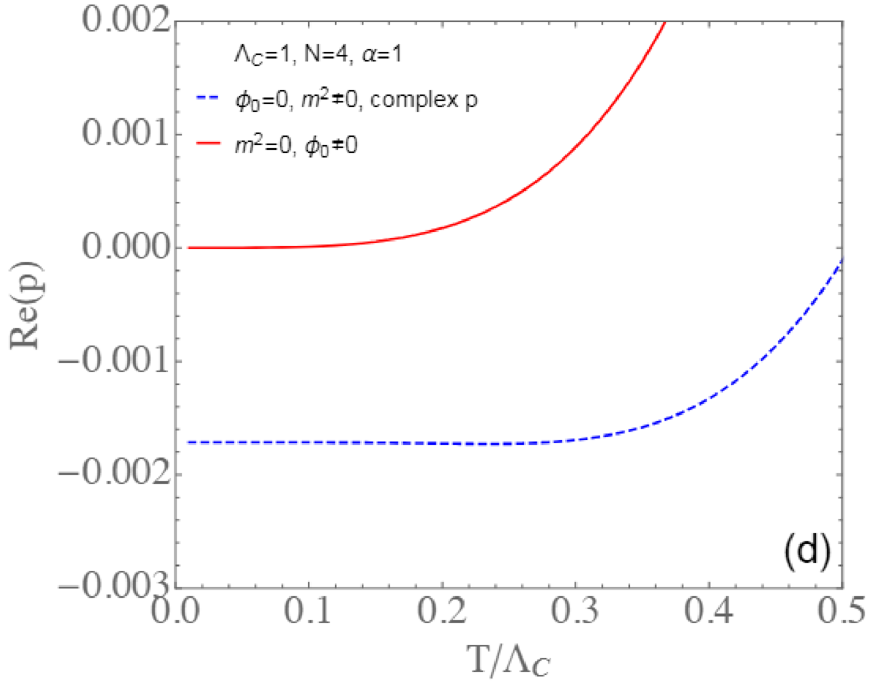
Phase structure

- First-order phase transition has implications for cosmology!



Phase structure

- For $\alpha \gtrsim 0.824$ there is SSB at all T; however, $\vec{\phi}_0$ is imaginary
  - Again, not the usual story



$$\vec{\phi}_0^2 = -\frac{N\alpha\Lambda_C^2}{16\pi^2} - \frac{NT^2}{12}$$

Su, **RW**, and Romatschke, in preparation

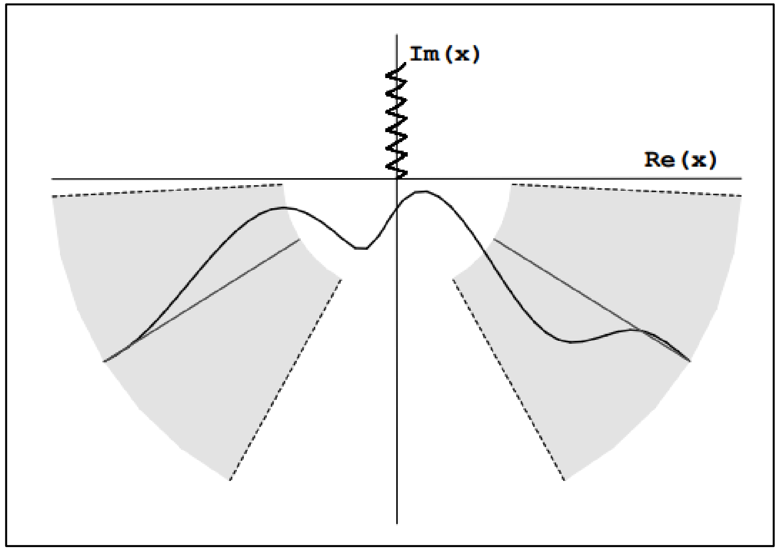
Non-Hermiticity

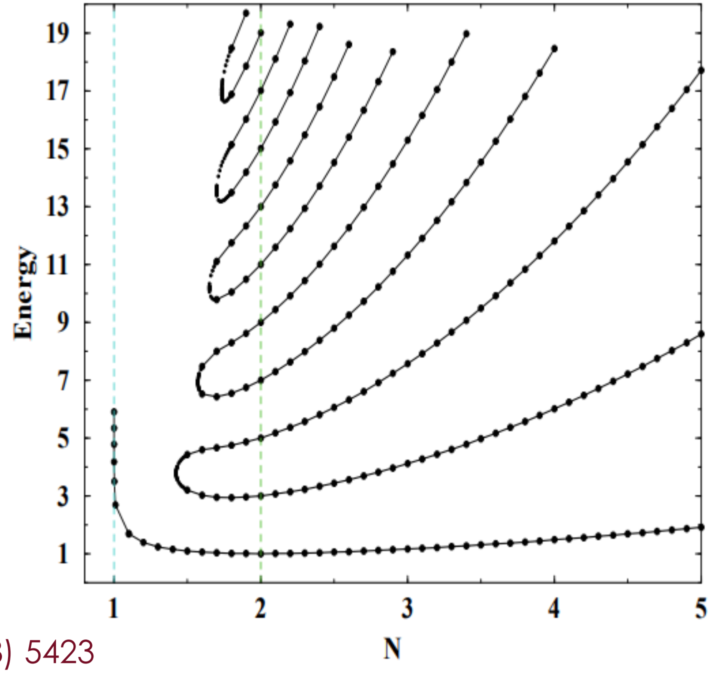
- $\langle \vec{\phi}_0^2 \rangle < 0$ at high temperatures
- Negative coupling constant
  - Path integral with $\lambda < 0$ is unbounded unless $\vec{\phi}$ is allowed to be complex (a non-Hermitian operator)
- How does renormalization produce a non-Hermitian theory?
  - We started out with what seemed like a Hermitian theory (the usual $O(N)$ model)
  - This is mysterious
- Is this theory really "sick"?

Non-Hermiticity

- Non-Hermitian theories can actually be physical!
- Bender and Boettcher, 1998, studied a class of non-Hermitian Hamiltonians with real and lower-bounded spectra, possessing an antilinear symmetry called $\mathcal{PT}$ symmetry: $x \to -x, \mathrm{i} \to -\mathrm{i}$

$$H = p^2 + x^2(\mathrm{i}x)^N$$





Bender and Boettcher, *PRL* 80 (1998) 5423

Non-Hermiticity

- Such non-Hermitian theories, with a discrete antilinear symmetry, called $\mathcal{PT}$ symmetry, and with real spectra, can have a well-defined inner product of quantum states and a notion of unitary!
  - Via what is called a $\mathcal{CPT}$ inner product of states

Bender, Brody, and Jones, *PRL* 89 (2002) 270401

$$\sin\alpha = \frac{r}{s}\sin\theta$$

- Example:

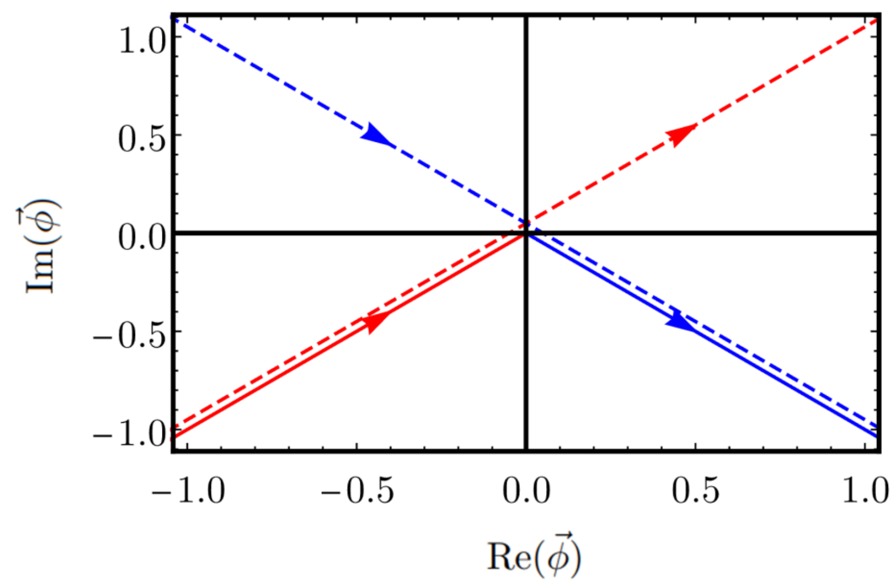
$$H = \begin{pmatrix} re^{\mathrm{i}\theta} & s \\ s & re^{-\mathrm{i}\theta} \end{pmatrix} \qquad \mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathcal{T}: \mathrm{i} \to -\mathrm{i} \qquad \mathcal{C} = \frac{1}{\cos\alpha}\begin{pmatrix} \mathrm{i}\sin\alpha & 1 \\ 1 & -\mathrm{i}\sin\alpha \end{pmatrix}$$

Bender, *Rept. Prog. Phys.* 70 (2007) 947

Non-Hermiticity

- Can make sense of the UV continuum limit of the $O(N)$ model with $\lambda < 0$ as a non-Hermitian $\mathcal{PT}$-symmetric theory, where $\mathcal{P}$ is $\vec{\phi} \to -\vec{\phi}$ and $\mathcal{T}$ is $\mathrm{i} \to -\mathrm{i}$.
  - $\vec{\phi}(x)$ is a non-Hermitian operator
  - Observables in the large-N expansion actually turn out to be the same as $\lambda > 0$ after renormalization

RW and Romatschke, another paper in progress….



$$\chi(x) \in \mathbb{R}, \vec{\eta}(x) \in \mathbb{R}^{N-1}$$

$$\vec{\phi} \cdot \hat{e} = \chi f(\chi)$$

$$\vec{\phi} - \hat{e}\vec{\phi} \cdot \hat{e} = \vec{\eta} f(\chi)$$

$$f(\chi) = \theta(\chi) e^{-\mathrm{i}\pi/4} + \theta(-\chi) e^{\mathrm{i}\pi/4}$$

RW, arXiv:2310.02516 [hep-th]

Stability of phases

- We check stability of dominant phases
- How? Hunt for tachyons   à la Coleman, Jackiw, and Politzer, *PRD* 10 (1974) 2491
  - Non-SSB phase for $\alpha \lesssim 0.824$ is stable at low temperatures, has a bound state ζ particle

Abbott, Kang, and Schnitzer, *PRD* 13 (1976) 2212;
Romatschke, *Int. J. Mod. Phys. A* 38 (2023) 28, 2350157;

- Need to find pole masses of $\vec{\phi}'\vec{\phi}'$ and $\zeta'\zeta'$ propagators including self-energies
  - Self-energies obtained by consistently resuming 1-loop diagrams
  - Called R2 resummation   Romatschke, *JHEP* 03 (2019) 149; Romatschke, *Mod. Phys. Lett. A* 35 (2020) 20050054
    Romatschke, *PRL* 127 (2021) 11, 111603

Stability of phases

- Propagators in SSB phase with self-energies at LO in large N:

$$\zeta'\zeta':\ D(k) = \frac{1}{N}\frac{1}{\frac{1}{8\lambda}+\Pi(k)}\left(1-\frac{\delta(k)}{V}\right)$$

$$\Pi(x) = \frac{1}{2}G_\perp(x)^2 + \frac{1}{N}\vec{\phi}_0^2 G_\parallel(x)$$

$$\text{Higgs:}\ \ G_\parallel(k) = \frac{1}{k^2+m^2+\Sigma_\parallel(k)}\left(1-\frac{\delta(k)}{V}\right)$$

$$\Sigma_\parallel(x) = D(x)\vec{\phi}_0^2$$

$$\text{"Goldstones":}\ \ G_\perp(k) = \frac{1}{k^2+m^2+\Sigma_\perp(k)}\left(1-\frac{\delta(k)}{V}\right)$$

$$\Sigma_\perp(x) = 0$$

Stability of phases

- In the Wick-rotated $\zeta'\zeta'$propagators in the SSB phase there is a branch point singularity on the imaginary axis $\mathrm{Re}(\omega_{\mathrm{Mink}}) = 0, \mathrm{Im}(\omega_{\mathrm{Mink}}) > 0$ at least at high temperatures
  - This is not a tachyon!
  - Dunno what you call this
  - Does not negatively impact calculations of transport, such as shear viscosity η/s

- We conclude that the SSB phase is in fact dynamically stable, but thermodynamically metastable at low T for $\alpha \lesssim 0.824$
  - At least, observables are well-behaved, if one grants non-Hermiticity. The model is neither "futile" nor "sick"

This will all be in Su, **RW**, and Romatschke, in preparation

Coupling to U(1) gauge field

- If there is no SSB at low temperatures for $\alpha \lesssim 0.824$ (including $M^2/\lambda < 0$), what happens to Higgs mechanism?
- How can masses for gauge bosons be generated?
- Consider the critical Abelian Higgs model with N scalar fields in lock-phase:
  - We discuss gauge symmetry and fixing in our paper Romatschke, Su, and Weller, arXiv:2405.00088 [hep-ph]

$$S[\vec{\phi}, A] = \int_{\beta,V} \mathrm{d}^4x \left(D_\mu^* \vec{\phi} D_\mu \vec{\phi} + \frac{\lambda}{N} \vec{\phi}^4 + \frac{1}{4} F_{\mu\nu}^2 \right)$$

$$D_\mu = \partial_\mu - \mathrm{i} \frac{e}{\sqrt{N}} A_\mu, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Mass generation without SSB!

- Can generate a mass for both the Higgs ϕ and the gauge boson A_μ in the ground state of the theory
  - There is no SSB in the ground state as shown by

Abbott, Kang, and Schnitzer, *PRD* 13 (1976) 2212;
Romatschke, *Int. J. Mod. Phys. A* 38 (2023) 28, 2350157;

- Do some "fast and loose" phenomenology:

$$\Lambda_C = \Lambda_{EW} \approx 135 \text{ GeV} \qquad \alpha_{EM}(\bar{\mu} = m_e) \approx \frac{1}{137}$$

$$m_A \approx 80.6 \text{ GeV}, \qquad m_{Higgs} \approx 222.6 \text{ GeV}$$

- In addition, a predicted Higgs-pair bound state and resonance ζ, which decay into two gauge bosons:
  - In principle can be seen at Large Hadron Collider (LHC)

$$m_{bound} \approx 400 \text{ GeV}, \qquad m_{res} \approx 581 \text{ GeV}, \qquad \Gamma_{res} \approx 350 \text{ GeV}$$

Mass generation without SSB!

- Read if interested:
  - Fewer parameters than Standard model
  - No bare mass parameters in Lagrangian
  - Can get "mass from nothing"!

Romatschke, Su, and Weller, arXiv:2405.00088 [hep-ph]

Mass From Nothing

Part I: The Abelian Higgs Model

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We study the Abelian Higgs model with multiple scalar fields, but without mass terms. Solving the model non-perturbatively order-by-order in the number of scalar fields, we find that radiative corrections generate masses for the scalar and gauge boson, without spontaneous symmetry breaking. The mass scales are set by the Λ-parameter of the electroweak running coupling, thereby naturally avoiding the hierarchy problem. No part of our calculation employs a weak-coupling expansion, and we find that the perturbative vacuum is metastable, and hence must decay to the stable non-perturbative vacuum of the theory, which we identify. Although the field content of our Lagrangian is standard, our results predict the existence of two heavy scalar resonances in addition to the Higgs. We believe that these predicted resonances will ultimately allow experimentalists to discriminate between our method and standard solutions of the Higgs model.

Conclusion

- $O(N)$ model at large N is neither “futile” nor “sick”
- Well-behaved equation of state
  - Indicating first-order phase transition to SSB at high temperatures
- There is no SSB at low temperatures!
  - Taken as a model of an independent Higgs field when $N = 4$, this throws into question whether the SSB Higgs vacuum is “stable” or preferred
  - The real stable vacuum is one without SSB
- The model exhibits non-Hermiticity
  - But this does not make it unacceptable as a model for physics
- Can generate masses for gauge bosons and for the scalar field without SSB!

Conclusion

- Thank you!