Magnetic Monopoles with Internal Structure

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Outline

The idea:

- Spectrum of certain gauge field theories contain magnetic monopoles ('t Hooft, Polyakov, 1974)
- In some of the theories the monopole solutions can be found even analytically
- Alas, number of these theories is rather limited. . .
- Our goal:
  - Enlarge the number of these theories (by resorting to effective field theories)
  - Perhaps the new exact solution will have interesting features. . .?

’t Hooft–Polyakov monopole
@
Georgi–Glashow model

(a short reminder)

Georgi–Glashow model

- Simplest model that allows for magnetic monopoles
  $\Rightarrow$ Prototypical, if not archetypal, example
- Spontaneously broken $SU(2)$ theory with adjoint (real triplet) scalar:

$$\mathcal{L} \;=\; \frac{1}{2}(D^\mu \boldsymbol{\phi})^2 - \frac{1}{4g^2}(\boldsymbol{F}^{\mu\nu})^2 - \frac{1}{4}\lambda\left(\boldsymbol{\phi}^2 - v^2\right)^2$$

  where

$$\begin{aligned} D^\mu \boldsymbol{\phi} &= \partial^\mu \boldsymbol{\phi} + \boldsymbol{A}^\mu \times \boldsymbol{\phi} \\ \boldsymbol{F}^{\mu\nu} &= \partial^\mu \boldsymbol{A}^\nu - \partial^\nu \boldsymbol{A}^\mu + \boldsymbol{A}^\mu \times \boldsymbol{A}^\nu \end{aligned}$$

- In the vacuum ($\boldsymbol{\phi}^2 = v^2$) the $SU(2)$ is broken down to the "electromagnetic" $U(1)$
- Spectrum:
  - Perturbative: Photon, $W^\pm$, Higgs boson
  - Non-perturbative: Magnetic monopole. . .

The monopole solution

Static, spherically symmetric classical solution with magnetic charge

- The solution is <u>guaranteed</u> to exist (and be stable) on topological reasons
  - Since $\pi_2\big(SU(2)/U(1)\big) = \mathbb{Z} \neq \{1\}$
- Asymptotically, the solution approaches the vacuum and:
  - $SU(2)$ is broken down to the electromagnetic $U(1)$
  - Electromagnetic field can be defined
  - Magnetic field happens to be non-vanishing!
  - $\Rightarrow$ magnetic charge $q_m = 4\pi$

Technically:

- The monopole solution is found using the "Hedgehog" Ansatz:

$$\boldsymbol{\phi}_a \; = \; v\frac{x_a}{r}H(r)\,, \qquad \boldsymbol{A}_a^i \; = \; \frac{1}{r^2}\varepsilon_{iab}x_b\big(K(r)-1\big)\,, \qquad \boldsymbol{A}_a^0 \; = \; 0$$

- The form factors $H(r)$, $K(r)$ are subject to <u>2^{nd}-order</u> ODEs
- $\Rightarrow$ Difficult! $\Rightarrow$ Solution can be found only numerically
- However, there is an exception...

BPS limit

The BPS limit:

- Limit of vanishing potential: $\lambda \to 0$
- But the boundary condition $\phi^2(\infty) = v^2$ is kept!

$\Rightarrow$ EOMs are just 1^{st}-order!

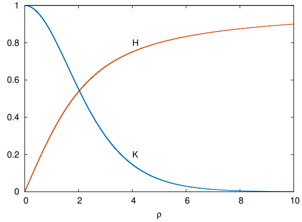
- Analytical solution exists ($\rho \equiv vgr$):

$$K = \frac{\rho}{\sinh\rho}, \qquad H = \coth\rho - \frac{1}{\rho}$$

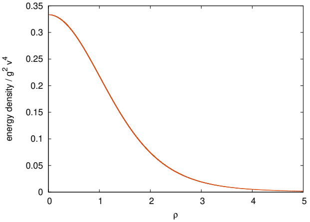
- Mass of the monopole:

$$M = \frac{4\pi v}{g}$$

Solution $K(\rho)$, $H(\rho)$:



Energy density $\mathcal{E}(\rho)$:



The lesson: The BPS limit is good – it makes the life **easier and happier!**

$\Rightarrow$ We will use it also in the following

Effective extension

The idea

Recall the Georgi–Glashow model:

$$\mathcal{L} \;=\; \frac{1}{2}(D^\mu \boldsymbol{\phi})^2 - \frac{1}{4g^2}(\boldsymbol{F}^{\mu\nu})^2 - V(\boldsymbol{\phi}^2)$$

The Lagrangian is a linear combination of

$$(D^\mu \boldsymbol{\phi})^2 \qquad \text{and} \qquad (\boldsymbol{F}^{\mu\nu})^2$$

$\longrightarrow$ The idea:

- Consider also the terms

$$(\boldsymbol{\phi}\cdot D^\mu \boldsymbol{\phi})^2 \qquad \text{and} \qquad (\boldsymbol{\phi}\cdot \boldsymbol{F}^{\mu\nu})^2$$

- Make the coefficients of this linear combination ϕ-dependent

In other words: We consider the most general model quadratic in $D^\mu \boldsymbol{\phi}$ and $\boldsymbol{F}^{\mu\nu}$

Generalization

⇒ Our generalized model:

$$\mathcal{L} = \frac{v^2}{2}\left[f_1^2\left(\frac{(D^\mu\boldsymbol{\phi})^2}{\boldsymbol{\phi}^2} - \frac{(\boldsymbol{\phi}\cdot D^\mu\boldsymbol{\phi})^2}{\boldsymbol{\phi}^4}\right) + f_3^2\,\frac{(\boldsymbol{\phi}\cdot D^\mu\boldsymbol{\phi})^2}{\boldsymbol{\phi}^4}\right] - \frac{1}{4g^2}\left[f_2^2\left((\boldsymbol{F}^{\mu\nu})^2 - \frac{(\boldsymbol{\phi}\cdot\boldsymbol{F}^{\mu\nu})^2}{\boldsymbol{\phi}^2}\right) + f_4^2\,\frac{(\boldsymbol{\phi}\cdot\boldsymbol{F}^{\mu\nu})^2}{\boldsymbol{\phi}^2}\right] - V(\boldsymbol{\phi}^2)$$

- f_i^2: Positive, dimensionless and gauge-invariant functions of $\boldsymbol{\phi}$
  - Technically: $\boldsymbol{\phi} = vH\boldsymbol{n}$ (where $\boldsymbol{n}^2 = 1$) $\Rightarrow$ $f_i^2 = f_i^2(H)$
- In fact, we consider a whole class of models:
  - Each set of the functions f_i^2 defines a particular theory
  - E.g., for $f_{1,3}^2 = H^2$ and $f_{2,4}^2 = 1$ we recover the Georgi–Glashow model
- Important: We modified only the interactions
  ⇒ Topology remains the same as in the Georgi–Glashow model
  ⇒ Monopole solutions are again present!
- (Unimportant: Negative powers of $\boldsymbol{\phi}$ and the projector-like structures are just for convenience – see next slides. . .)

The BPS limit

How to achieve the BPS limit?

- Georgi–Glashow model: Only $V \to 0$
- The generalized model: Also $V \to 0$, but f_i must satisfy

$$f_3 f_4 \;=\; H \frac{\mathrm{d}}{\mathrm{d}H}\left(f_1 f_2\right)$$

Then the Bogomolny bound (lower energy bound of a static configuration) is saturated, when

- The EOM of the $\underline{1^{\mathrm{st}}\text{-order}}$ is satisfied:

$$D^i \boldsymbol{\phi} \;=\; \frac{H}{g}\left[\frac{f_2}{f_1}\left(\boldsymbol{B}^i - \frac{\boldsymbol{\phi}\cdot\boldsymbol{B}^i}{\boldsymbol{\phi}^2}\boldsymbol{\phi}\right) + \frac{f_4}{f_3}\frac{\boldsymbol{\phi}\cdot\boldsymbol{B}^i}{\boldsymbol{\phi}^2}\boldsymbol{\phi}\right] \qquad \text{(where } \boldsymbol{B}^i \equiv -\tfrac{1}{2}\epsilon^{ijk}\boldsymbol{F}^{jk}\text{)}$$

- By definition, if this eq. is satisfied, the energy density is a total derivative:

$$\mathcal{E} \;=\; \partial^i\left(\frac{f_1 f_2}{gH}\boldsymbol{\phi}\cdot\boldsymbol{B}^i\right)$$

The BPS limit: Spherical symmetry

- Recall the spherically-symmetric “hedgehog” Ansatz:

$$\boldsymbol{\phi}_a \;=\; v\frac{x_a}{r}H(r)\,, \qquad \boldsymbol{A}_a^i \;=\; \frac{1}{r^2}\varepsilon_{iab}x_b\big(K(r)-1\big)\,, \qquad \boldsymbol{A}_a^0 \;=\; 0$$

- The EOMs follow as ($\rho \equiv vgr$):

$$\begin{aligned} \partial_\rho(\log K) &= -\frac{f_1}{f_2} \\ \partial_\rho(\log H) &= \frac{1-K^2}{\rho^2}\frac{f_4}{f_3} \end{aligned}$$

- 1st order $\Rightarrow$ There is a hope that, perhaps, they could be easily solvable. . .

Two classes of exact solutions

Indeed! There are (at least) two classes when the EOMs can be solved analytically:

1 If $\frac{f_3}{f_4} = H \frac{\mathrm{d}}{\mathrm{d}H}\left(\frac{f_1}{f_2}\right)$:

$$K = \frac{\rho}{\sinh\rho}$$

$$H = \left(\frac{f_1}{f_2}\right)^{-1}(\kappa) \qquad \text{where } \kappa \equiv \coth\rho - \frac{1}{\rho}$$

⟶ The "'t Hooft–Polyakov" class

2 If $f_1 = f_2$:

$$K = \xi \exp(-\rho)$$

$$H = \left(\int \frac{\mathrm{d}H}{H}\frac{f_3}{f_4}\right)^{-1}(1-\lambda) \qquad \text{where } \lambda \equiv \frac{1}{\rho} - \xi^2\left[\frac{\mathrm{e}^{-2\rho}}{\rho} + 2\mathrm{Ei}(-2\rho)\right]$$

⟶ The "ξ" class

3 Perhaps some other(s). . . ?

The “’t Hooft–Polyakov” class

The “’t Hooft–Polyakov” class

If $\frac{f_3}{f_4} = H \frac{\mathrm{d}}{\mathrm{d}H}\left(\frac{f_1}{f_2}\right)$:

$$K = \frac{\rho}{\sinh \rho}$$

$$H = \left(\frac{f_1}{f_2}\right)^{-1}(\kappa) \qquad \text{where } \kappa \equiv \coth \rho - \frac{1}{\rho}$$

- This class is a direct generalization of the ’t Hooft–Polyakov solution (included here as a special case $f_{1,3} = H$ and $f_{2,4} = 1$)
- However, this class contains monopoles with some novel features
- Let’s see an example. . .

An example: Power-function model

- The simplest example (two-parametric class of theories with $n \geq m$):

$$f_1^2 = H^{n+m}\,, \qquad f_3^2 = nmH^{n+m}\,,$$
$$f_2^2 = H^{n-m}\,, \qquad f_4^2 = \frac{n}{m}H^{n-m}\,.$$

- The Lagrangian:

$$\mathcal{L} = \frac{1}{2}\left(\frac{\phi}{v}\right)^{n+m-2}\left[(D^\mu \boldsymbol{\phi})^2 + (nm-1)\frac{(\boldsymbol{\phi}\cdot D^\mu \boldsymbol{\phi})^2}{\boldsymbol{\phi}^2}\right] - \frac{1}{4g^2}\left(\frac{\phi}{v}\right)^{n-m}\left[(\boldsymbol{F}^{\mu\nu})^2 + \left(\frac{n}{m}-1\right)\frac{(\boldsymbol{\phi}\cdot \boldsymbol{F}^{\mu\nu})^2}{\boldsymbol{\phi}^2}\right]$$

- Solution:

$$K = \frac{\rho}{\sinh \rho}\,, \qquad H = \sqrt[m]{\kappa} \qquad \text{(where } \kappa \equiv \coth\rho - \tfrac{1}{\rho}\text{)}$$

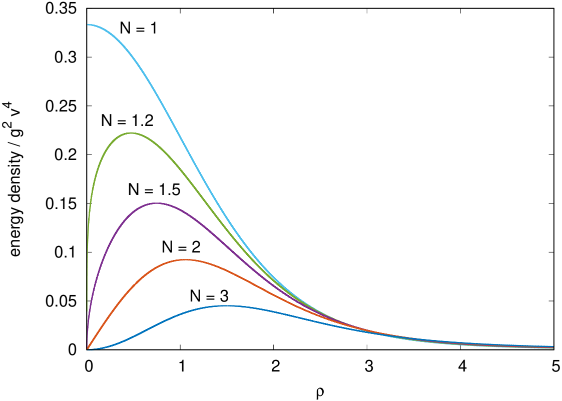
- Energy density:

$$\mathcal{E} = \kappa^N\left[2\kappa\frac{K^2}{\rho^2} + \frac{N}{\kappa}\frac{(1-K^2)^2}{\rho^4}\right]v^4 g^2 \qquad \text{where} \quad N \equiv \frac{n}{m}$$

Power-function model: Energy density

Where is the monopole's energy concentrated?

- For $N = 1$:
  - Most of the energy is in the center of the monopole
  - This is the 't Hooft–Polyakov monopole

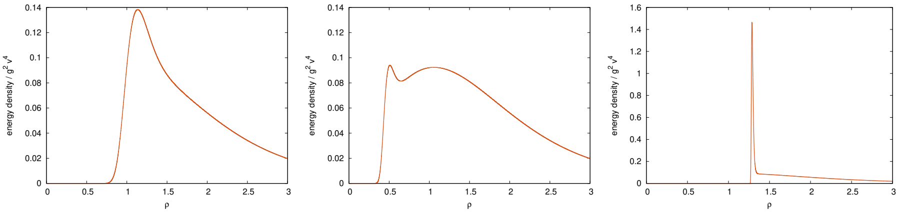


- For $N > 1$:
  - Energy density vanishes in the center of the monopole!
  - The energy is stored in a *spherical shell* around the center of the monopole
  - $\Rightarrow$ "Hollow" monopoles
  - Seems to be in fact quite typical behavior (also in other models...)

Other models – other features

In other models there can be other novel features:

- Wider and more pronounced “cavity” in the center of “hollow monopole”
  (The energy density falls off *exponentially* in the origin)
- Two or more local maxima of the energy density
  (Spherical shell of energy around the monopole center is structured, with several “sub-shells”)
- Sharper and more pronounced energy shell



(All these density energy plots correspond to exact solutions)

The “ξ” class

The "ξ" class

If $f_1 = f_2$:

$$
\begin{aligned}
K &= \xi \exp(-\rho) \\
H &= \left(\int \frac{\mathrm{d}H}{H} \frac{f_3}{f_4} \right)^{-1} (1-\lambda) \qquad \text{where } \lambda \equiv \tfrac{1}{\rho} - \xi^2 \left[\tfrac{\mathrm{e}^{-2\rho}}{\rho} + 2\mathrm{Ei}(-2\rho) \right]
\end{aligned}
$$

(where $\mathrm{Ei}(x) = -\int_{-x}^{\infty} \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t$ is the *exponential integral*)

Notice the presence of a parameter ξ:

- Not present in the Lagrangian: Emerges as a constant of integration
- Regularity of energy density: $\xi \in (-1, 1)$ (otherwise unconstrained)

The parameter ξ is physical (i.e., *measurable*):

- It appears in a physical quantity: The energy density
- It controls the profile of the energy density
- However, it doesn't change the total energy: Monopoles with different ξ have the same mass

$\Rightarrow$ ξ is an internal degree of freedom (or moduli space parameter) of the monopole!

An example: Power-function model

A particular example (a representative of a larger family):

- Lagrangian:

$$\mathcal{L} = \frac{1}{2}\left(\frac{\phi}{v}\right)^6 \left\{ (D^\mu \boldsymbol{\phi})^2 + \left[16\left(\frac{\phi}{v}\right)^{-2} - 1\right] \frac{(\boldsymbol{\phi}\cdot D^\mu \boldsymbol{\phi})^2}{\boldsymbol{\phi}^2} \right\} - \frac{1}{4g^2}\left(\frac{\phi}{v}\right)^8 \left\{ (\boldsymbol{F}^{\mu\nu})^2 + \left[4\left(\frac{\phi}{v}\right)^{2} - 1\right] \frac{(\boldsymbol{\phi}\cdot \boldsymbol{F}^{\mu\nu})^2}{\boldsymbol{\phi}^2} \right\}$$

- Solution:

$$K = \xi \mathrm{e}^{-\rho}\,,$$

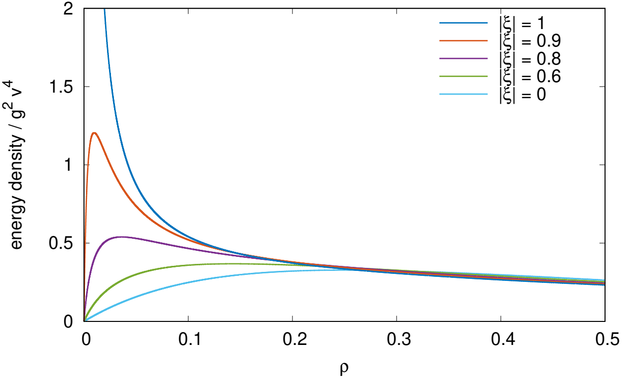
$$H = \left[1 + \frac{1 - \xi^2 \mathrm{e}^{-2\rho}}{\rho} - 2\xi^2 \mathrm{Ei}(-2\rho)\right]^{-1/2}$$

- Energy density:

$$\mathcal{E} = H^8 \left[2\frac{K^2}{\rho^2} + 4H^2 \frac{(1-K^2)^2}{\rho^4}\right] v^4 g^2$$

$\longrightarrow$ depends on ξ through K and H

The energy density



Indeed:

- The parameter ξ controls the “shape” of the monopole
- For $|\xi| \geq 1$ the energy density is singular

Summary & Outlook

Summary:

- Considered a class of effective $SU(2)$ models that admit monopole solutions
- Constructed the BPS limit
- Found exact monopole solutions with interesting properties
  - Interesting/weird/bizarre energy density profiles
  - New internal degree of freedom ξ: Controls the "shape" of the monopole, but doesn't change its total energy

Outlook (short term): Are there any deeper/physical reasons for...

- ... the existence of the hollow cavity in the monopoles?
- ... the occurrence of ξ? Perhaps some symmetry?

Outlook (long term):

- SUSY? Dyons? Higher magnetic charges? Multi-monopole solutions? Dynamics? ...?

Reference: *Phys.Rev.D 107 (2023) 12, 12*, `arXiv:2303.15602`