Soft Off-Shell Recursion Relations for Pions

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ICHEP 2024, Prague

20.07.2024

[2401.04731],[24xx.xxxx] CB, Karol Kampf, Jiří Novotný, Jaroslav Trnka

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$\longrightarrow$ **Soft Factor Expansion** (SFE) of pion amplitudes.

Hidden Structure: Mixed Amplitudes

- Closer inspection of Adler Zero reveals hidden structure in soft limit. Cachazo et.al. [1604.03893]
  Taking $p_i \to \lambda p_i$ and then $\lambda \to 0$ yields

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$$A_n \xrightarrow{s_{13} = s_{14} = \cdots = s_{1n-2} = 0} A_4 \times M_3^{3\phi} \times M_{n-1}^{3\phi}$$

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Since amplitudes $M_n^{3\phi}$ are everywhere: can we use them to **compute** pion amplitudes?

Pion amplitudes and X-variables

- Scalar amplitudes are rational functions of Lorentz-invariants

$$X_{ij} = (p_i + p_{i+1} + \dots p_{j-1})^2 = s_{i,j-1}, \qquad X_{i,i+1} = p_i^2 = 0, \qquad X_{ij} = X_{ji}.$$

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- Simplest NLSM amplitudes:

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- Equivalently think of A_n as functions of **labels**,

$$A_n = A_n(123 \ldots n{-}1\, n).$$

Soft Limit in X-variables

- At level of X-variables define soft limit (e.g. $p_n \to 0$) as formal **replacement of labels**

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- Ex.: At 6 points,

$$\begin{aligned} A_6(123456) \xrightarrow{6\mapsto 1} & -X_{15}M_5^{3\phi}(12345) - X_{12}M_5^{3\phi}(23451) \\ &= A_6(123451), \end{aligned}$$

  where $M_5^{3\phi}(1^\phi 2^\pi 3^\pi 4^\phi 5^\phi) = 1 - \frac{X_{13}+X_{24}}{X_{14}} - \frac{X_{24}+X_{35}}{X_{25}}$.

Label Soft Theorem at n Points

- Generally, at n-points, we find the **label soft theorem**,

$$A_n(1\dots n) \xrightarrow{n\mapsto 1} -X_{1,n-1}M^{3\phi}_{n-1}(1\dots n{-}1) - X_{12}M^{3\phi}_{n-1}(2\dots n{-}1\,1)$$
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- Suggests to decompose the amplitude as

$$A_n(1\dots n) = -X_{1,n-1}M^{3\phi}_{n-1}(1\dots n{-}1) - X_{2n}M^{3\phi}_{n-1}(2\dots n{-}n) + R_n$$

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- **Remainder function** R_n crucially satisfies

$$R_n(12\ldots n) \xrightarrow{n\mapsto 1} 0.$$

Structure of Remainder Function

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$$S_{n,k} = X_{1k} - X_{kn},$$

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- Coefficients $C_{n,k}$ are given in terms of **lower-point NLSM and mixed amplitudes**,

$$C_{n,k} = \begin{cases} \frac{1}{X_{1k}} A_k(1\dots k) M^{3\phi}_{n-k+2}(k\dots n), & \text{if } k \text{ even,} \\ -\frac{1}{X_{kn}} M^{3\phi}_k(1\dots k) A_{n-j+2}(k\dots n), & \text{if } k \text{ odd.} \end{cases}$$

The Soft Factor Expansion I

- Plugging in yields **all-order formula** for NLSM amplitudes

$$A_n = -\, X_{1,n-1} M^{3\phi}_{n-1}(1\ldots n{-}1) - X_{2n} M^{3\phi}_{n-1}(2\ldots n{-}n)$$

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&\qquad + \frac{\bar{S}_{n,2j+1}}{X_{2j+1,n}} M^{3\phi}_{2j+1}(1\ldots 2j{+}1) A_{n-2j}(2j{+}1\ldots n) \Bigg\},
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with $\bar{S}_{n,j} \equiv -S_{n,j}$. $\rightarrow$ **Soft Factor Expansion** (SFE)

- Efficiently packages amplitude into $\mathcal{O}(n)$ terms.
- Manifests label soft theorem ($\Leftrightarrow$ Adler Zero) term by term for $p_n \rightarrow 0$.

The Soft Factor Expansion II

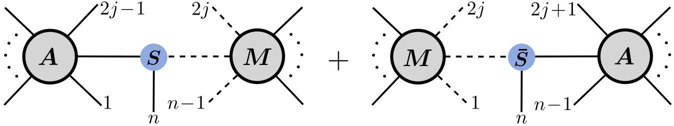
- **Soft Factor Expansion:** (SFE)

$$A_n = -X_{1,n-1}M^{3\phi}_{n-1} - X_{2n}M^{3\phi}_{n-1} + R_n,$$

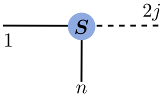
$$R_n = \sum_{j=2}^{\frac{n-2}{2}} \left\{ \frac{S_{n,2j}}{X_{1,2j}} A_{2j} \times M^{3\phi}_{n-2j+1} + \frac{\bar{S}_{n,2j+1}}{X_{2j+1,n}} M^{3\phi}_{2j+1} \times A_{n-2j} \right\}.$$

- Observation: R_n takes form of **cubic vertex expansion**,

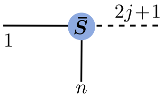
$$R_n = \sum_{j=2}^{\frac{n-2}{2}}$$



- Effective cubic vertices:



$= S_{n,2j}X_{2j,n},$



$= \bar{S}_{n,2j+1}X_{1,2j+1}.$

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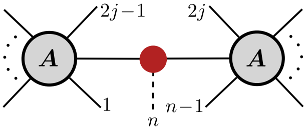

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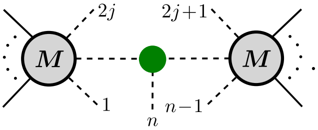

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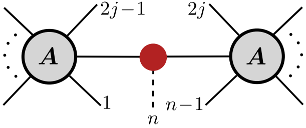

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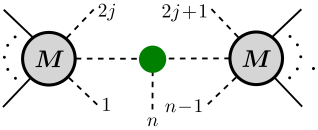

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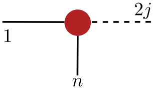
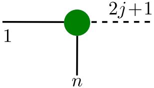


$= 1$,



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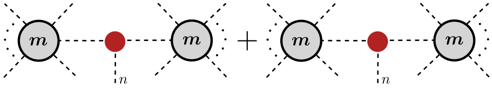
$$m_n \simeq \sum_j$$



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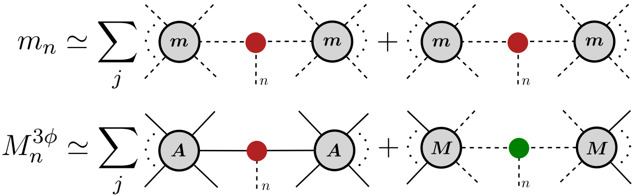

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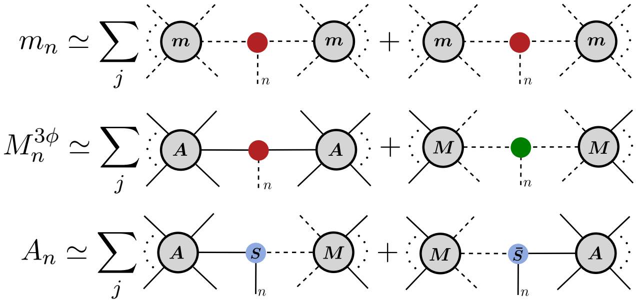
$$1 = M_3^{3\phi} \xrightarrow{\text{SFE}} A_4 \xrightarrow{\text{CVE } 3\phi} M_5^{3\phi} \xrightarrow{\text{SFE}} A_6 \longrightarrow \dots$$

- Highly efficient: each step requires summing only $\mathcal{O}(n)$ terms.
- Analogous to cubic Berends-Giele recursion relations for $\mathrm{tr}\phi^3$-theory amps. m_n,



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- General decomposition continues to hold for L-loop integrands,

$$A_n^{(L)}(1\dots n) = -X_{1,n-1}M_{n-1}^{(L)3\phi}(1\dots n{-}2\, n{-}1) - X_{2,n}M_{n-1}^{(L)3\phi}(2\dots n{-}1\, n) + R_n^{(L)}.$$

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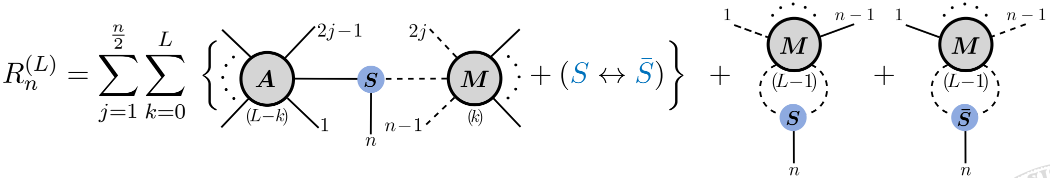

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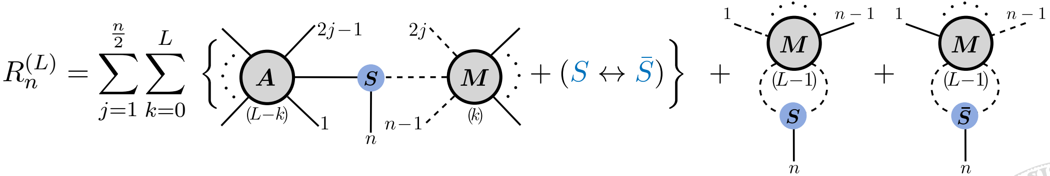

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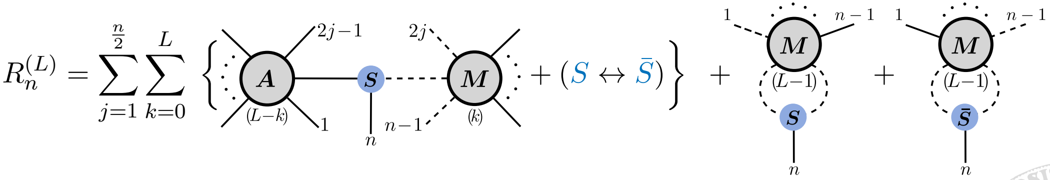

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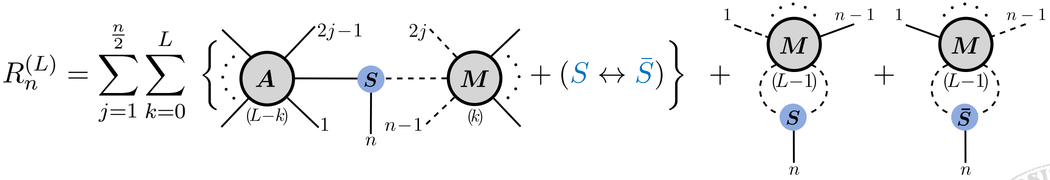

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Thank you!