# Relations between gravitons and gluons from twistor space

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Based on 2406.04539 with T. Adamo

# Gluon and graviton scattering at tree-level

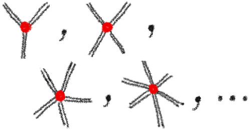
- We consider tree-level scattering of gluons and gravitons in Minkowski space
- Amplitudes are constructed from Feynman rules of the appropriate theory
- Gluon amplitudes can be further partitioned according to the colour-ordering of the generators of the colour group

$$\mathcal{A}_n = \sum_{\rho\in S_{n-1}} \mathrm{Tr}(\mathrm{T}^{\rho(1)}\cdots\mathrm{T}^{\rho(n)})\,\mathcal{A}_n[\rho]$$

where the relation to the structure constants are e.g. $\mathrm{i}f^{abc} = \mathrm{Tr}(\mathrm{T}^a\mathrm{T}^b\mathrm{T}^c) - \mathrm{Tr}(\mathrm{T}^b\mathrm{T}^a\mathrm{T}^c)$.

YM:

GR:



# Two organizing principles of tree-level amplitudes

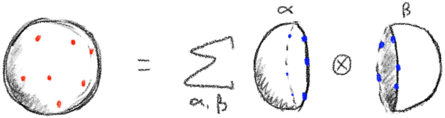
- Double copy

$$\text{gravity} = \text{gauge} \otimes \text{gauge}$$

- External helicity configuration (in 4d)
  - Spin of external particle projected against its momentum

# What is the double copy?

- The original double copy relation was discovered by Kawai-Lewellen-Tye (KLT) in 1986, relating closed and open string amplitudes



- Taking the field theory limit this amounts to

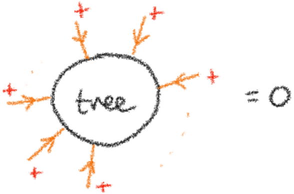
$$\mathcal{M}^{\text{GR}}_{\text{tree}}(1,\ldots,n) = \sum_{\substack{\alpha\in S_{n-3}\\ \beta\in\tilde{S}_{n-3}}} \mathcal{A}^{\text{YM}}_{\text{tree}}(\alpha)\ \underbrace{S[\alpha|\beta]}_{\text{KLT kernel}}\ \mathcal{A}^{\text{YM}}_{\text{tree}}(\beta).$$

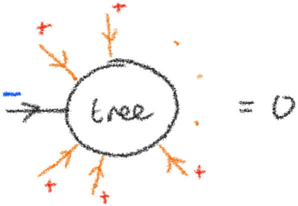
$S_{n-3}, \tilde{S}_{n-3}$ are two bases of $(n-3)!$ colour-orderings

- Exists in many guises: colour-kinematics duality (BCJ), CHY, classical (Kerr-Schild), etc.

# External helicity configurations

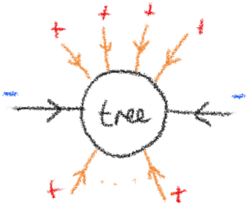
- The complexity of graviton and gluon amplitudes is graded by helicity
- For example





- These can be viewed as statements about the consistency and integrability of the self-dual sector of the two theories

# Maximally helicity violating (MHV) amplitudes



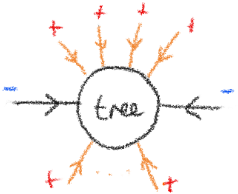
- In terms of *spinor-helicity* variables to describe massless momenta $p_\mu(\sigma^\mu)^{\alpha\dot{\alpha}} = p^{\alpha\dot{\alpha}} = |p\rangle^{\dot{\alpha}}\,[p|^{\alpha}$, amplitudes at *all multiplicity* take very simple forms [Parke-Taylor:'86, Hodges:'12]

$$\mathcal{A}^{\text{YM}}_{\text{MHV}}(123\ldots n) = \delta^4(\cdots)\,\frac{\langle i\,j\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

$$\mathcal{M}^{\text{GR}}_{\text{MHV}}(n) = \delta^4(\cdots)\,\langle i\,j\rangle^8\,\det{}'(\mathbb{H}_{ij})$$

# Description using twistor theory

- The theory of twistor space ($\mathbb{PT} \subset \mathbb{CP}^3$) trivialises self-dual backgrounds in gravity and Yang-Mills. We can treat the positive helicity particles as a self-dual background [Penrose:'76, Ward:'77]
- Negative helicity particles can be viewed as perturbations on this background. The MHV amplitudes can then be read off from the two-point correlation function [Mason-Skinner:'08]



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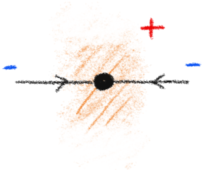

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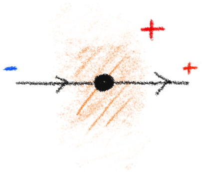

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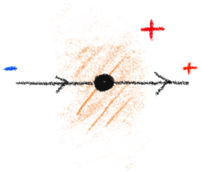
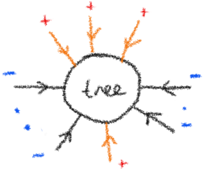


- For more negative helicity particles, use twistor string theory [Nair:'88, Witten:'03, Berkovits:'04; Skinner:'12] to derive all-multiplicity formulae (which can be proven using recursion relations)

# All-multiplicity tree-level formulae for arbitrary helicity



- At N$^{d-1}$MHV with $d+1$ negative helicity particles $\tilde{\mathbf{g}}$ (or $\tilde{\mathbf{h}}$ for gravity)

$$\mathcal{A}^{\mathrm{YM}}_{n,d}[\rho] = \int \mathrm{d}\mu_d \, |\tilde{\mathbf{g}}|^4 \, \mathrm{PT}_n[\rho] \qquad \mathcal{M}^{\mathrm{GR}}_{n,d} = \int \mathrm{d}\mu_d \, |\tilde{\mathbf{h}}|^8 \, \det{}'(\mathbb{H}) \, \det{}'(\mathbb{H}^\vee)$$

[Witten:'04; Roiban-Spradlin-Volovich:'04] [Cachazo-Skinner:'12]

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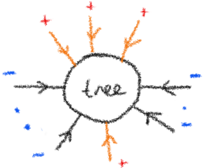

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- The integral is over maps $\mathcal{Z} : \mathbb{CP}^1 \to \mathbb{PT}$ of degree $d$ and $n$ marked points on $\mathbb{CP}^1$. $\mathrm{d}\mu_d$ contains the external kinematics
- Here $|\tilde{\mathbf{g}}| = \prod_{i<j\in\tilde{\mathbf{g}}}(ij)$, $\mathrm{PT}_n[\rho] = \frac{1}{(\rho(1)\rho(2))\ldots(\rho(n)\rho(1))}$ and $(i\,j) = z_i - z_j$
- $\mathrm{det}'(\mathbb{H})$ and $\mathrm{det}'(\mathbb{H}^\vee)$ are reduced determinants over $\mathbf{h}$ (positive helicity) and $\tilde{\mathbf{h}}$ respectively

$$\text{YM amplitude} \xrightarrow[\text{Double copy}]{\qquad\qquad} \text{GR amplitude}$$

$$\begin{tikzcd} \text{YM amplitude} \arrow[r, "\text{Double copy}"'] \arrow[d, "\text{helicity grading}"'] & \text{GR amplitude} \arrow[d, "\text{helicity grading}"] \\ \mathcal{A}_{n,d}[\alpha] = \int \mathrm{d}\mu_d\, \mathcal{I}_n^{\mathrm{YM}}[\alpha] & \mathcal{M}_{n,d} = \int \mathrm{d}\mu_d\, \mathcal{I}_n^{\mathrm{GR}} \end{tikzcd}$$

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Double copy on non-trivial backgrounds??

Further understanding of double copy?

# Helicity-graded double copy (1)

- We use tools from graph theory to find the relation

$$\det{}'(\mathbb{H})\,\det{}'(\mathbb{H}^{\vee}) = \sum_{\substack{a\tilde{\rho}b\rho \\ \tilde{\omega}^{T}ab\omega}} \mathrm{PT}[a\tilde{\rho}b\rho]\, \underbrace{S_{n,d}[\tilde{\rho},\rho|\tilde{\omega},\omega]}_{\substack{\text{KLT kernel} \\ \text{in twistor space}}}\, \mathrm{PT}[\tilde{\omega}^{T}ab\omega]$$

where

$$S_{n,d}[\tilde{\rho},\rho|\tilde{\omega},\omega] = \mathcal{D}[\tilde{\omega},\omega]\left[\sum_{\tilde{T}\in\mathcal{T}^{a}_{\tilde{\rho},\tilde{\omega}}}\prod_{(i\to j)}\tilde{\phi}_{ij}\right]\times\left[\sum_{T\in\mathcal{T}^{b}_{\rho,\omega}}\prod_{(i\to j)}\phi_{ij}\right]$$

- Remark: the basis of orderings now has $(n-d-2)!\times(d)!$ elements — split by chirality. This is in contrast to the usual KLT double copy which is over a basis of orderings of size $(n-3)!$

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# Helicity-graded double copy (2)

- The spacetime momentum kernel is the matrix inverse of the amplitudes $m_n$ of another massless theory: biadjoint scalar theory [Cachazo-He-Yuan:'13]
- We prove (using recursion relations in twistor space) that the matrix inverse of our object, treated as an integrand for a scalar theory, also has this property

$$m_n[a\tilde{\rho}b\rho|\tilde{\omega}^T ab\omega] = \int \mathrm{d}\mu_d \, S^{-1}_{n,d}[\tilde{\rho}, \rho|\tilde{\omega}, \omega]$$

  This gives a representation of BAS amplitudes in twistor space

- Remark: due to chirality, only specific colour orderings are permitted, which encode the dependence on degree $d$

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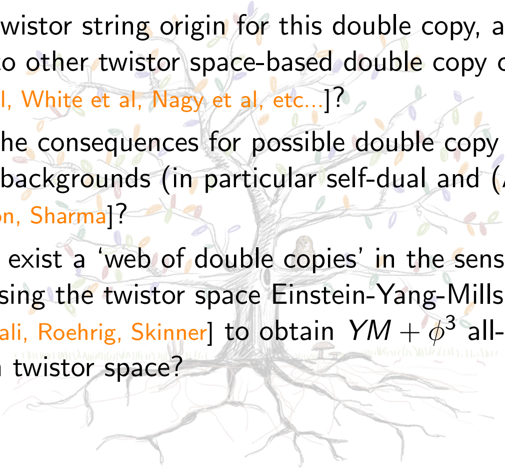
$$\boxed{\mathcal{I}^{\text{GR}}_{n,d} = \sum_{\substack{a\tilde{\rho}b\rho \\ \tilde{\omega}^T ab\omega}} \mathcal{I}^{\text{YM}}_{n,d}[a\tilde{\rho}b\rho] \, S_{n,d}[\tilde{\rho},\rho|\tilde{\omega},\omega] \, \mathcal{I}^{\text{YM}}_{n,d}[\tilde{\omega}^T ab\omega]}$$

# Outlook



- Is there a twistor string origin for this double copy, and could it have a relation to other twistor space-based double copy constructions [Borsten et al, White et al, Nagy et al, etc...]?
- What are the consequences for possible double copy constructions on non-trivial backgrounds (in particular self-dual and (A)dS) [Adamo, Bogna, Mason, Sharma]?
- Does there exist a 'web of double copies' in the sense of CHY? For example, using the twistor space Einstein-Yang-Mills (EYM) formula [Adamo, Casali, Roehrig, Skinner] to obtain $YM + \phi^3$ all-multiplicity formulae in twistor space?

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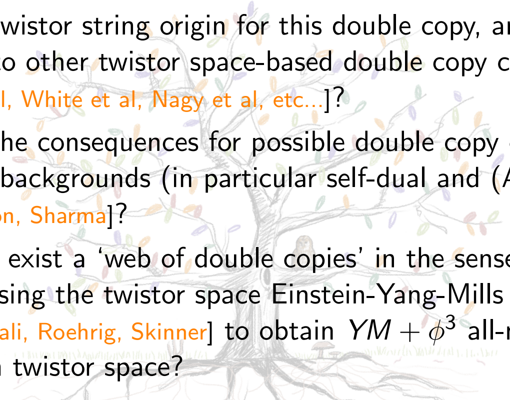


Děkuji vám!

# Back-up slides

$$\text{Gravity:} \quad |\tilde{\mathbf{h}}|^8 \det{}'(\mathbb{H}) \det{}'(\mathbb{H}^\vee)$$

- Hodges matrix has entries for $i, j \in \mathbf{h}$ (+ hel.)

$$\mathbb{H}_{ij} = \frac{[\frac{\partial}{\partial\mu(\sigma_i)} \frac{\partial}{\partial\mu(\sigma_j)}]}{(ij)}, \qquad \mathbb{H}_{ii} = -\sum_{\substack{j\in\mathbf{h} \\ j\neq i}} \mathbb{H}_{ij} \prod_{l\in\tilde{\mathbf{h}}} \frac{(jl)}{(il)}$$

- Dual Hodges matrix has entries for $i, j \in \tilde{\mathbf{h}}$ (- hel.)

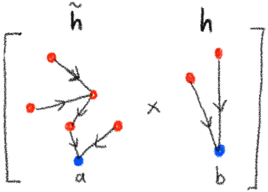
$$\mathbb{H}^\vee_{ij} = \frac{\langle\lambda(\sigma_i)\,\lambda(\sigma_j)\rangle}{(ij)}, \qquad \mathbb{H}^\vee_{ii} = -\sum_{\substack{j\in\tilde{\mathbf{h}} \\ j\neq i}} \mathbb{H}^\vee_{ij} \prod_{k\in\tilde{\mathbf{h}}\setminus\{i,j\}} \frac{(ki)}{(kj)}$$

- The reduced determinants are

$$\det{}'(\mathbb{H}) = \frac{|\mathbb{H}^b_b|}{|\tilde{\mathbf{h}}\cup\{b\}|^2}, \qquad \det{}'(\mathbb{H}^\vee) = \frac{|\mathbb{H}^{\vee\,a}_a|}{|\tilde{\mathbf{h}}\setminus\{a\}|^2}$$

$$S_{n,d}[\rho,\tilde{\rho}|\omega,\tilde{\omega}] = \mathcal{D}[\omega,\tilde{\omega}] \left[ \sum_{\tilde{T}\in\mathcal{T}^{a}_{\tilde{\rho},\tilde{\omega}}} \prod_{(i\to j)} \tilde{\phi}_{ij} \right] \times \left[ \sum_{T\in\mathcal{T}^{b}_{\rho,\omega}} \prod_{(i\to j)} \phi_{ij} \right]$$

$$= \mathcal{D}(\omega,\tilde{\omega}) \sum_{\tilde{T},T}$$



$$\phi_{ij} \coloneqq [\partial_{\mu}(\sigma_i)\,\partial_{\mu}(\sigma_i)](ij) \prod_{l\in\tilde{\mathbf{h}}\setminus\{a,y\}} (il)(jl), \qquad i,j\in\mathbf{h},$$

$$\tilde{\phi}_{ij} \coloneqq \frac{\langle\lambda(\sigma_i)\,\lambda(\sigma_j)\rangle}{(ij)} \prod_{k\in(\tilde{\mathbf{h}}\cup\{b,t\})\setminus\{i,j\}} \frac{1}{(ki)(kj)} \qquad i,j\in\tilde{\mathbf{h}}$$