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Solving a Feynman integral depending on two elliptic curves: the 5 mass Kite family

or how to solve an integral by (almost) never integrating

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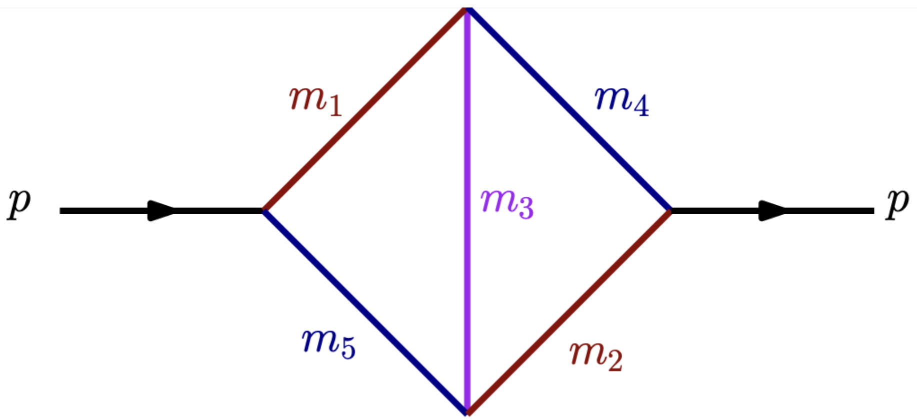
Feynman Integrals (FI)

- **Applications**: **Perturbative** expansion of **QCD**, **Black Hole** Scattering, **EFT** of Large Scale-Structures...

- Most solved **FI**s evaluate to **multiple polylogarithms (MPLs)**, From 2-loops this is isn't enough **[Bloch, Vanhove, 1309.5865]**



- Most general 2-point, 2-loop **FI**
- Dependent on **elliptic MPLs**
- Mixes two different **elliptic curves**
- The first **elliptic FI** involving > 3 scales

Differential Equations [Kotikov, 1991]

- State of the art is to set up a set of **MIs** and find an associated Deq:

$$\mathrm{d}\mathbf{MIs} = \mathcal{A}(D, p^2, m_i^2, \mathrm{d}p^2, \mathrm{d}m_i^2)\,\mathbf{MIs}; \quad D = 2 - 2\epsilon$$

- We try to find the "**canonical form**" to this equation through the gauge transformation $\mathbf{U} \cdot \mathbf{MIs} := \mathbf{J}$ s.t. [Henn, 1304.1806]:

$$\mathrm{d}\mathbf{J} = \epsilon \underbrace{\tilde{\mathcal{A}}(p^2, m_i^2, \mathrm{d}p^2, \mathrm{d}m_i^2)}_{\epsilon \text{ independent}} \mathbf{J} \quad \rightarrow \quad \mathbf{J} = \mathbb{P}\exp\left[\epsilon \int_\gamma \tilde{\mathcal{A}}\right] \mathbf{J}_0$$

- **U** will depend on the Periods of **both** elliptic curves:

$$\psi_1(p^2, m_1, m_2, m_3) \propto \int \mathrm{d}x\,((x - r_1)(x - r_2)(x - r_3)(x - r_4))^{-1/2}$$

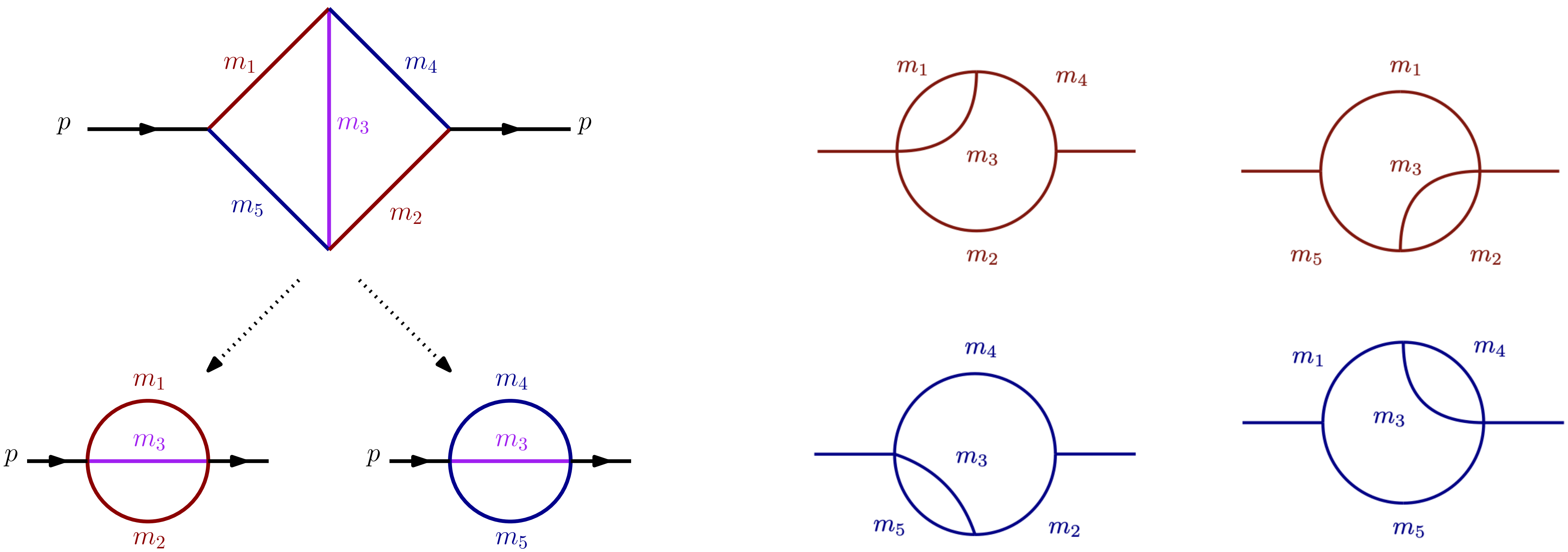
$$\psi_1(p^2, m_3, m_4, m_5) \propto \int \mathrm{d}x\,((x - q_1)(x - q_2)(x - q_3)(x - q_4))^{-1/2}$$

The Kite integral family

- The Kite integral family has **30 Master Integrals**, most notably:

$$\mathbf{MIs} = \Bigg(\underbrace{\ldots, I_{11100}, I_{21100}, I_{12100}, I_{11200}, I_{00111}, I_{00211}, I_{00121}, I_{00112}}_{\text{existing results } \textbf{[Bogner, Müller-Stach, Weinzierl, 1907.01251]}},$$

$$\underbrace{I_{11110}, I_{11101}, I_{01111}, I_{10111}, I_{11111}}_{\text{our results}} \Bigg)$$



The Kite integral family

- The differential equation organises into **blocks**[1]:

$$
\mathrm{d}\begin{pmatrix} I_{11100} \\ I_{21100} \\ I_{12100} \\ I_{11200} \\ I_{00111} \\ I_{00211} \\ I_{00121} \\ I_{00112} \\ I_{11110} \\ I_{11111} \end{pmatrix} = \begin{pmatrix}
\mathcal{A}_{1,1} & \mathcal{A}_{1,2} & \mathcal{A}_{1,3} & \mathcal{A}_{1,4} & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathcal{A}_{2,1} & \mathcal{A}_{2,2} & \mathcal{A}_{2,3} & \mathcal{A}_{2,4} & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathcal{A}_{3,1} & \mathcal{A}_{3,2} & \mathcal{A}_{3,3} & \mathcal{A}_{3,4} & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathcal{A}_{4,1} & \mathcal{A}_{4,2} & \mathcal{A}_{4,3} & \mathcal{A}_{4,4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathcal{A}_{5,5} & \mathcal{A}_{5,6} & \mathcal{A}_{5,7} & \mathcal{A}_{5,8} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathcal{A}_{6,5} & \mathcal{A}_{6,6} & \mathcal{A}_{6,7} & \mathcal{A}_{6,8} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathcal{A}_{7,5} & \mathcal{A}_{7,6} & \mathcal{A}_{7,7} & \mathcal{A}_{7,8} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathcal{A}_{8,5} & \mathcal{A}_{8,6} & \mathcal{A}_{8,7} & \mathcal{A}_{8,8} & 0 & 0 \\
\mathcal{A}_{9,1} & \mathcal{A}_{9,2} & \mathcal{A}_{9,3} & \mathcal{A}_{9,4} & 0 & 0 & 0 & 0 & \mathcal{A}_{9,9} & 0 \\
\mathcal{A}_{10,1} & \mathcal{A}_{10,2} & \mathcal{A}_{10,3} & \mathcal{A}_{10,4} & \mathcal{A}_{10,5} & \mathcal{A}_{10,6} & \mathcal{A}_{10,7} & \mathcal{A}_{10,8} & \mathcal{A}_{10,9} & \mathcal{A}_{10,10}
\end{pmatrix} \cdot \begin{pmatrix} I_{11100} \\ I_{21100} \\ I_{12100} \\ I_{11200} \\ I_{00111} \\ I_{00211} \\ I_{00121} \\ I_{00112} \\ I_{11110} \\ I_{11111} \end{pmatrix}
$$

[1] we only display some of the sectors

The Eyeball integral family

$$\textbf{Ansatz:}\ \mathrm{d}\underbrace{\left(\frac{I_{11110}}{I^{(0)}_{11110}} + F\tilde{I}_{11100}\right)}_{\text{new master Integral } \tilde{I}_{11110}} = \epsilon\,(\dots)$$

- where $I^{(0)}_{11110}$ is the **Maximal Cut**, determining F is the actual challenge and requires the definition of the **Moduli space**:

$$(p^2, m_1^2, m_2^2, m_3^2, m_4^2, m_5^2) \to (\underbrace{\tau, z_1, z_2, z_3}_{\text{known result}}, \underbrace{z_4, z_5}_{\text{new method!}})$$

- with a **Jacobian**:

$$\begin{pmatrix} \mathrm{d}\tau \\ \mathrm{d}\vec{z} \end{pmatrix} = \begin{pmatrix} \frac{\partial \tau}{\partial p^2} & \frac{\partial \tau}{\partial \vec{m}^2} \\ \frac{\partial \vec{z}}{\partial p^2} & \frac{\partial \vec{z}}{\partial \vec{m}^2} \end{pmatrix} \begin{pmatrix} \mathrm{d}p^2 \\ \mathrm{d}\vec{m}^2 \end{pmatrix}$$

The Eyeball integral family

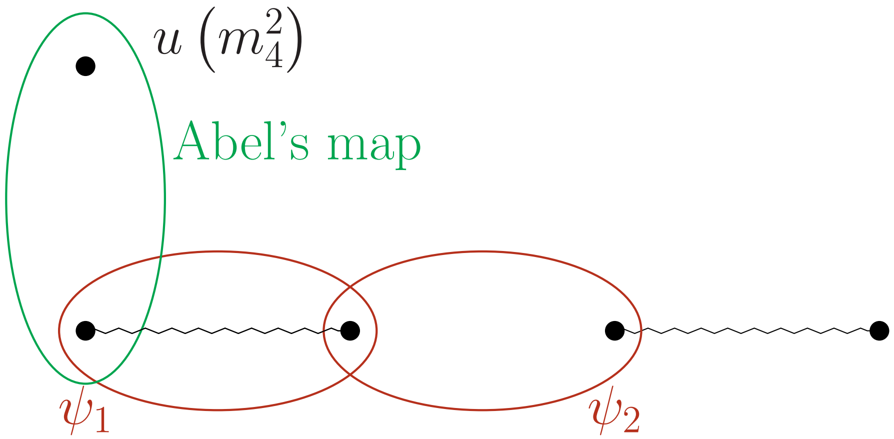
- F can be shown to be expressed as (with $f^{(\cdots)}$ some rational function):

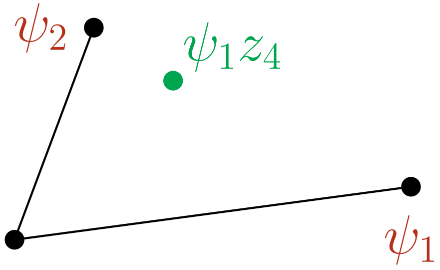
$$F = \left(f^{(m_1^2)} \frac{\partial m_1^2}{\partial \tau} + f^{(m_2^2)} \frac{\partial m_2^2}{\partial \tau} + f^{(p^2)} \frac{\partial p^2}{\partial \tau} + f^{(m_4^2)} \frac{\partial m_4^2}{\partial \tau} \right) \psi_1 \frac{\partial \tau}{\partial p^2}$$

- We should find how m_4 maps to the torus! We find a natural candidate:

$$\int_0^{m_4^2} \mathrm{d}m_4'^2 \, I_{11110}^{(0)} = \int_0^{m_4^2} \frac{\mathrm{d}m_4'^2}{\sqrt{P_4(m_4'^2)}} = \psi_1 z_4$$

- This turns out to be **Abel's map** (inverse of $\wp(z)$) for our torus!





The Kite integral family

- The **Kite** integral follows the following differential equation:

$$\mathrm{d}I_{11111} = \mathcal{A}^{(0)}_{10,10} I_{11111} + \mathcal{A}^{(0)}_{10,1} \tilde{I}_{11100} + \mathcal{A}^{(0)}_{10,4} \tilde{I}_{00111} + \epsilon(\dots)$$

- Analogously to the **Eyeballs**, make the ansatz:

$$\mathrm{d}\underbrace{\left(\frac{I_{11111}}{I^{(0)}_{11111}} + F_1 \tilde{I}_{11100} + F_2 \tilde{I}_{00111} \right)}_{\text{new master Integral } \tilde{I}_{11111}} = \epsilon\,(\dots)$$

- F_1 and F_2 can be determine as before since we can compute $\dfrac{\partial m_4^2}{\partial \tau^{(11100)}}$, $\dfrac{\partial m_5^2}{\partial \tau^{(11100)}}$, $\dfrac{\partial m_4^2}{\partial \tau^{(00111)}}$ and $\dfrac{\partial m_5^2}{\partial \tau^{(00111)}}$

The solution

- The **Kite** integral family follows the following **canonical** differential equation:

$$\mathrm{d}\begin{pmatrix} \vdots \\ \\ \tilde{I}_{11111} \end{pmatrix} = \epsilon \tilde{\mathcal{A}} \begin{pmatrix} \vdots \\ \\ \tilde{I}_{11111} \end{pmatrix}$$

- The entries of $\tilde{\mathcal{A}}$ are either **rational functions** with simple poles or **"elliptic" functions** $g^{(\alpha)}\left(\sum_i a_i z_i, \tau\right)$ on **either** torus with simple poles which are related to θ-functions through:

$$\sum_{\alpha=0}^{\infty} \eta^{\alpha-1} g^{(\alpha)}(z,\tau) = \pi \frac{\theta_1'(0,\tau)\theta_1(\pi(z+\eta),\tau)}{\theta_1(\pi z,\tau)\theta_1(\pi\eta,\tau)}$$

Conclusion and Outlook

- Opens the door to study more **multi-scale FIs** beyond **MPLs**
- Understanding how to find **Moduli space** $\mathcal{M}_{1,5}$ is **crucial**
- Structure of differential equation **never mixes elliptic curves**, what would happen for **FIs** in which they do? **[Müller, Weinzierl, 2205.04818]**
- Characterising the **Moduli** of more complicated geometries is an open problem!

Thank you for your attention !