On the initial singularity and extendibility of flat quasi-de Sitter spacetimes

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with Eric Ling and Jerome Quintin

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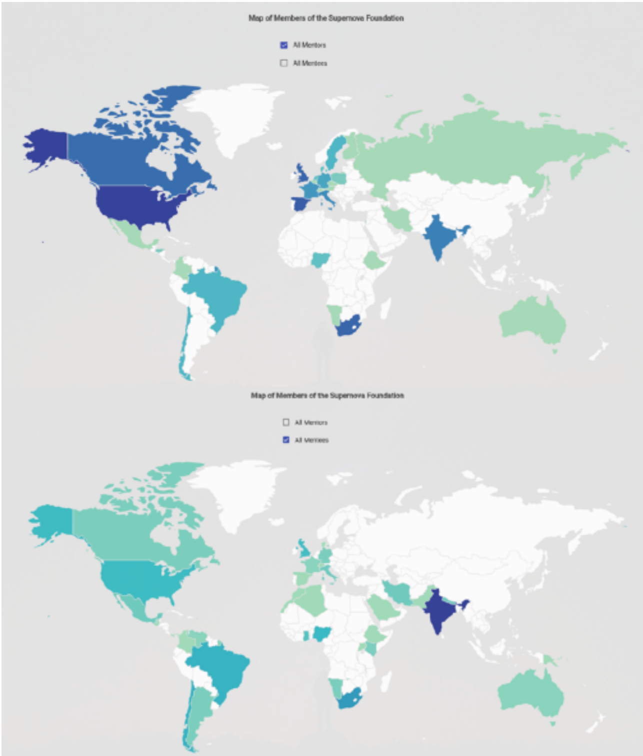
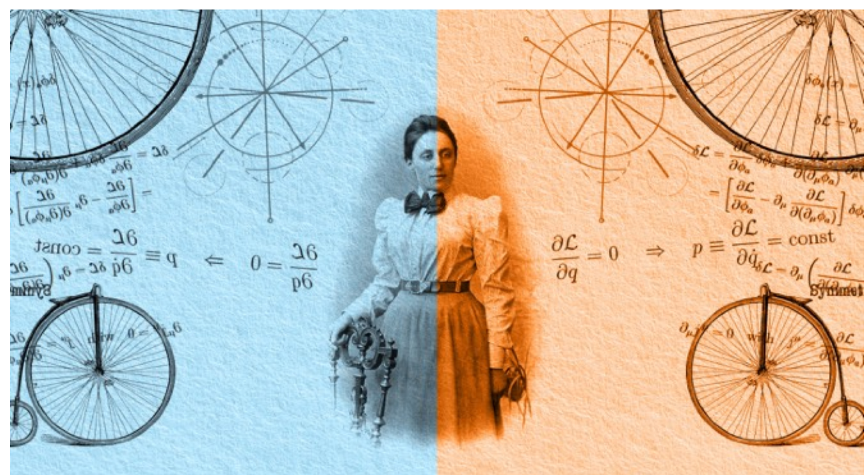
Woman. Life. Freedom.





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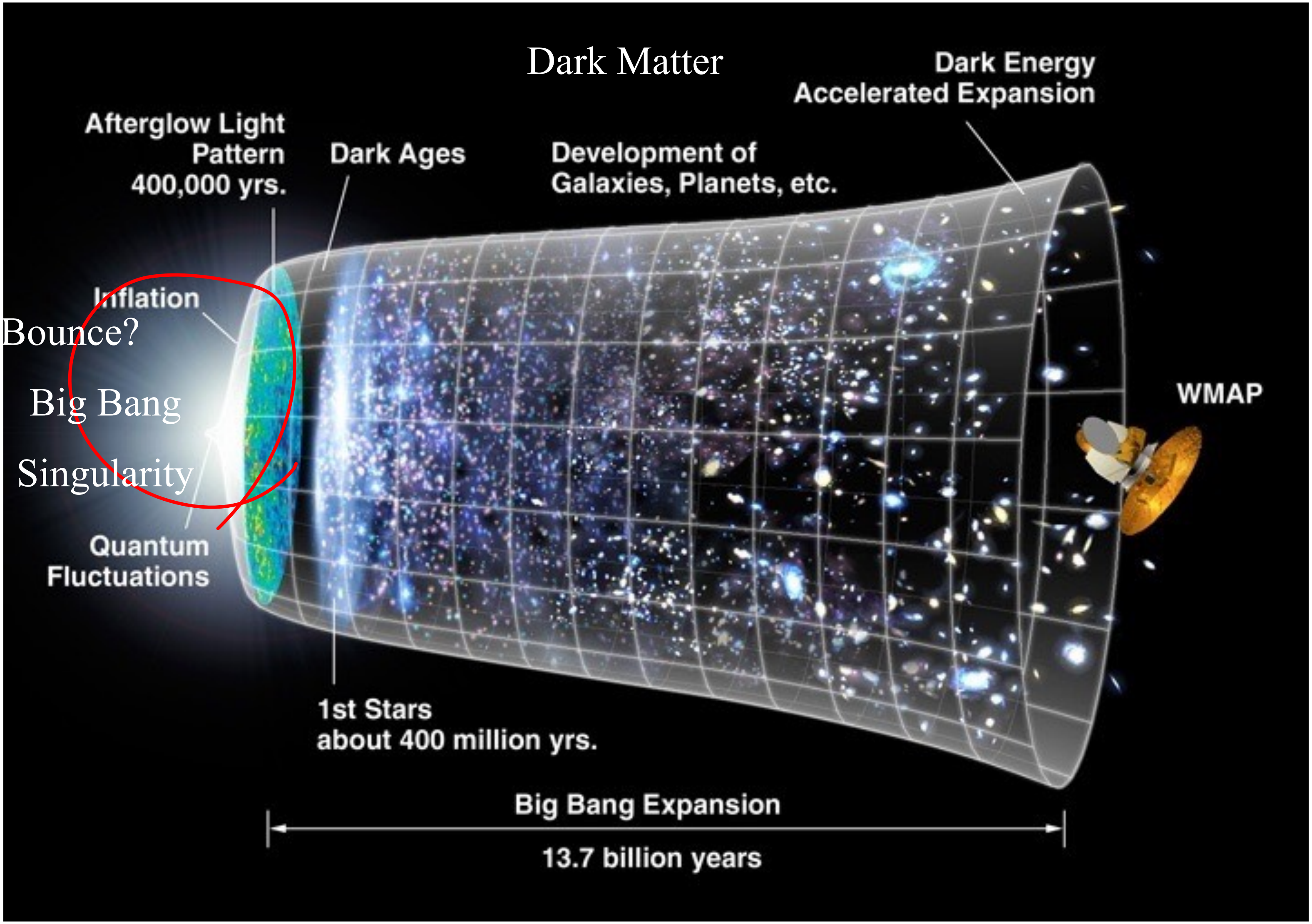
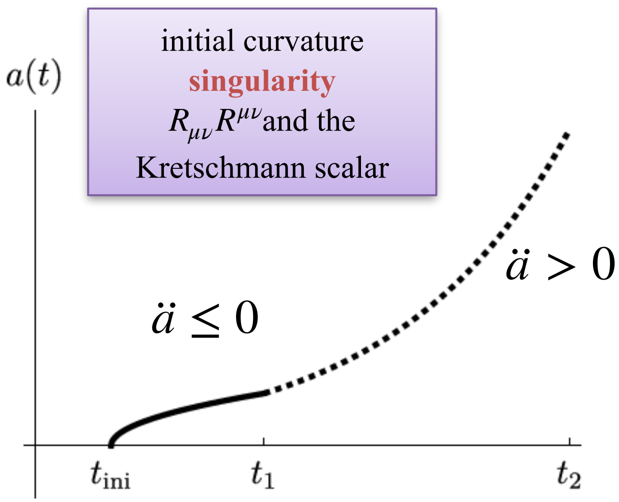
The Jigsaw Puzzle

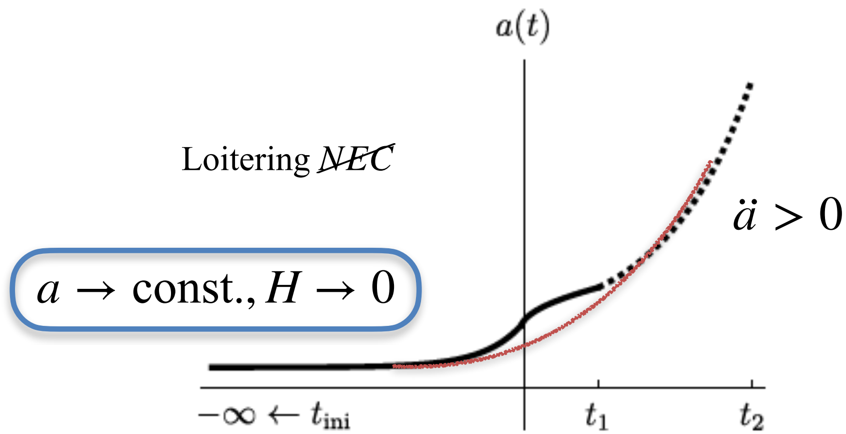


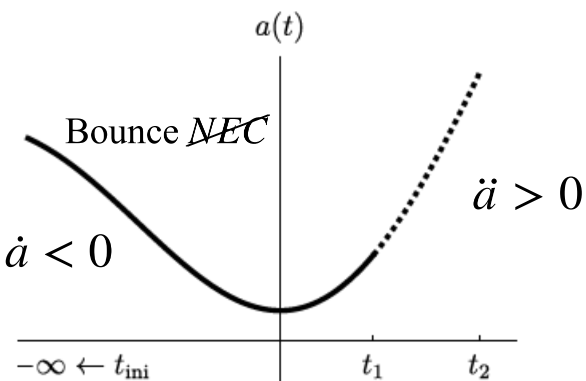
What is inflation?

$$ds^2 = -dt^2 + a(t)^2(dr^2 + r^2 d\Omega^2) \quad \text{with} \quad \dot{a} > 0, \;\; \ddot{a} > 0$$

Possible pre-inflationary phases:

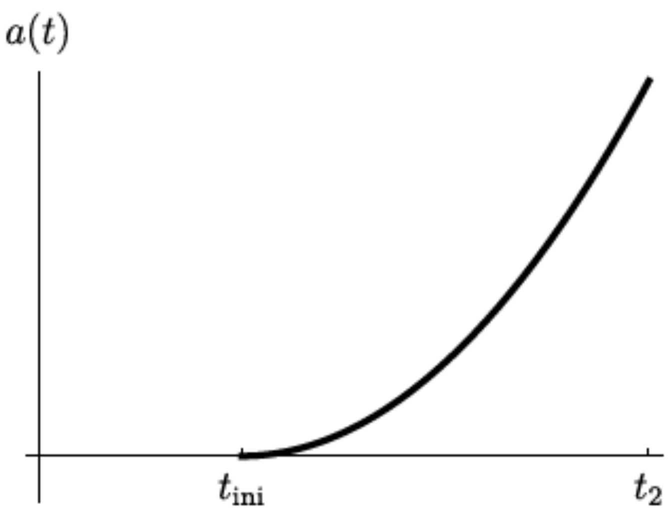


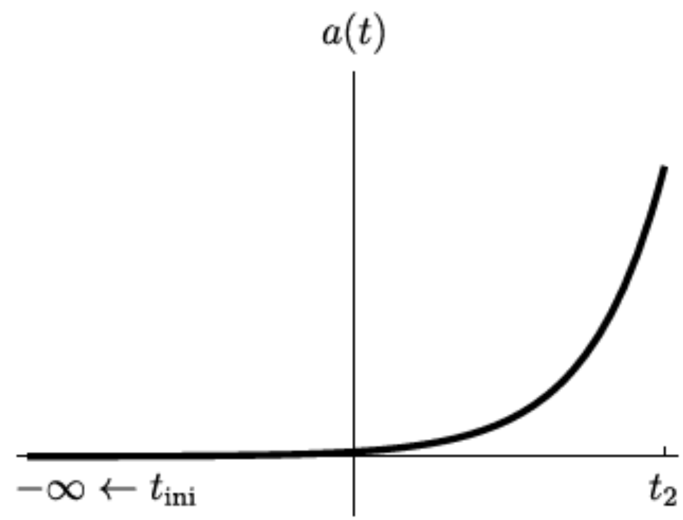




often requires new physics

Inflating from the “very beginning” ($H_{in} \neq 0$) :

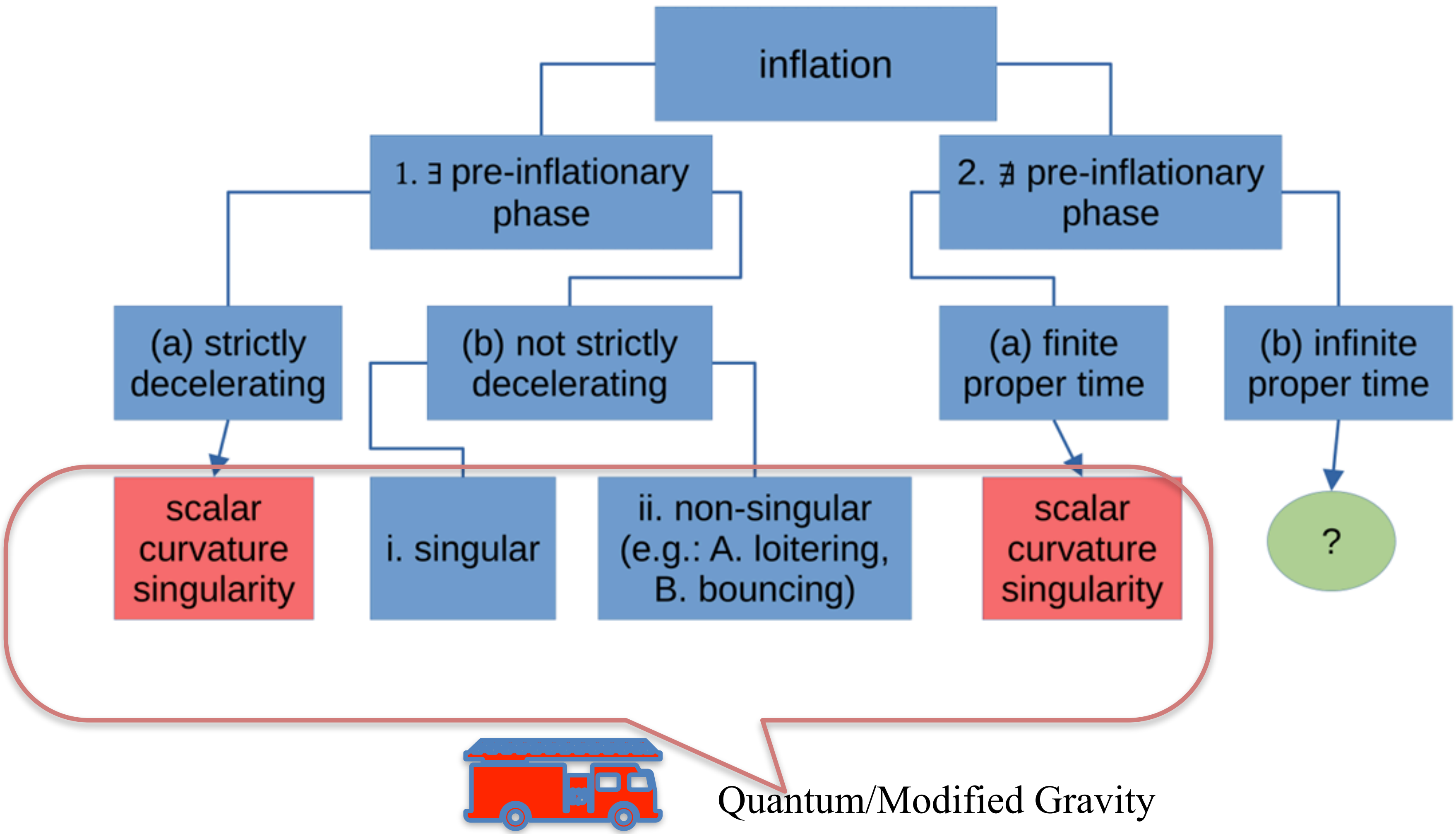




a finite initial time $t_{ini} \rightarrow$ curvature singularity
($R_{\mu\nu}R^{\mu\nu}$ and the Kretschmann scalar)

Unclear what happens as
$t_{ini} \rightarrow -\infty$

Does inflation address Big Bang Singularity?



Borde-Guth-Vilenkin theorem

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Inflationary Spacetimes Are Incomplete in Past Directions

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Many inflating spacetimes are likely to violate the weak energy condition, a key assumption of singularity theorems. Here we offer a simple kinematical argument, requiring no energy condition, that a cosmological model which is inflating—or just expanding sufficiently fast—must be incomplete in null and timelike past directions. Specifically, we obtain a bound on the integral of the Hubble parameter over a past-directed timelike or null geodesic. Thus inflationary models require physics other than inflation to describe the past boundary of the inflating region of spacetime.

Proof of BGV (for null in flat FLRW):

$$ds^2 = -dt^2 + a(t)^2(dr^2 + r^2 d\Omega^2)$$

Consider an affine parameter λ of a null geodesic in FLRW and some reference time t_f,

$$d\lambda = \frac{a(t)}{a_f} dt$$

$$H = \frac{d \ln a}{dt} \implies \int_{\lambda(t_i)}^{\lambda(t_f)} H(\lambda) d\lambda = 1 - \frac{a_i}{a_f} \leq 1 \qquad \text{for } 0 \leq a_i < a_f$$

$$\implies H_{av}(t_i) \equiv \frac{1}{\lambda(t_f) - \lambda(t_i)} \int_{\lambda(t_i)}^{\lambda(t_f)} H(\lambda) d\lambda \leq \frac{1}{\lambda(t_f) - \lambda(t_i)}$$

Without the loophole

$$\text{let } c \in \mathbb{R} > 0 \text{ if } \forall t_i < t_f, \; 0 < c \leq H_{av}(t_i) \quad \text{then} \quad \lambda(t_f) - \lambda(t_i) \leq \frac{1}{c}$$

The past geodesics have a finite affine length $\implies$ Geodesic Incompleteness

Does it always imply some kind of singularity?

Coordinate singularities

Roughly, a spacetime contains a coordinate singularity when a defining set of coordinates fails to capture all the geometry of the spacetime, and another set of coordinates exposes this geometry.

Flat dS:

$$ds^2 = -dt^2 + e^{2t}(dr^2 + r^2 d\Omega^2)$$

Extendible

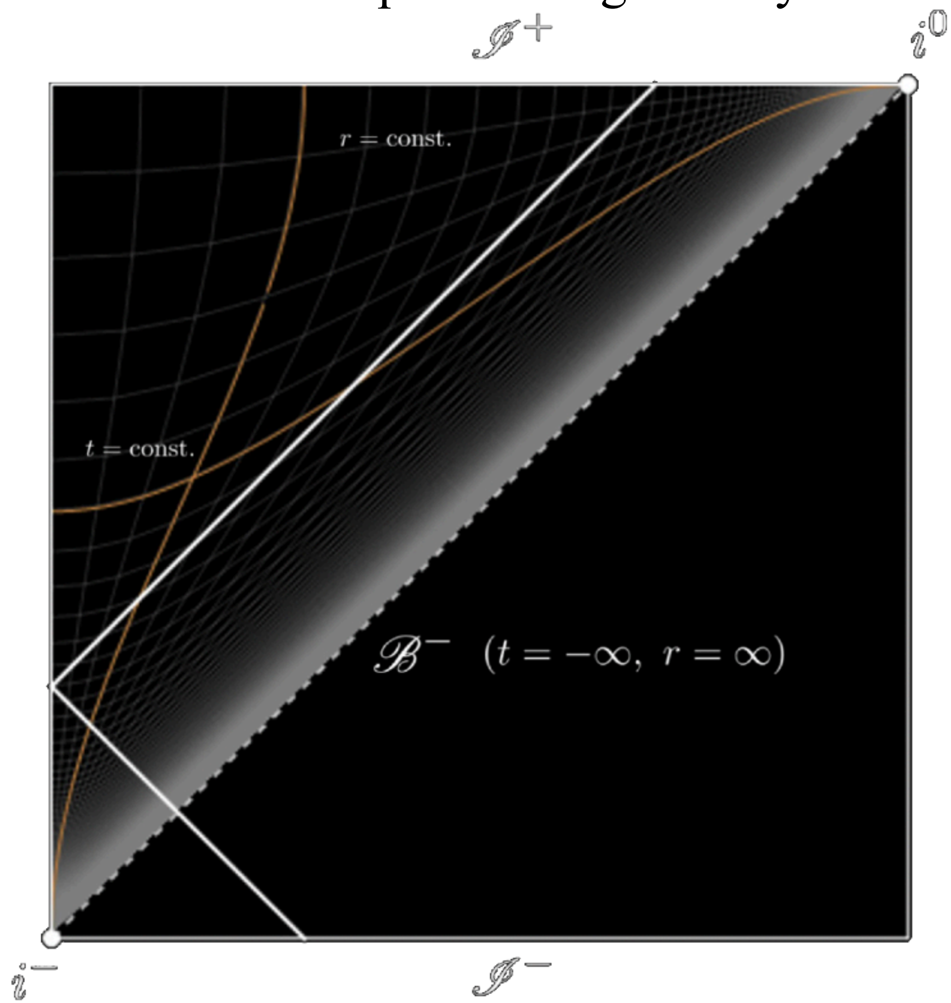
Closed dS

$$ds^2 = -dt^2 + \cosh^2(t) d\Omega^2_{(3)}$$

Conformal dS

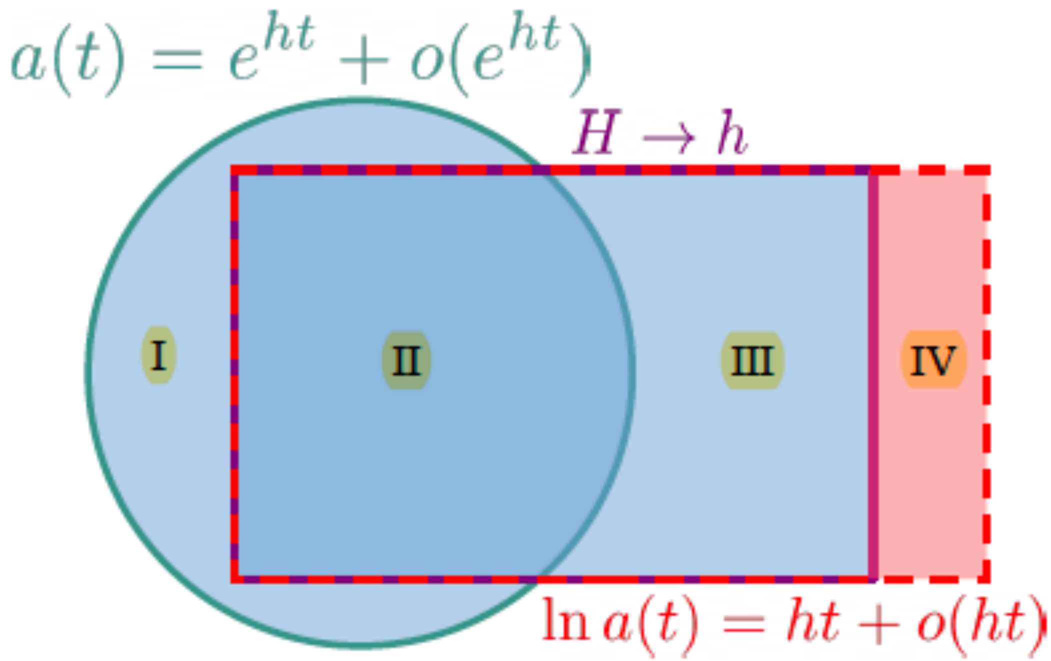
$$ds^2 = \frac{1}{\cos^2 T}(-dT^2 + d\Omega^2_{(3)})$$

Flat de Sitter spacetimes conformally embed into the Einstein static universe



But Inflation is "quasi de Sitter" not exact de Sitter!

So what if Inflation is close enough to dS?



"Eddington-Finkelstein coordinates for quasi dS"

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Fix any $t_{\mathrm{L}} \in (-\infty, t_{\mathrm{max}})$ then define conformal time to be

$$\eta(t) = -\int_{t}^{t_{\mathrm{L}}} \frac{d\tilde{t}}{a(\tilde{t})}$$

then introduce new coordinates (λ, v) as functions of (t, r):

$$\lambda(t) := \int_{-\infty}^{t} d\tilde{t}\, a(\tilde{t}) \quad \text{and} \quad v(t, r) := \eta(t) + r\,,$$

(λ is the affine parameter of null geodesics, which are characterized by $v =$ constant)

Yoshida & Quintin 2018

the coordinates (λ, v) are a diffeomorphism from $(-\infty, t_{\max}) \times (0,\infty)$ onto

$$U := \left\{(\lambda, v) \mid \lambda \in (0,\lambda_{\max}) \text{ and } v \in \big(\eta(\lambda), \infty\big)\right\}.$$

$$ds^2 = -2\, d\lambda\, dv + a(\lambda)^2 dv^2 + a(\lambda)^2 (v - \eta(\lambda))^2 d\Omega^2$$

for dS:

$$ds^2 = -2\, d\lambda\, dv + \lambda^2 dv^2 + (\lambda v + 1)^2 d\Omega^2, \quad a = \lambda > 0$$

For

$$a(t) \overset{t\to-\infty}{=} e^{ht} + o(e^{ht})$$

or

$$H(t) := \frac{\dot{a}(t)}{a(t)} \overset{t\to-\infty}{\to} h$$

$\Longrightarrow a \to 0$ and $a(t)\eta(t) \to -1/h$ as $t \to -\infty$.

$\Longrightarrow C^0$ extendable

then a continuous extension (M_{ext}, g_{ext}) of (M, g) is given by

$$M_{ext} = M \cup M_{\lambda\leq 0}, \qquad M_{\lambda\leq 0} := \{(\lambda, v) \mid \lambda \leq 0 \text{ and } v \in \mathbb{R}\} \times \mathbb{S}^2$$

and

$$g_{\text{ext}} = \begin{cases} -2\,d\lambda\,dv + a(\lambda)^2 dv^2 + a(\lambda)^2 (v - \eta(\lambda))^2 d\Omega^2 & \text{on } M \\ -2\,d\lambda\,dv + h^{-2} d\Omega^2 & \text{on } M_{\lambda\leq 0}\,. \end{cases}$$

Generally $a \to 0^+$, $\eta \to -\infty$, $\lambda \to 0^+$ as $t \to -\infty$ and c^k extendibility of the metric requires

$$a^2, a^2\eta, a^2\eta^2 \in C^k$$

if $\dfrac{\dot{H}}{a^2}$ converges to a finite limit as $t \to -\infty$, then $\exists$ C^2 extension

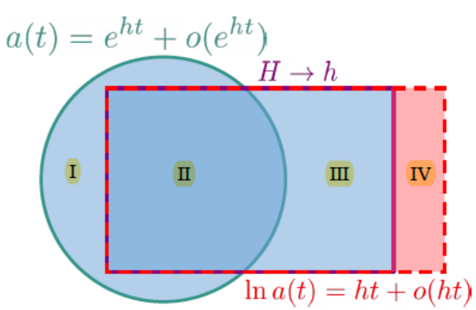
$\Longrightarrow$ extension of geodesics

Toy example:

$$a(t) = e^t + \sin^2(e^{-3t})e^{2t} \implies \lim_{t\to-\infty} \frac{a(t)}{e^t} = 1 \implies \exists\ C^0 \text{ extension}$$

But $H = \frac{\dot{a}}{a}$ does not have a limit as $t \to -\infty$! $\implies$ curvature singularity

coordinate singularities and curvature singularities are not mutually exclusive

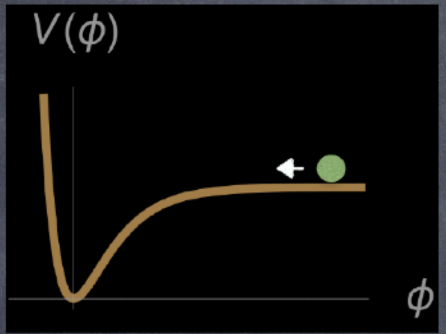


if $\frac{\dot{H}}{a^2}$ converges to a finite limit as $t \to -\infty$, then $\exists\ C^2$ extension

extension of geodesics

Starobinsky

$$V(\varphi) = \frac{3}{4}m^2(1 - e^{-\sqrt{2/3}\varphi})^2$$



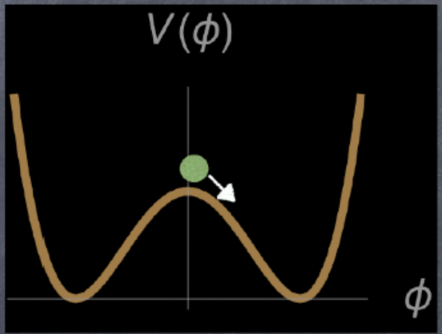
$$a(t) \simeq a_e e^{\frac{m}{2}t}\left(1 - \frac{2}{3}me^{-\sqrt{\frac{2}{3}}\varphi_e t}\right)$$

$$H \to \frac{m}{2}, \dot{H} \to 0$$

$$\dot{H}/a^2 \to -\infty$$

Small-field

$$V(\varphi) = V_0\left(1 - \left(\frac{\varphi}{2m}\right)^2\right)$$



$$a(t) \simeq a_e e^{\sqrt{\frac{V_0}{3}}t - \frac{\varphi_e^2}{8}\exp(\frac{2}{m^2}\sqrt{\frac{V_0}{3}}t)}$$

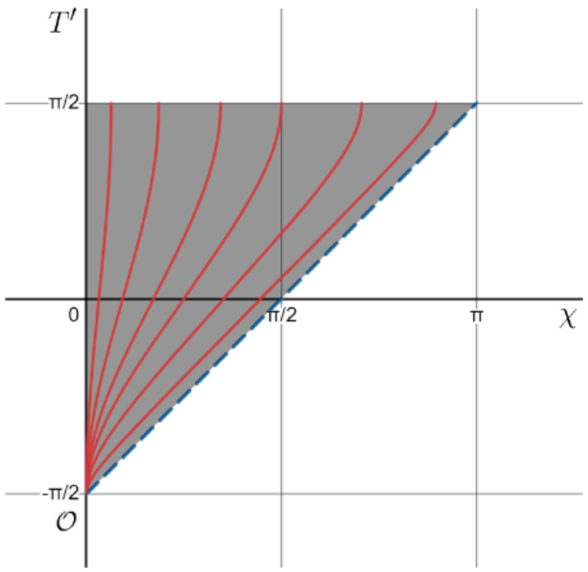
$$H \to \sqrt{\frac{V_0}{3}}, \dot{H} \to 0$$

$\dot{H}/a^2$ **smooth in** a

Credit Quintin

Conformal embeddings of quasi-de Sitter spacetimes into the Einstein static universe

Suppose $a(t) = e^{ht} + o(e^{ht})$ or $H(t) \to h$ as $t \to -\infty$ for some $h \in \mathbb{R}_{>0}$, then



Flat FLRW spacetime (M, g): $M = (-\infty, t_{\max}) \times \mathbb{R}^3$

$$g = -dt^2 + a(t)^2 h_{\mathbb{E}}$$

Embed $\downarrow$ $g = \omega^2 \hat{g}$ $\uparrow$ Extend

Flat dS $(\hat{M}, \hat{g})$: $\hat{M} = \mathbb{R} \times \mathbb{R}^3$ and $\hat{g} = -d\hat{t}^2 + e^{2h\hat{t}} h_{\mathbb{E}}$,

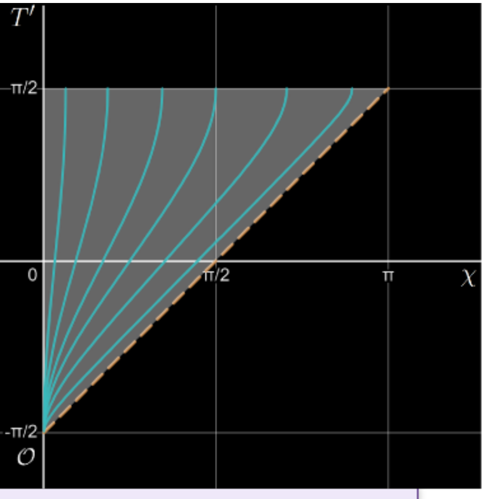
Embed $\downarrow$ $\hat{g} = \Omega^2 \tilde{g}$ $\uparrow$ Extend

The Einstein static $(\widetilde{M}, \tilde{g})$: $\widetilde{M} = \mathbb{R} \times \mathbb{S}^3$ and $\tilde{g} = -(dT')^2 + h_{\mathbb{S}^3}$, $\implies \exists\ C^0$ conformal emb.

existence of C^0 extendibility similar to before

the extension is different: closed (global) de Sitter

The cosmological constant appears as an initial condition



Proposition (G.G., E. Ling, J. Quintin '23)

Under appropriate assumptions of the scale factor, a $k = 0$ FLRW spacetime is past-asymptotically de Sitter if and only if

$$\rho(-\infty) = -p(-\infty).$$

Beyond homogeneity and isotropy

Let $(\widetilde{M}, \widetilde{g})$ be a C^0 conformal extension of (M, g) with conformal factor Ω such that $M = I^{+}_{\tilde{g}}(\mathcal{O}, \widetilde{M})$ for some point $\mathcal{O} \in \partial_0^- M$ and

- (M, g) solves the Einstein equations with a perfect fluid (u, ρ, p)
- Integral curves of u have past endpoint $\mathcal{O}$ within $\widetilde{M}$. Moreover, the vector field along each integral curve of $\frac{1}{\Omega}u$ extends continuously to a $\widetilde{g}$-timelike vector at $\mathcal{O}$.
- ρ and p and $\Omega^2 \mathrm{Ric}_g$ extend continuously to $M \cup \{\mathcal{O}\}$. $\Omega^2 \mathrm{Ric}_g$is the Ricci tensor for (M, g).
- $(\widetilde{M}, \widetilde{g})$ is strongly causal at $\mathcal{O}$.

Then the continuous extensions of ρ and p satisfy $\widetilde{p} = -\widetilde{\rho}$ at $\mathcal{O}$.

Conclusion

- Geometrical formulation of Inflationary spacetime “close enough” to dS in the asymptotic past.
- Theorems on asymptotic conditions to guarantee the extendibility of Inflationary backgrounds in the past.
- Found evidence even beyond FLRW, Λ appears as an initial condition 😰