Imperial College London

Stochastic effective theory for scalar fields in de Sitter spacetime

Arttu Rajantie

CATCH-22+2, DIAS

2 May 2024

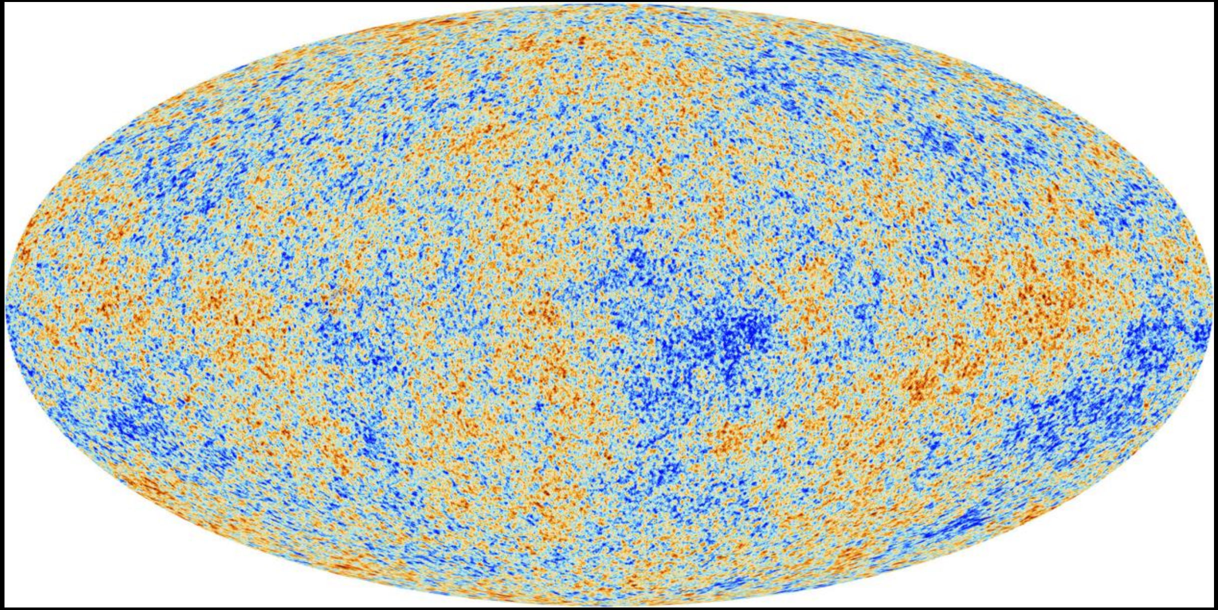
In collaboration with:
Archie Cable, Jose Eliel Camargo-Molina, Mariana Carrillo Gonzalez, Sebastian Cespedes, Liina Jukko, Diana López Nacir, Tommi Markkanen, Gonzalo Santa Cruz Moreno, Stephen Stopyra, and Tommi Tenkanen

Based on:

- T. Markkanen, AR and T. Tenkanen, PRD98(2018)123532.
- T. Markkanen, AR, S. Stopyra and T. Tenkanen, JCAP08(2019)001.
- T. Markkanen and AR, JCAP03(2020)049.
- A. Cable and AR, PRD104(2021)103511.
- J.E. Camargo-Molina and AR, PRD107(2023)103504.
- J.E. Camargo-Molina, M. Carrillo Gonzalez and AR, PRD107(2023)063533.
- A. Cable and AR, PRD106(2022)123522.
- A. Cable and AR, PRD109(2024)045017.

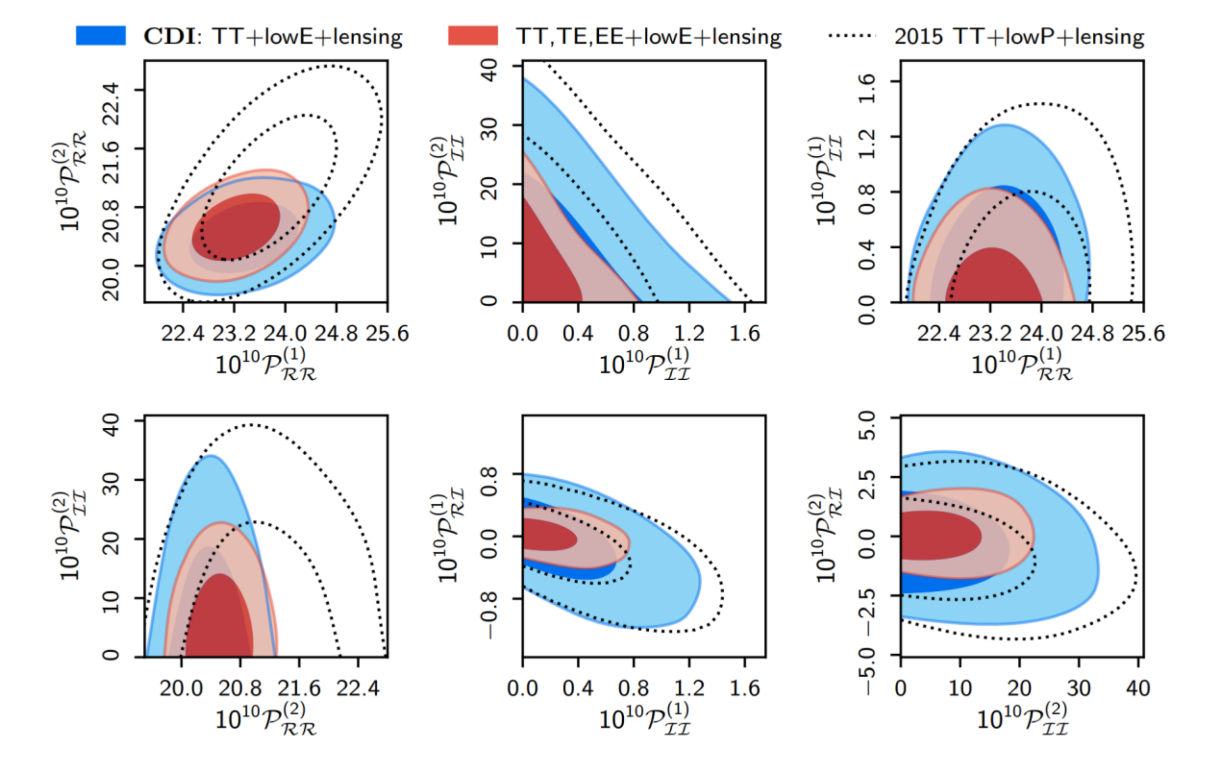
Scalar Fields in the Early Universe

- Inflaton field:
  - Primordial curvature perturbations
  - Large scale structure
  - Cosmic microwave background temperature anisotropy
- "Spectator" fields (e.g. the Higgs?):
  - Isocurvature perturbations



Scalar Fields in the Early Universe

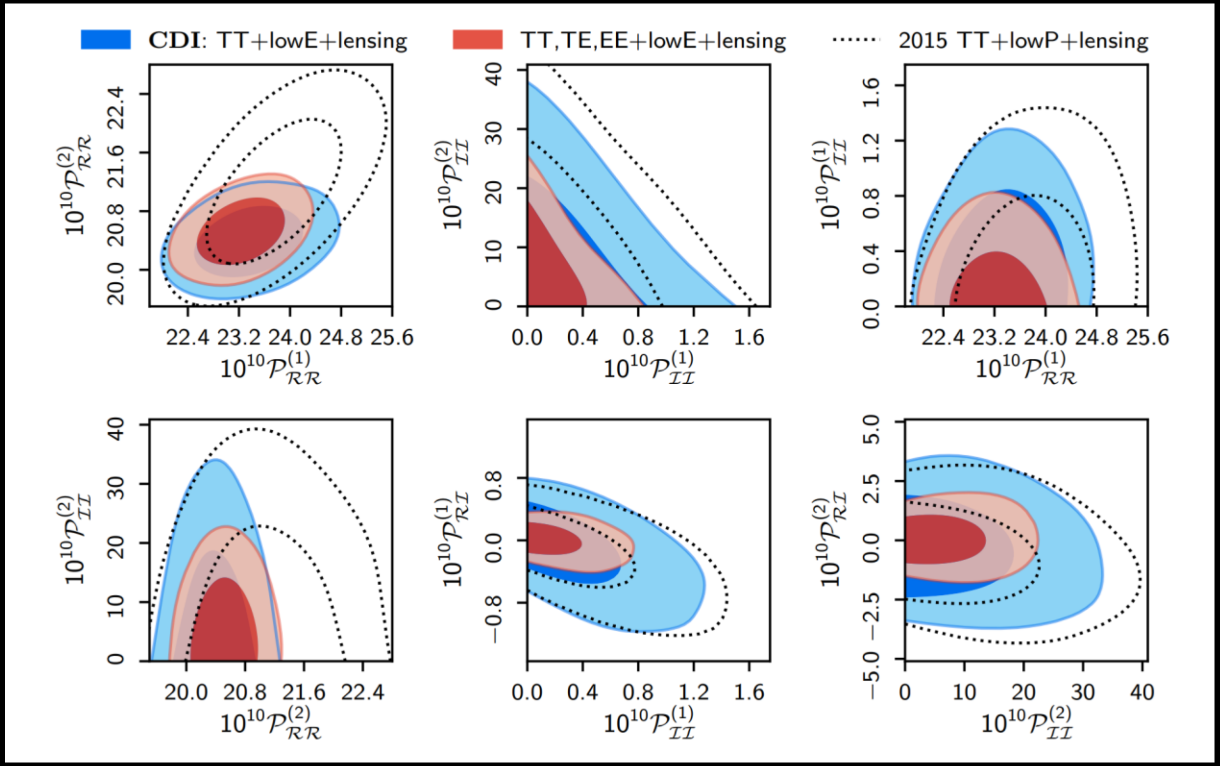
- Inflaton field:
  - Primordial curvature perturbations
  - Large scale structure
  - Cosmic microwave background temperature anisotropy
- “Spectator” fields (e.g. the Higgs?):
  - Isocurvature perturbations



Imperial College
London

Scalar Fields in the Early Universe

- Inflaton field:
  - Primordial curvature perturbations
  - Large scale structure
  - Cosmic microwave background temperature anisotropy
- "Spectator" fields (e.g. the Higgs?):
  - Isocurvature perturbations



- Given by equal-time correlation functions
  $\langle\phi(t,\vec{r})\phi(t,\vec{r}')\rangle$, $\quad \langle f(\phi(t,\vec{r}))f(\phi(t,\vec{r}'))\rangle$
  on massively superhorizon scales
  $$|\vec{x}-\vec{x}'| = a(t)|\vec{r}-\vec{r}'| \sim e^{50}H^{-1}$$
  at the end of inflation

Free Scalar Field in de Sitter

- De Sitter spacetime $ds^2 = dt^2 - a(t)^2 d\vec{r}^2;\ a(t) = e^{2Ht}$
- Scalar field correlator (Chernikov&Tagirov 1968):

$$\langle\phi(t,\vec{r})\phi(t,\vec{r}\,')\rangle \sim \frac{H^2}{16\pi^2}\frac{\Gamma(2\nu)\Gamma\left(\frac{3}{2}-\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)}\left(\frac{H|\vec{x}-\vec{x}\,'|}{2}\right)^{-3+2\nu} \sim \frac{3H^4}{8\pi^2 m^2}\left(H|\vec{x}-\vec{x}\,'|\right)^{-\frac{2m^2}{3H^2}},$$

where $|\vec{x}-\vec{x}\,'| = a(t)|\vec{r}-\vec{r}\,'|$ is the physical distance and $\nu = \sqrt{\frac{9}{4}-\frac{m^2}{H^2}}$

- If $m \lesssim H$, correlations on cosmological scales

Interacting Scalar Field

- One-loop correction for $0 < m \ll H$:

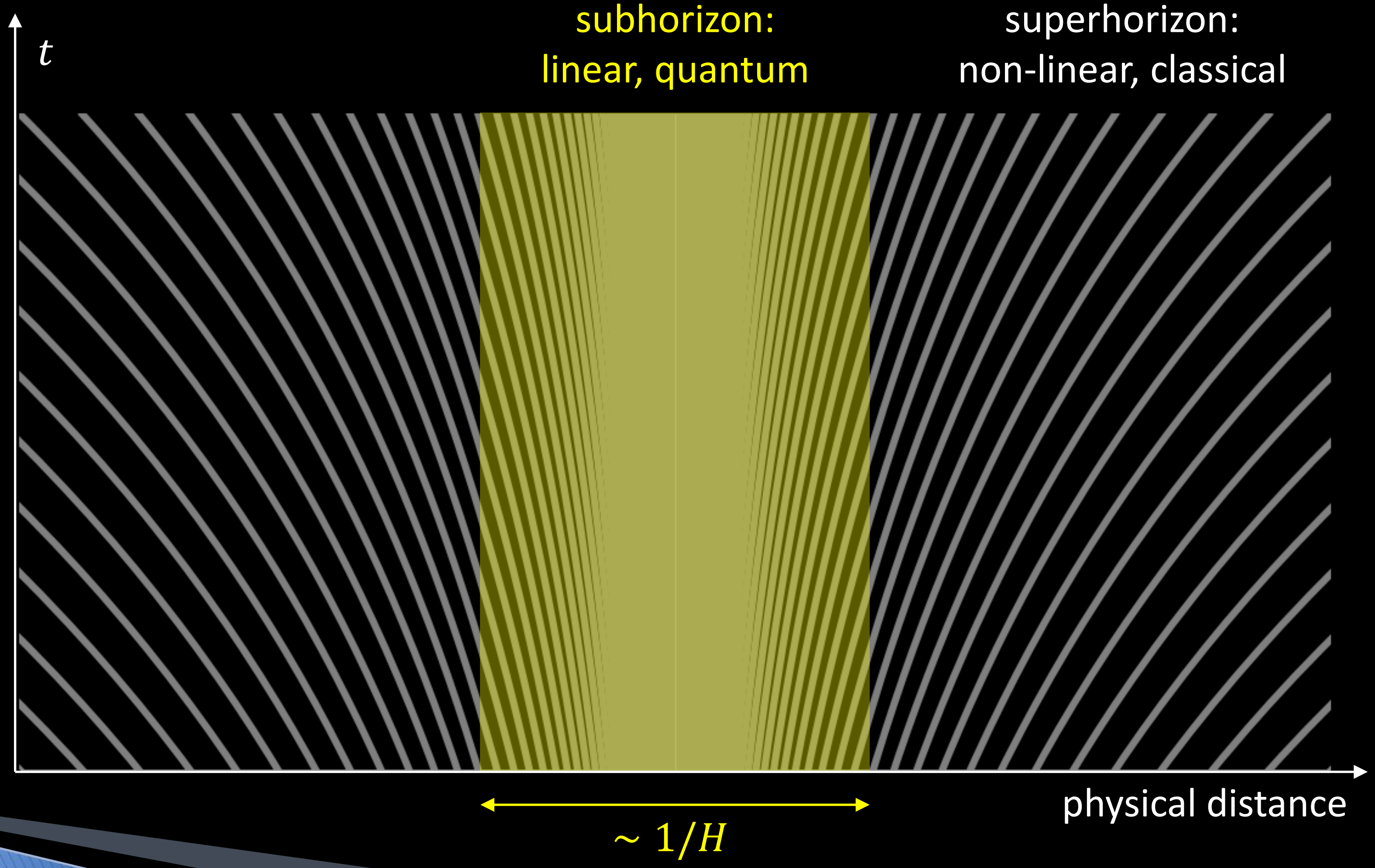
$$m^2 \to m_{\text{eff}}^2 = m^2 + \frac{9\lambda H^4}{8\pi^2 m^2} + O(\lambda^2)$$

- Expansion parameter $\frac{\lambda H^4}{m^4}$

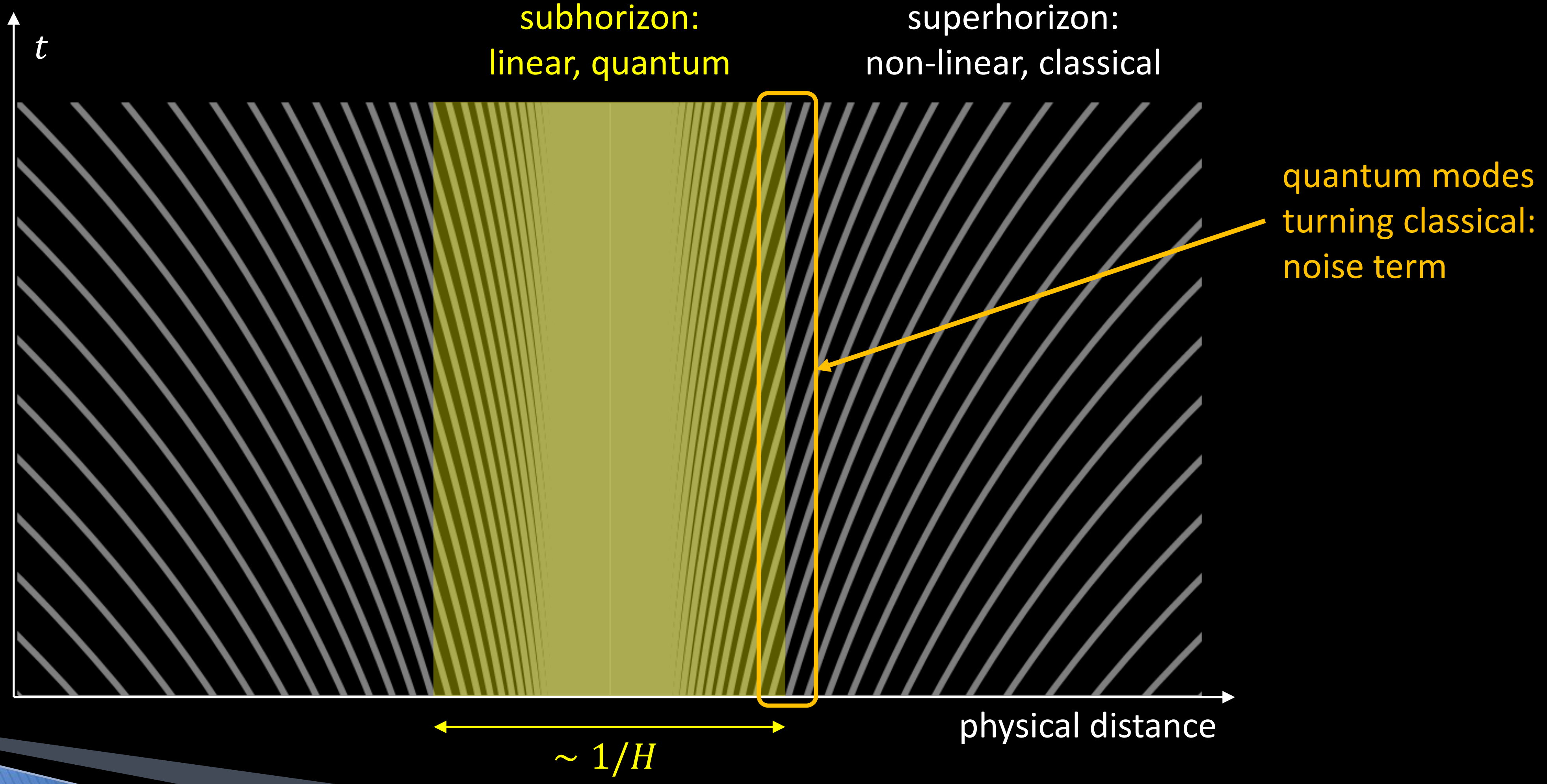
  $\Rightarrow$ Perturbation theory breaks down when $m \lesssim \lambda^{1/4} H$ - Infrared problem
- Consider, e.g., the Higgs with $m = 125\ \text{GeV}, \lambda \approx 0.1$
- One way around: Stochastic Theory (Starobinsky 1982, Starobinsky&Yokoyama 1994)

Comoving modes

Imperial College London



Comoving modes



Stochastic Theory

$$\ddot{\phi} + 3H\dot{\phi} - e^{-2Ht}\vec{\nabla}^2\phi + V'(\phi) = f\,[\delta\hat{\phi}]$$

- Approximations (Starobinsky&Yokoyama 1994):
  - Long-wavelength $e^{-2Ht}\vec{\nabla}^2\phi \to 0$
  - Overdamped $\ddot{\phi} \to 0$
  - Stochastic $f\,[\delta\hat{\phi}] \to 3H\xi$

$\Rightarrow$ Langevin equation: $\dot{\phi} + \frac{1}{3H}V'(\phi) = \xi,\ \langle\xi(t)\xi(t')\rangle = \frac{H^3}{4\pi^2}\delta(t-t')$

(But what is $V(\phi)$ beyond tree level?)

Stochastic Theory

- Fokker-Planck equation for the probability distribution $P(t;\phi)$:

$$\frac{\partial P(t;\phi)}{\partial t} = \left(\frac{V''(\phi)}{3H} + \frac{V'(\phi)}{3H}\frac{\partial}{\partial\phi} + \frac{H^3}{8\pi^2}\frac{\partial^2}{\partial\phi^2}\right)P(t;\phi)$$

- Separation of variables: Linearly independent solutions

$$P_n(t;\phi) = e^{-\Lambda_n t} e^{-\frac{4\pi^2 V(\phi)}{3H^4}} \psi_n(\phi),$$

where Λ_n and ψ_n are eigenvalues and eigenfunctions of the eigenvalue equation

$$\left[\frac{1}{2}\frac{\partial^2}{\partial\phi^2} - \frac{1}{2}\left(v'(\phi)^2 - v''(\phi)\right)\right]\psi_n(\phi) = -\frac{4\pi^2\Lambda_n}{H^3}\psi_n(\phi), \qquad v(\phi) = \frac{4\pi^2}{3H^4}V(\phi)$$

- Note:
  - Similarity with Schrödinger equation: Actually supersymmetric quantum mechanics

Correlation Functions

- Equilibrium correlator of any local function $f(\phi)$ between two points $\vec{r}$ and $\vec{r}'$ at time t:

$$\langle f(\phi(t,\vec{r}))f(\phi(t,\vec{r}'))\rangle = \sum_n \langle 0|f|n\rangle^2 (|\vec{x}-\vec{x}'|H)^{-\frac{2\Lambda_n}{H}}$$

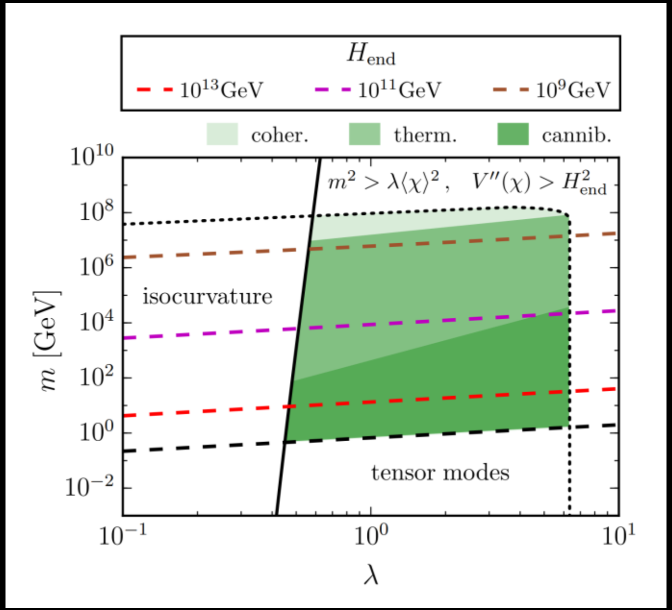
where $\langle m|f|n\rangle = \int d\phi\, \psi_m(\phi)\, f(\phi)\, \psi_n(\phi)$

- To find the **exact** long-distance asymptotics, it is enough to find the lowest eigenvalues Λ_n and eigenfunctions ψ_n

Spectator Dark Matter

- Markkanen, AR & Tenkanen 2018:
  Nearly massless scalar field ϕ, potential $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4$
- Fluctuations produced during inflation → dark matter
- Astrophysics ⇒ Current dark matter density

$$\rho_{DM} \approx \frac{mT_0^3}{M_{Pl}^{3/2}} \frac{|\phi|^{3/2}}{X_f(|\phi|)}$$



- Asymptotic correlator given by $n = 2$ because ρ_{DM} is an even function:

$$\langle \rho_{DM}(t,\vec{r})\rho_{DM}(t,\vec{r}\,')\rangle \sim \langle 0|\rho_{DM}|2\rangle^2 (H|x-x'|)^{-2\Lambda_2}$$

- Isocurvature amplitude

$$\mathcal{P}_{JJ}(k_*) \approx 2\Lambda_2 \langle 0|\rho_{DM}|2\rangle^2 e^{-2\Lambda_2 N_*} \sim 0.43\sqrt{\lambda}\mathrm{e}^{-0.58\sqrt{\lambda}N_*}, \qquad N_* \sim 60$$

- Constraint (Planck 2018): $\mathcal{P}_{JJ}(k_*) < 8.8 \times 10^{-11}$

Vacuum Transitions

- Potential with two metastable minima, e.g.,

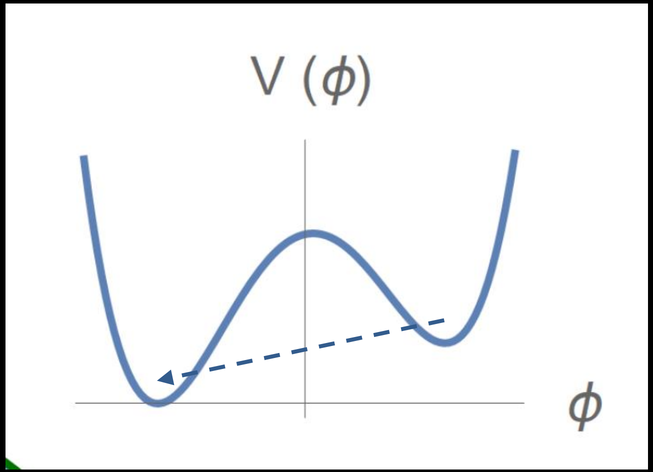
$$V(\phi) = \mu^3\phi - \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4$$

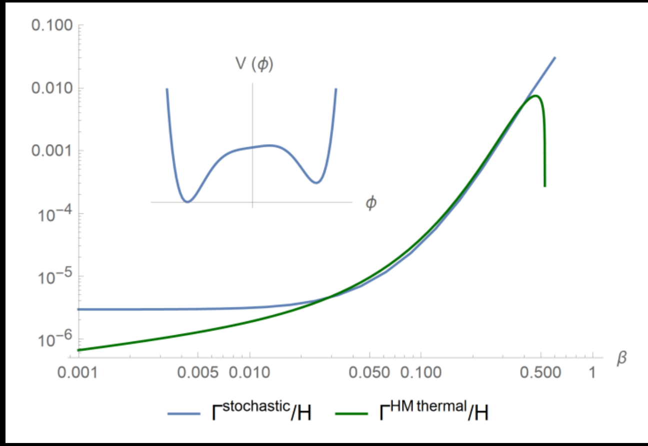
- Camargo-Molina and AR 2022:
  Transition rate per time given by the lowest non-zero eigenvalue

$$\Gamma = \Lambda_1$$

- Corresponds to a Hawking-Moss transition
  - Instanton result (Camargo-Molina,Carrillo Gonzalez & AR 2022):

$$\Gamma_{\mathrm{HM}} = \frac{\kappa}{2\pi}\left|\frac{\det S''(\phi_{\mathrm{top}})}{\det S''(\phi_{\mathrm{FV}})}\right|^{-1/2} e^{-\frac{8\pi^2\Delta V}{3H^4}}$$





(Camaargo-Molina et al 2022)

Quantitative Comparison

- When the saddle-point approximation is valid, the stochastic and instanton results agree at one loop if the potential in the Langevin equation is the **constraint effective potential**,

$$V_{\text{eff}}(\bar{\phi}) \equiv -\frac{1}{\mathcal{V}}\ln\int d\phi\,\delta\left(\frac{1}{\mathcal{V}}\int d^4x\phi-\bar{\phi}\right)e^{-S[\phi]}$$

(Note: different effective potentials are equivalent in Minkowski, but not in de Sitter!)

- Is that generally the case, i.e., do we actually have $\dot{\bar{\phi}}+\frac{1}{3H}V'_{\text{eff}}(\bar{\phi})=\xi$?

THE CONSTRAINT EFFECTIVE POTENTIAL

L. O'RAIFEARTAIGH, A. WIPF and H. YONEYAMA

School of Theoretical Physics, Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland

Received 19 December 1985

Beyond Free and Overdamped Approximations

- What if $m \sim H$?
- Cable&AR 2021: Second-order stochastic system:

$$\dot{\phi} = \pi + \xi_\phi$$
$$\dot{\pi} = -3H\pi - V'(\phi) + \xi_\pi$$

- Noise matrix

$$\begin{pmatrix} \langle \xi_\phi(t)\xi_\phi(t')\rangle & \langle \xi_\phi(t)\xi_\pi(t')\rangle \\ \langle \xi_\pi(t)\xi_\phi(t')\rangle & \langle \xi_\pi(t)\xi_\pi(t')\rangle \end{pmatrix} = \begin{pmatrix} \sigma^2_{\phi\phi} & \sigma^2_{\phi\pi} \\ \sigma^2_{\pi\phi} & \sigma^2_{\pi\pi} \end{pmatrix} \delta(t-t')$$

- Matching:
  - Compute correlation functions perturbatively in both QFT and stochastic theory
  - Determine stochastic parameters by demansing that they agree

Parameters of the Stochastic Theory

- For $m_R^2 \ll H^2$, we obtain (Cable and Rajantie 2023)

$$m_S^2 = m_R^2(M) + \lambda \frac{3H^2}{4\pi^2}\left(\gamma_E + \ln\frac{M}{2aH}\right) + \mathcal{O}\left(\lambda m_R^2\right)$$
$$\lambda_S = \lambda_R + O\left(\lambda^2 \frac{H^2}{m_R^2}\right)$$
$$\sigma_{\phi\phi}^2 = \frac{H^3}{4\pi^2} + \lambda \frac{H^5(3\ln 4 - 8)}{32\pi^4 m_R^2} + O\left(\lambda H^3\right)$$
$$\sigma_{\pi\phi}^2 = \sigma_{\pi\pi}^2 = 0 + O(\lambda^2)$$

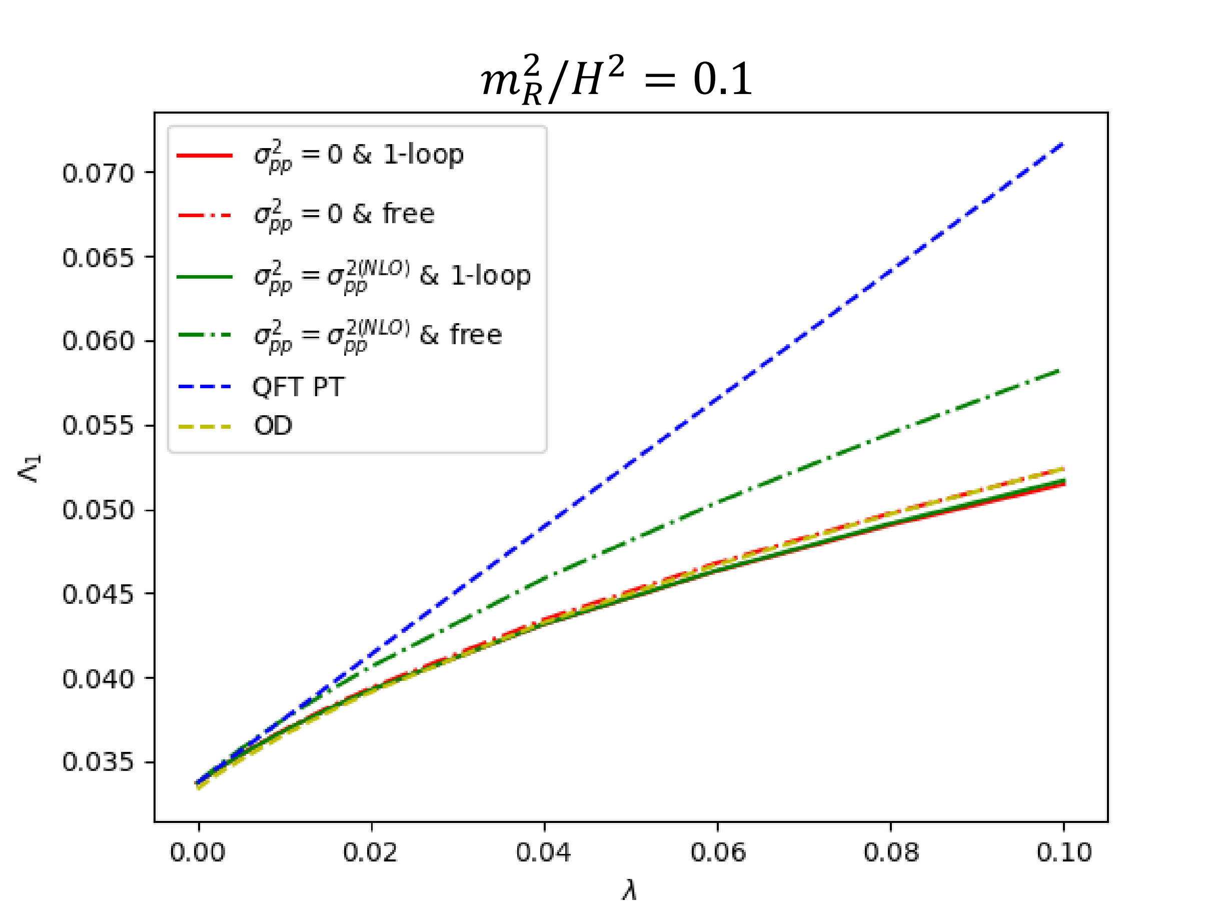
- Notes:
  - Dependence on the renormalisation scale M cancels, as it must
  - Expansion parameter $\lambda H^2/m^2$, not $\lambda H^4/m^4$ as when computing observables

Spectral Index

- Exponent of the asymptotic correlator

$$\langle \phi(t,\vec{r})\phi(t,\vec{r}')\rangle \sim (H|\vec{x}-\vec{x}'|)^{-2\Lambda_1}$$

- Relative error:
  - One-loop QFT: $\mathcal{O}\left(\lambda \frac{H^4}{m^4}\right)^2$
  - 1st-order Stochastic: $\mathcal{O}\left(\frac{m^2}{H^2}\right), \mathcal{O}(\lambda \frac{H^2}{m^2})$
  - 2nd-order Stochastic: $\mathcal{O}\left(\lambda \frac{H^2}{m^2}\right)^2$



Summary

- Stochastic effective theory: Powerful non-perturbative approach to scalar fields in de Sitter
- Asymptotic long-distance correlators from spectral expansion:
  - Isocurvature constraints for dark matter models
  - Vacuum transition rate
- Starobinsky-Yokoyama theory:
  - 1st order stochastic system: Requires $m \ll H$
  - Parameters determined at tree level
- 2nd order stochastic theory:
  - Parameters determined by matching long-distance correlators at one loop
  - Relative error $\mathcal{O}\left(\lambda \frac{H^2}{m^2}\right)^2$