Confinement in Schwinger model at finite temperature and θ

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H. Ohata,
"Monte Carlo study of Schwinger model without the sign problem,"
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Outline

- Introduction
- Lattice bosonized Schwinger model
  –methodology
- Confinement at finite temperature and θ
  –result
- Summary and outlook

Introduction

Strong interaction is described by QCD:

$$S_{\rm QCD} = \int d^4x \, \frac{1}{2} \mathrm{tr}\big(G_{\mu\nu} G_{\mu\nu}\big) + \sum_f \overline{q}_f \big(\gamma_\mu D_\mu + m_f\big) q_f$$

Gauge principle and renormalizability almost uniquely determine the action.

The only possible extension to QCD is the θ term :

$$i\theta Q, \quad Q = \int d^4x \, \frac{g^2}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathrm{tr}\big(G_{\mu\nu} G_{\rho\sigma}\big)$$

Unlike $S_{\rm QCD}$, Q is antisymmetric under the CP transformation.
$\Longrightarrow$
The real parameter θ determines the degree of CP symmetry breaking in QCD.
The CP symmetry is strictly preserved in QCD for some reason:

$$|\theta| \lesssim 3 \times 10^{-12}$$ Abel et al., '20

Interesting properties of the θ term

$$i\theta Q, \quad Q = \int d^4x \, \frac{g^2}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathrm{tr}\left(G_{\mu\nu} G_{\rho\sigma}\right)$$

- θ term is total derivative $\Longrightarrow$ purely quantum effects
- Q takes an integer $\Longrightarrow$ θ has 2π periodicity
  $\theta = \pi$ is the most distinct point from our vacuum $\theta = 0$.
- CP transformation: $\theta \to -\theta$
  $\Longrightarrow$ CP symmetry is not explicitly broken at $\theta = \pi$.

θ term is interesting, but...

By examining the quantum effects of the θ term, we can gain insights into the quantum aspects of the system and deepen the understanding of our vacuum $\theta = 0$.
However, investigating the effects of the θ term is difficult by the Monte Carlo method due to the sign problem.

Monte Carlo method and sign problem

$$\langle \hat{O} \rangle_T = \mathrm{tr}\big(\hat{O}\exp(-\hat{H}/T)\big) \big/ \mathrm{tr}\big(\exp(-\hat{H}/T)\big)$$
$$= \text{(Eliminate operator by inserting completeness relations)}$$
$$= \int D\phi\, O \exp(-S_E) \bigg/ \int D\phi \exp(-S_E)$$

An imaginary term like $i\pi\partial_\tau\phi$ appears in the exponential. In the case of the scalar theory

$$-i\pi\partial_\tau\phi + \frac{1}{2}\pi^2 = \underbrace{\frac{1}{2}(\pi - i\partial_\tau\phi)^2}_{\text{integrated out}} + \frac{1}{2}(\partial_\tau\phi)^2 \quad \Longrightarrow \quad S_E \in \mathbb{R}$$

The expectation value can be approximated by sampling configurations with the probability $\exp(-S_E)$.

Sign problem

If S_E has an imaginary part (e.g., at finite θ), $\exp(-S_E)$ cannot be a probability. Monte Carlo method is not applicable.

Purpose of this talk

Purpose of this talk

Investigate the confining properties of the Schwinger model (QED in 1 + 1 dims) at finite temperature and θ using the Monte Carlo method.

$$S_E[A_\mu, \psi, \overline{\psi}]_{g,m,\theta} = \int d^2x \left[\frac{1}{4} F_{\mu\nu}F_{\mu\nu} + \overline{\psi}(\not{\partial} + g\not{A} + m)\psi \right] + i\theta Q,$$

$$Q := \int d^2x \frac{g}{4\pi}\epsilon_{\mu\nu}F_{\mu\nu} = \int d^2x \frac{g}{2\pi} E \in \mathbb{Z}.$$

Similarities with QCD

- Chiral anomaly: $\partial_\mu j_5^\mu = \frac{g}{\pi} E$
- Confinement (discussed later)
- θ term → sign problem

Properties originating from low dimensionality

- Gauge coupling has mass dimension.
- Equivalent bosonized form (We exploit this feature)

Lattice bosonized Schwinger model

Bosonization Coleman, '75, Mandelstam, '75

Hamiltonian of the Schwinger model

$$H = \int dx \left[-i\psi^\dagger(\partial_x - igA^1)\gamma^5\psi + \frac{g^2}{2}\left(\frac{E}{g} + \frac{\theta}{2\pi}\right)^2 + m\bar{\psi}\psi \right]$$

Gauss law: $\partial_x E/g = \psi^\dagger\psi$

The Dirac fermion in 1 + 1 dimensions can be described by the scalar field:

$$-\bar{\psi}i\partial_x\gamma^1\psi = \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x\phi)^2,$$

$$\bar{\psi}\psi = -\frac{e^\gamma}{2\pi}\mu\mathcal{N}_\mu \cos(2\sqrt{\pi}\phi), \quad \mu : \text{regularization scale}$$

After bosonization, the Gauss law can be solved locally

$$\partial_x E/g = \psi^\dagger\psi = \partial_x\phi/\sqrt{\pi}$$

$$\Longrightarrow$$

$$E/g = \phi/\sqrt{\pi}$$

Bosonized Schwinger model

Bosonized Schwinger model Coleman, Jackiw, and Susskind, '75

$$H = \int dx \, \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x \phi) + \frac{g^2}{2\pi}\left(\phi + \frac{\theta}{2\sqrt{\pi}}\right)^2 - \frac{e^\gamma mg}{2\pi^{3/2}} \mathcal{N}_{g/\sqrt{\pi}} \cos(2\sqrt{\pi}\phi), \quad \phi/\sqrt{\pi} = E/g$$

$\mathcal{N}_\mu$ denotes normal ordering with respect to the creation and annihilation operators defined as

$$\phi(x) =: \int_{-\infty}^{\infty} \frac{dk}{2\pi}\left[\frac{1}{2\sqrt{k^2+\mu^2}}\right]^{1/2} \left(a(k,\mu)e^{-ikx} + a^\dagger(k,\mu)e^{ikx}\right),$$

$$\pi(x) =: -i\int_{-\infty}^{\infty} \frac{dk}{2\pi}\left[\frac{\sqrt{k^2+\mu^2}}{2}\right]^{1/2} \left(a(k,\mu)e^{-ikx} - a^\dagger(k,\mu)e^{ikx}\right)$$

For the path-integral rep. at $m \neq 0$, $\mathcal{N}_\mu$ must be removed.

Removing normal ordering Coleman, '75; Bender et al., '85

Using Wick's theorem,

$$\mathcal{N}_\mu \exp\left(i2\sqrt{\pi}\phi\right) = \exp(2\pi\Delta(x=0;\mu)) \exp\left(i2\sqrt{\pi}\phi\right)$$

$\Delta(x;\mu)$: Feynman propagator for the scalar field of mass μ

Continuum

Feynman propagator regularized with a UV cutoff Λ:

$$\Delta(x;\mu;\Lambda) := \Delta(x;\mu) - \Delta(x;\Lambda) = \frac{1}{2\pi}\ln\frac{\Lambda}{\mu} + \mathcal{O}(x^2)$$

$$\Longrightarrow \mathcal{N}_\mu \exp\left(i2\sqrt{\pi}\phi(x)\right) = (\Lambda/\mu) \exp\left(2\sqrt{\pi}\phi(x)\right)$$

Lattice

$$\mathcal{N}_{g/\sqrt{\pi}} \exp\left(i2\sqrt{\pi}\phi_x\right) = C(ag) \exp\left(i2\sqrt{\pi}\phi_x\right)$$

$$C(ag) := \exp\left\{2\pi\Delta_{\mathrm{latt}}\left(0; g/\sqrt{\pi}; 1/a\right)\right\} \simeq 10/ag$$

Lattice bosonized Schwinger model

Euclidean action of the lattice bosonized Schwinger model

$$S_E = \sum_{\tau=0}^{L_\tau-1}\sum_{x=0}^{L_x-1} \frac{1}{2}(\partial_\tau \phi_{x,\tau})^2 + \frac{1}{2}(\partial_x \phi_{x,\tau})^2 + \frac{(ag)^2}{2\pi}\left(\phi_{x,\tau} + \frac{\theta}{2\sqrt{\pi}}\right)^2$$

$$- \frac{e^\gamma}{2\pi^{3/2}}\frac{m}{g}(ag)^2 C(ag)(\simeq 10/ag) \cos\left(2\sqrt{\pi}\phi_{x,\tau}\right) \in \mathbb{R}$$

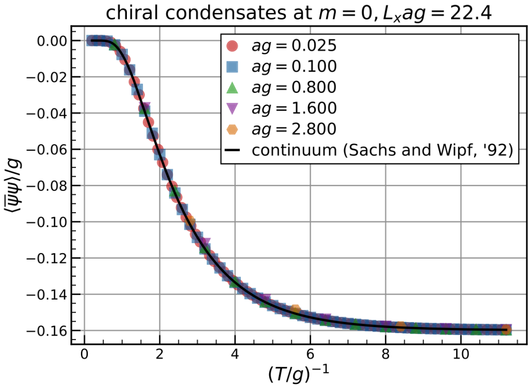
Advantages

- Monte Carlo simulation without encountering the sign problem
- Low-cost Configuration generation using the heat-bath algorithm
- Chiral anomaly is exactly preserved on a lattice.
- First convergence to the continuum limit (next slide)

Analytical chiral condensate at $m = 0$ Ohata, '23

$$\left\langle \overline{\psi}\psi \right\rangle_{\text{latt}} = -\frac{e^{\gamma}}{2\pi^{3/2}} g C(ag) \left\langle \cos(2\sqrt{\pi}\phi) \right\rangle_{\text{free},L_x,L_\tau}$$

$$= -\frac{e^{\gamma}}{2\pi^{3/2}} g \exp\left[-2\pi\left\{\Delta_{\text{latt}}(ag)_{L_x,L_\tau} - \Delta_{\text{latt}}(ag)\right\}\right].$$



VEV of the chiral condensate is reproduced at any ag.
$\Longleftrightarrow$ Chiral anomaly is exactly preserved on a lattice.
Fast convergence to the continuum limit even at $T \neq 0$

Confinement at finite temperature and θ

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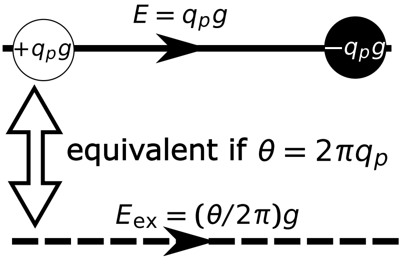
$$H = \int dx \, \frac{g^2}{2}\left(\frac{E}{g} + \frac{\theta}{2\pi}\right)^2 + \cdots$$

θ is an external electric field:

$$\frac{E_{\text{ex}}}{g} = \frac{\theta}{2\pi}$$



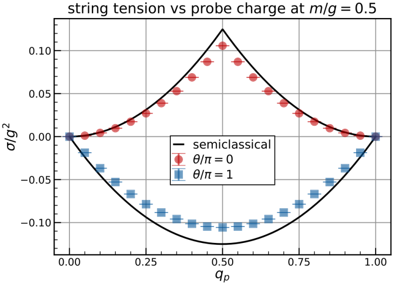
String tension can be obtained from the difference in the free energy densities (after taking $L_x a \to \infty$)

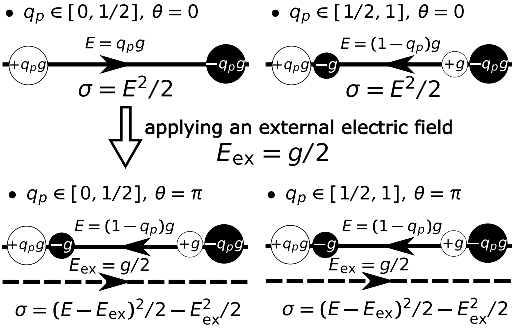
$$\sigma(q_p, \theta) = f(2\pi q_p + \theta) - f(\theta)$$
$$= \frac{-1}{L_x L_\tau a^2} \ln \left\langle \exp\left[-\frac{(ag)^2}{\sqrt{\pi}} \sum_{x,\tau} q_p \left(\phi_{x,\tau} + \frac{\theta + \pi q_p}{2\sqrt{\pi}}\right)\right]\right\rangle_\theta$$

Integer charge is screened by the creation of a dynamical charge pair due to the 2π periodicity of θ.

String tension of probe charge $q_p \in [0, 1]$

$ag = 0.2,\ L_x \times L_\tau = 112 \times 56$





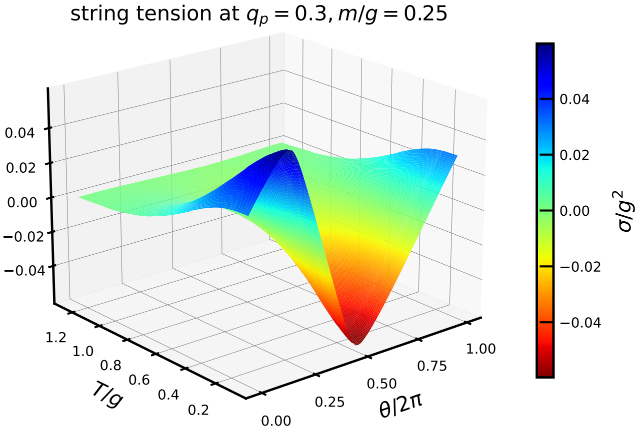
Semiclassical estimate:

$$\sigma/g^2 = \begin{cases} \frac{1}{2} q_p^2, & q_p \in \left[0, \frac{1}{2}\right],\ \theta = 0, \\ \frac{1}{2}(1 - q_p)^2, & q_p \in \left[\frac{1}{2}, 1\right],\ \theta = 0, \\ -\frac{1}{2} q_p (1 - q_p), & q_p \in [0, 1],\ \theta = \pi. \end{cases}$$

See Misumi et al., ’19; Honda et al., ’22 for explanation of negative string tension for integer probe charge in the charge-N (N > 1) Schwinger models in terms of the Z_N 1-form symmetry

String tension σ in (T, θ) plane

Using reweighing method, (T, θ) plane can be covered densely



- At low temperature, as $\theta = 0 \to \pi$,
  confining→deconfining→inversely confining
- At high temperature, always deconfining

Summary and outlook

Summary and outlook

Summary

- The Schwinger model with the θ term can be simulated by the Monte Carlo method using bosonization.
- The lattice bosonized Schwinger model was verified through the comparison with previous results.
- Confining properties in the Schwinger model at finite temperature and θ were quantitatively revealed.

Please refer to H. Ohata, PTEP 2024, 013B02 (2024), arXiv:2311.04738 for the determination of the phase diagram at $\theta = \pi$ in the temperature and fermion mass plane.

Outlook

- Application to finite density system
- Application to more nontrivial models

Backup

Previous studies

The Schwinger model at finite θ has been studied mainly by three methods.

- Tensor network in the Hamiltonian formalism
  Apply the tensor network method to the 2^{L_x} dimensional spin Hamiltonian formulation of the Schwinger model
  Byrnes et al., 02; Banuls et al., '13; Buyens et al., '14; many other works

- Quantum algorithm (mostly in the Hamiltonian formalism)
  Apply the quantum algorithm to the spin Hamiltonian
  For example, obtain the ground state via adiabatic state preparation
  Kuhn et al., 14; Chakraborty et al., '22; Honda et al., '22; many other works

- Tensor network in the path-integral formalism
  Evaluate the path-integral deterministically using the Grassmann tensor renormalization group
  Shimizu and Kuramashi, '14; Shimizu and Kuramashi, '14; Akiyama et al., '24

Spin representation of Schwinger model

Hamiltonian of the Schwinger model

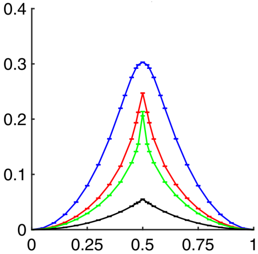
$$H = \int dx \left[-i\psi^\dagger(\partial_x - igA^1)\gamma^5\psi + \frac{g^2}{2}\left(\frac{E}{g} + \frac{\theta}{2\pi}\right)^2 + m\bar{\psi}\psi \right]$$

Gauss law: $\partial_x E/g = \psi^\dagger\psi$

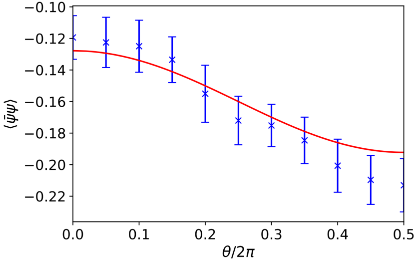
staggered fermion + Gauss law + Jordan-Wiger transformation

$\Longrightarrow$ 2^{L_x} dimensional spin Hamiltonian

Tensor Network,
$m/g = 0.125, 0.25, 0.3, 0.5$
Buyens et al., '17



Quantum computing, $m/g = 0.1$
Chakraborty et al., '22



The origin of sign problem at finite θ

Hamiltonian of the Schwinger model

$$H = \int dx \left[-i\psi^\dagger (\partial_x - igA^1)\gamma^5 \psi + \frac{1}{2}\left(E + \frac{g\theta}{2\pi}\right)^2 + m\bar{\psi}\psi \right]$$

Gauss law: $\partial_x E/g = \psi^\dagger \psi$

$$S_E = \int d^2x - iE\partial_\tau A^1 + \frac{1}{2}\left(E + \frac{g\theta}{2\pi}\right)^2 + \cdots$$

$$= \int d^2x \underbrace{\frac{1}{2}\left\{E + \left(\frac{g\theta}{2\pi} - i\partial_\tau A^1\right)\right\}^2}_{\text{integrated out}} + \frac{1}{2}\left(\partial_\tau A^1 + i\frac{g\theta}{2\pi}\right)^2 + \cdots$$

$$= \int d^2x \frac{1}{2}\left(\partial_\tau A^1\right)^2 + \underbrace{i\frac{g\theta}{2\pi}\partial_\tau A^1}_{\text{purely imaginary}} + \cdots$$

After bosonization, the Gauss law can be solved locally, and S_E remains real.

Generating Monte Carlo configurations

Heat bath algorithm

Start with an initial field configuration $\{\phi_{x,\tau}\}$

1. focus on $\phi_{x,\tau}$ at some site (x,τ)
2. update $\phi_{x,\tau}$ while fixing the rest (heat bath)
3. repeat 1 and 2 for all sites

Repeating the sweep many times, the field configuration $\{\phi_{x,\tau}\}$ starts to distribute with $P(\{\phi_{x,\tau}\}) \propto \exp(-S_E(\{\phi_{x,\tau}\}))$.

$$P(\phi_{x,\tau}) \propto \exp\left\{-2I(ag)\left(\phi_{x,\tau} - \frac{\overline{\phi}_{x,\tau}}{I(ag)}\right)^2\right\}$$

$$\times \exp\left\{\frac{e^\gamma}{2\pi^{3/2}}(m/g)(ag)^2 C(ag)\cos\left(2\sqrt{\pi}\phi_{x,\tau} - \theta\right)\right\},$$

$\overline{\phi}_{x,\tau} := (\phi_{x,\tau+1} + \phi_{x,\tau-1} + \phi_{x+1,\tau} + \phi_{x-1,\tau})/4, I(ag) := 1 + (ag)^2/4\pi.$

Generate a Gaussian random number, apply the rejection sampling

Chiral anomaly in the bosonized form

Bosonized form of the chiral current:

$$j_5^\mu = \overline{\psi}\gamma^5\gamma^\mu\psi = \begin{cases} \frac{1}{\sqrt{\pi}}\pi, & \mu = 0, \\ -\frac{1}{\sqrt{\pi}}\partial_x\phi, & \mu = 1. \end{cases}$$

Bosonized form of the Schwinger model at $m = \theta = 0$

$$H = \int dx \, \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x\phi)^2 + \frac{g^2}{2\pi}\phi^2.$$

Time evolution

$$\dot{\pi} = -\frac{\delta H}{\delta\phi} = \partial_x^2\phi - \frac{g^2}{\pi}\phi,$$

The conservation law of the chiral current is broken as

$$\partial_\mu j_5^\mu = \frac{1}{\sqrt{\pi}}\left(\dot{\pi} - \partial_x^2\phi\right) = \frac{g}{\pi}E.$$

This relation holds also on a lattice with no $\mathcal{O}(a)$ correction.

Chiral symmetry in Schwinger model

At $m = 0$, the action has the chiral symmetry

$$U(1)_\mathrm{V} \times U(1)_\mathrm{A} \times SU(N_f)_\mathrm{V} \times SU(N_f)_\mathrm{A}.$$

$U(1)_\mathrm{A}$ is explicitly broken by the chiral anomaly.

Spontaneous continuous symmetry breaking is prohibited in $1+1$ dims model (except the Higgs mechanism). Coleman, '73

- $N_f \geq 2$
  $\left\langle \overline{\psi}\psi \right\rangle \neq 0 \Longrightarrow$ spontaneous $SU(N_f)_\mathrm{A}$ symmetry breaking, which contradicts Coleman's theorem.
- $N_f = 1$
  We don't have $SU(N_f)_\mathrm{A}$ symmetry from the beginning.
  $\Longrightarrow$
  $\left\langle \overline{\psi}\psi \right\rangle \neq$ does not contradict Coleman's theorem.