

# Restoration of residual gauge symmetries due to topological defects and color confinement in the Lorenz, Maximal Abelian gauges

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- In the Lorenz gauge, KO criterion is also derived as **the condition of the restoration of the residual gauge symmetry** with the special choice of the gauge transformation function  $\omega(x)$ . (Hata(1982))
- What is the residual gauge symmetry?

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- In this talk, we reconsider the condition of the restoration of the residual local gauge symmetry by using finite gauge transformation.

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- The restoration condition of the residual local gauge symmetry in the Lorenz gauge
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- Summary

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- The Lagrangian density in the Lorenz gauge is given by

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{\text{inv}} + \mathcal{L}_{\text{GF+FP}}, \\ \mathcal{L}_{\text{inv}} &= -\frac{1}{4} \mathcal{F}_{\mu\nu}^A \mathcal{F}^{\mu\nu A} + \mathcal{L}_{\text{matter}}(\psi, D_\mu \psi), \\ \mathcal{L}_{\text{GF+FP}} &= -i\delta_B \left\{ \bar{\mathcal{C}}^A \left( \partial^\mu \mathcal{A}_\mu^A + \frac{\alpha}{2} \mathcal{B}^A \right) \right\}. \end{aligned} \quad (3)$$

# The restoration condition of the residual local gauge symmetry in the Lorenz gauge

- We consider the **finite** local gauge transformation  $U(x)$

$$\begin{aligned}\delta_U \mathcal{A}_\mu(x) &= \Omega_\mu(x) + U(x) \mathcal{A}_\mu(x) U^\dagger(x) - \mathcal{A}_\mu(x), \quad \boxed{\Omega_\mu(x) := ig^{-1} U(x) \partial_\mu U^\dagger(x)}, \\ \delta_U \mathcal{B}(x) &= U(x) \mathcal{B}(x) U^\dagger(x) - \mathcal{B}(x), \quad \delta_U \mathcal{C}(x) = U(x) \mathcal{C}(x) U^\dagger(x) - \mathcal{C}(x), \quad \delta_U \bar{\mathcal{C}}(x) = U(x) \bar{\mathcal{C}}(x) U^\dagger(x) - \bar{\mathcal{C}}(x).\end{aligned}\tag{4}$$

For  $U(x)$  to be a residual symmetry for the Lorenz gauge,  $U(x)$  should satisfy

$$\mathcal{D}^\mu \tilde{\Omega}_\mu = \partial^\mu \tilde{\Omega}_\mu(x) - ig U[\mathcal{A}^\mu, \tilde{\Omega}_\mu] U^\dagger = 0, \quad (\tilde{\Omega}_\mu(x) := -ig^{-1} U^\dagger(x) \partial_\mu U(x)).\tag{5}$$

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- We can calculate the Noether current  $\mathcal{J}_U^\mu$  associated with (4) and the divergence of this current is equal to the transformation of the Lagrangian density under the transformation (4)

$$\begin{aligned} \partial_\mu \mathcal{J}_U^\mu &= \delta_U \mathcal{L} = \delta_U \mathcal{L}_{\text{GF+FP}} = -i \delta_U \delta_B \left[ \text{tr} \left\{ \bar{\mathcal{C}} \left( \partial^\mu \mathcal{A}_\mu + \frac{\alpha}{2} \mathcal{B} \right) \right\} \right] = -i \delta_B \delta_U \left[ \text{tr} \left\{ \bar{\mathcal{C}} \left( \partial^\mu \mathcal{A}_\mu + \frac{\alpha}{2} \mathcal{B} \right) \right\} \right] \\ &= -i \delta_B [\text{tr} \{ \mathcal{D}_\mu \bar{\mathcal{C}} ig^{-1} U^\dagger \partial^\mu U \}] = i \delta_B [(\mathcal{D}_\mu \bar{\mathcal{C}})^A \tilde{\Omega}^{\mu A}]. \end{aligned} \quad (6)$$



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→ The local gauge current  $\mathcal{J}_U^\mu$  is conserved in the physical state space. ((6) is BRST transformation  $\delta_B$  exact)

# The restoration condition of the residual local gauge symmetry in the Lorenz gauge

- The restoration condition of the residual symmetry is written as the condition of **the disappearance of the massless pole**. In the one-gluon sector  $\Phi = \mathcal{A}_\nu^B(y)$ , we focus on the following Ward-Takahashi (WT) identity

$$i\partial_\mu^x \langle \mathcal{J}_U^\mu(x)\Phi \rangle = \delta^D(x-y) \langle \delta^U \mathcal{A}_\nu^A(x) \rangle + i \langle \partial_\mu \mathcal{J}_U^\mu(x) \mathcal{A}_\nu^A(y) \rangle. \quad (7)$$

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- Performing Fourier transformation, we obtain

$$\int d^D x e^{ip(x-y)} \partial_\mu^x \langle \mathcal{J}_U^\mu(x)\mathcal{A}_\nu^A(y) \rangle = \langle \delta^U \mathcal{A}_\nu^A(y) \rangle + i \int d^D x e^{ip(x-y)} \langle \delta^U \mathcal{L}(x)\mathcal{A}_\nu^A(y) \rangle. \quad (8)$$

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- From (6), the second term of (8) is reduced to

$$\begin{aligned} & i \int d^D x e^{ip(x-y)} \langle \delta^U \mathcal{L}(x) \mathcal{A}_\nu^B(y) \rangle \\ &= - \int d^D x e^{ip(x-y)} \tilde{\Omega}^{\mu A}(x) \langle \delta(\mathcal{D}_\mu \bar{\mathcal{C}})^A(x) \mathcal{A}_\nu^B(y) \rangle = - \int d^D x e^{ip(x-y)} \tilde{\Omega}^{\mu A}(x) \langle (\mathcal{D}_\mu \bar{\mathcal{C}})^A(x) \delta \mathcal{A}_\nu^B(y) \rangle \\ &= - \int d^D x e^{ip(x-y)} \tilde{\Omega}^{\mu A}(x) \left[ \frac{\partial_\mu^x \partial_\nu^x}{\partial_x^2} \delta^D(x-y) + \left( g_{\mu\rho} - \frac{\partial_\mu^x \partial_\rho^x}{\partial_x^2} \right) \langle g(\mathcal{A}^\rho \times \bar{\mathcal{C}})^A(x) (\mathcal{D}_\nu \mathcal{C})^B(y) \rangle \right], \end{aligned} \quad (9)$$

where we have used  $\langle \delta_B[\dots] \rangle = 0$  and Schwinger-Dyson equation  $\int d\mu \frac{\delta}{\delta\Phi(x)} e^{iS} F = 0$ .

# The restoration condition of the residual local gauge symmetry in the Lorenz gauge

- We obtain the restoration condition of the residual local gauge symmetry in the one-gluon sector  $\mathcal{A}_\nu^B(y)$

$$I_\nu^B = \lim_{p \rightarrow 0} \int d^D x e^{ip(x-y)} \left\{ \left( \Omega_\nu^B(x) - \tilde{\Omega}^{\mu B}(x) \frac{\partial_\mu^x \partial_\nu^x}{\partial_x^2} \right) \delta^D(x-y) + \tilde{\Omega}^{\mu A}(x) \left( g_{\mu\nu} - \frac{\partial_\mu^x \partial_\nu^x}{\partial_x^2} \right) u^{AB}(x-y) \right\}$$

$$\begin{cases} = 0 & \text{restoration} \\ \neq 0 & \text{no restoration} \end{cases} \quad (10)$$

where we have defined the Kugo-Ojima (KO) function  $u^{AB}$  in the configuration space

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- In the non-compact gauge theory, it is enough to consider the infinitesimal gauge transformation and residual symmetry  $\Omega^{\mu A}(x) = \tilde{\Omega}^{\mu A}(x) = \delta^{AC} b^\mu$  ( $C$  is arbitrary component of the Lie algebra).

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- (10) is reduced to Kugo-Ojima criterion as shown by Hata(1982)

$$0 = \lim_{p^2 \rightarrow 0} [\delta^{AB} + u^{AB}(p^2)]. \quad (12)$$

# The restoration condition of the residual local gauge symmetry in the Lorenz gauge

- We restrict the topological configurations satisfying

$$\begin{aligned} 0 &= \lim_{p \rightarrow 0} \int d^D p e^{ipx} [\Omega^{\mu B}(x) - \tilde{\Omega}^{\mu B}(x)] \frac{\partial_\mu^x \partial_\nu^x}{\partial_x^2} \delta^D(x-y) \\ &= \lim_{p \rightarrow 0} \int \frac{d^D k}{(2\pi)^2} e^{-i(p-k)y} [\Omega^{\mu B}(p-k) - \tilde{\Omega}^{\mu B}(p-k)] \frac{k_\mu k_\nu}{k^2}. \end{aligned} \quad (13)$$



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- Unlike the KO criterion (12),  $\delta$  and  $u$  have different factors,  $\Omega^\mu$  and  $\tilde{\Omega}^\mu$ .

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- We restrict the topological configurations satisfying

$$\begin{aligned} 0 &= \lim_{p \rightarrow 0} \int d^D p e^{ipx} [\Omega^{\mu B}(x) - \tilde{\Omega}^{\mu B}(x)] \frac{\partial_\mu^x \partial_\nu^x}{\partial_x^2} \delta^D(x-y) \\ &= \lim_{p \rightarrow 0} \int \frac{d^D k}{(2\pi)^2} e^{-i(p-k)y} [\Omega^{\mu B}(p-k) - \tilde{\Omega}^{\mu B}(p-k)] \frac{k_\mu k_\nu}{k^2}. \end{aligned} \quad (13)$$

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- This topological configuration is the residual symmetry.
- If  $h(x^2) \equiv 1$ , (15) includes  $D = 2$ : Abrikosov-Nielsen-Olesen vortex,  $D = 3$ : Wu-Yang monopole,  $D = 4$ : Alfaro-Fubini-Furlan meron but **the Euclidean action is divergent**.

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→ In order for such configurations to give a finite Euclidean action and to contribute to the path integral,  $h(x^2) \rightarrow 0$  moderately as  $x \rightarrow 0$

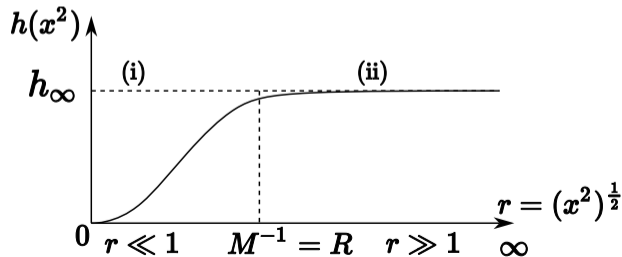
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- To obtain the finite  $S_E$ ,  $h(x^2)$  must approach 0 of order  $r^{2\delta}$  ( $r = (x^2)^{\frac{1}{2}}$ ,  $\delta > 1/4$ ) in  $r \ll 1$  and it must approach  $h_\infty$  (a finite value) rapidly in  $r \gg 1$ . ( $D = 2$ : Kondo(2018),  $D = 3$ : Nishino et al. (2018))



# The restoration condition of the residual local gauge symmetry in the Lorenz gauge

- We put  $q := p - k$

$$\begin{aligned} \Omega_\mu^B(q) &:= \int d^D x e^{iqx} \Omega_\mu^B(x), \quad \tilde{\Omega}_\mu^B(q) := \int d^D x e^{iqx} \tilde{\Omega}_\mu^B(x) \\ \lim_{p \rightarrow 0} \int \frac{d^D k}{(2\pi)^D} &\left( \Omega^{\mu B}(q) g_{\mu\nu} - \tilde{\Omega}^{\mu B}(q) \frac{k_\mu k_\nu}{k^2} \right). \end{aligned} \quad (17)$$

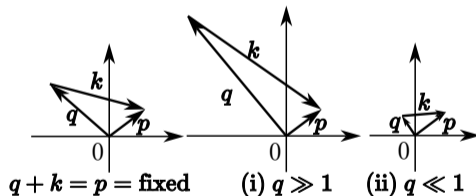


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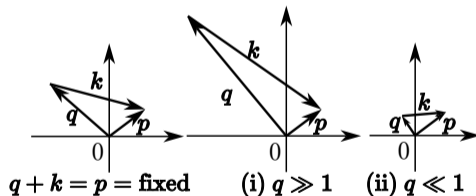


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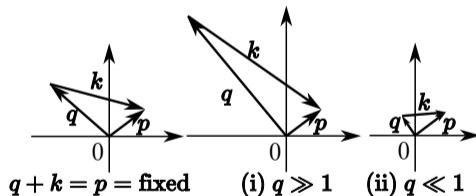
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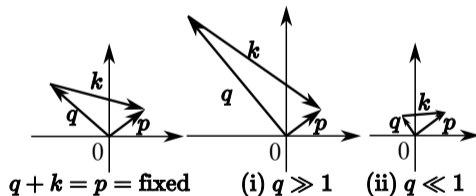
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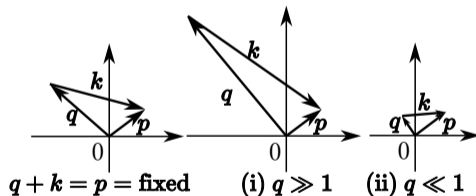
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→  $\delta$  play the role of **avoiding the UV divergence** ( $\delta > 1/4$ ).

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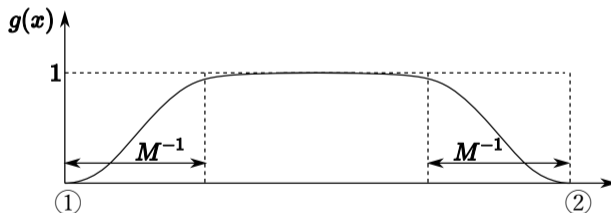
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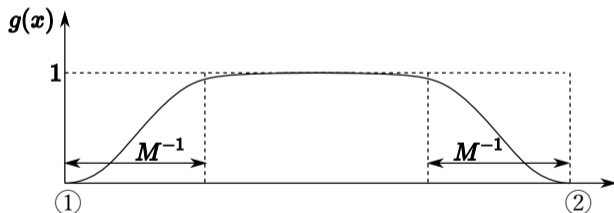


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- If we introduce a topological configuration with multiple defects, these results are consistent with the confinement picture where the vacuum **condensation of topological defects** leads to confinement.

# The restoration condition of the residual local gauge symmetry in the Maximal Abelian gauge

- We decompose the Lie-algebra valued quantity to the diagonal Cartan part and remaining off-diagonal part.
- The gauge field  $\mathcal{A}_\mu(x) = \mathcal{A}_\mu^A T_A$  with the generators  $T_A$  ( $A = 1, \dots, N^2 - 1$ ) of the Lie algebra  $su(N)$  has the decomposition:

$$\mathcal{A}_\mu(x) = \mathcal{A}_\mu^A(x) T_A = a_\mu^j(x) H_j + A_\mu^a(x) T_a, \quad (23)$$

where  $H_j$  are the Cartan generators and  $T_a$  are the remaining generators of the Lie algebra.

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- The gauge field  $\mathcal{A}_\mu(x) = \mathcal{A}_\mu^A T_A$  with the generators  $T_A$  ( $A = 1, \dots, N^2 - 1$ ) of the Lie algebra  $su(N)$  has the decomposition:

$$\mathcal{A}_\mu(x) = \mathcal{A}_\mu^A(x) T_A = a_\mu^j(x) H_j + A_\mu^a(x) T_a, \quad (23)$$

where  $H_j$  are the Cartan generators and  $T_a$  are the remaining generators of the Lie algebra.

- We choose **the Maximal Abelian (MA) gauge**

$$(\mathcal{D}^\mu[a]A_\mu(x))^a := \partial^\mu A_\mu^a + gf^{ajb} a^{\mu j} A_\mu^b(x) = 0. \quad (24)$$

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- The GF+FP term of the Lagrangian in the MA gauge is given as below

$$\mathcal{L}_{\text{GF+FP}} = -i\delta_B \left\{ \bar{C}^a \left( (\mathcal{D}^\mu[a]A_\mu)^a + \frac{\alpha}{2} B^a \right) + \bar{c}^j (\partial^\mu a_\mu^j + \frac{\alpha}{2} b^j) \right\}. \quad (26)$$

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- Assuming Abelian dominance, we can calculate the divergence of the Noether current  $\mathcal{J}_U^\mu$  as in the Lorenz gauge:

$$\partial_\mu \mathcal{J}_U^\mu = \delta_U \mathcal{L} = \delta_U \mathcal{L}_{\text{GF+FP}} = -i\delta_U \delta_B \left[ \text{tr} \left\{ \bar{\mathcal{C}} \left( \partial^\mu \mathcal{A}_\mu + \frac{\alpha}{2} \mathcal{B} \right) + g \bar{C}^a (a^\mu \times A_\mu)^a \right\} \right] \rightarrow i\delta_B \partial_\mu \bar{c}^j \tilde{\Omega}^{\mu j}. \quad (27)$$

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- $U$  is residual local gauge symmetry, it must satisfy

$$\partial^\mu \Omega_\mu^j(x) = 0. \quad (28)$$

# The restoration condition of the residual local gauge symmetry in the Maximal Abelian gauge

- The restoration condition of the residual local gauge symmetry in the one diagonal gluon sector  $a_\nu^j(y)$  is obtained as in the Lorenz gauge:

$$I_\nu^j = \lim_{p \rightarrow 0} \int d^D x e^{ip(x-y)} \left( \Omega^{\mu j}(x) g_{\mu\nu} - \tilde{\Omega}^{\mu j}(x) \frac{\partial_\mu^x \partial_\nu^x}{\partial_x^2} \right) \delta^D(x-y) \quad (29)$$

$$= \lim_{p \rightarrow 0} \int \frac{d^D k}{(2\pi)^D} e^{-i(p-k)y} \left( \Omega^{\mu j}(p-k) g_{\mu\nu} - \tilde{\Omega}^{\mu j}(p-k) \frac{k_\mu k_\nu}{k^2} \right) \begin{cases} = 0 & \text{restoration} \\ \neq 0 & \text{no restoration} \end{cases} \quad (30)$$

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- If the condition:

$$\lim_{p \rightarrow 0} \int d^D p e^{ipx} [\Omega^{\mu j}(x) - \tilde{\Omega}^{\mu j}(x)] \frac{\partial_\mu^x \partial_\nu^x}{\partial_x^2} \delta^D(x-y) = 0 \quad (31)$$

is satisfied, (30) is reduced to

$$I_\nu^j = \lim_{p \rightarrow 0} \int \frac{d^D k}{(2\pi)^D} e^{-i(p-k)y} \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \Omega^{\mu j}(p-k). \quad (32)$$

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- Similar to the discussion of the condition (13), this restoration condition is satisfied for the same  $\Omega_\mu, \tilde{\Omega}_\mu$  (indeed, it satisfy the condition of residual gauge symmetry (28)) by **correcting the topological configuration to contribute to path integrals and introducing multiple pairs of topological defects.**

- We gave **the generalization of the color confinement criterion** in the Lorenz gauge, the MA gauge as the residual symmetry restoration conditions for the one gauge field sector with **the finite gauge transformation**.
- We discussed the restoration of the residual local gauge symmetry **by correcting the topological configuration to contribute to path integrals**.
- The axial-like gauge is under way.

- To give the complete discussion for confinement, it is necessary to consider interactions among various fields, including different kinds of fields such as the ghost and gauge field.
- A physical picture of symmetry restoration due to topological configurations contributing to the path integral is provided. Therefore, based on this picture, it is necessary to discuss the condensation of topological defects in arbitrary dimensions.
- The concrete calculation in the axial-like gauge.



We calculate the expected value contained in the integrand. If we break it down into a longitudinal part and a transverse part, we get

$$\begin{aligned}
 & \langle (\mathcal{D}_\mu \bar{\mathcal{C}}(x))^B \delta \mathcal{A}_\nu^A(y) \rangle \\
 &= g_{\mu\rho} \langle (\mathcal{D}^\rho \bar{\mathcal{C}})^B(x) \delta \mathcal{A}_\nu^A(y) \rangle \\
 &= \frac{\partial_x^\mu \partial_x^\rho}{\partial_x^2} \langle (\mathcal{D}^\rho \bar{\mathcal{C}})^B(x) \delta \mathcal{A}_\nu^A(y) \rangle + \left( g_{\mu\rho} - \frac{\partial_x^\mu \partial_x^\rho}{\partial_x^2} \right) \langle (\mathcal{D}^\rho \bar{\mathcal{C}})^B(x) (\mathcal{D}_\nu \mathcal{C})^A(y) \rangle \\
 &= \frac{\partial_x^\mu \partial_x^\rho}{\partial_x^2} \langle (\mathcal{D}^\rho \bar{\mathcal{C}})^B(x) (\mathcal{D}_\nu \mathcal{C})^A(y) \rangle + \left( g_{\mu\rho} - \frac{\partial_x^\mu \partial_x^\rho}{\partial_x^2} \right) \langle g(\mathcal{A}^\rho \times \bar{\mathcal{C}})^A(x) (\mathcal{D}_\nu \mathcal{C})^A(y) \rangle, \quad (33)
 \end{aligned}$$

where we have used  $\left( g_{\mu\rho} - \frac{\partial_x^\mu \partial_x^\rho}{\partial_x^2} \right) \partial_x^\rho = 0$ .

If we note the left derivative  $\frac{\delta S}{\delta \mathcal{C}^B(x)} = -i(\mathcal{D}^\rho \partial_\rho \bar{\mathcal{C}})^B(x)$ ,  $\frac{\delta S}{\delta \mathcal{B}^B(x)} = \partial_\rho \mathcal{A}^{\rho B}(x) + \alpha \mathcal{B}^B(x)$ , the part of (33) that excludes  $\frac{\partial^\mu}{\partial x^\mu}$  in the first term is

$$\begin{aligned}
 & \partial_\rho^x \langle (\mathcal{D}^\rho \bar{\mathcal{C}})^B(x) \delta \mathcal{A}_\nu^A(y) \rangle \\
 &= \int d\mu e^{iS} \partial_\rho (\mathcal{D}^\rho \bar{\mathcal{C}})^B(x) \delta \mathcal{A}_\nu^A(y) \\
 &= \int d\mu e^{iS} [(\mathcal{D}^\rho \partial_\rho \bar{\mathcal{C}})^B(x) \delta \mathcal{A}_\nu^A(y) + (g \partial_\rho \mathcal{A}^\rho \times \bar{\mathcal{C}})^B(x) \delta \mathcal{A}_\nu^A(y)] \\
 &= \int d\mu e^{iS} \left[ i \frac{\delta S}{\delta \mathcal{C}^A(x)} \delta \mathcal{A}_\nu^A(y) + \left( g \left( \frac{\delta S}{\delta \mathcal{B}} - \alpha \mathcal{B} \right) \times \bar{\mathcal{C}} \right)^B(x) \delta \mathcal{A}_\nu^A(y) \right]. \quad (34)
 \end{aligned}$$

From Schwinger-Dyson equation  $\int d\mu \frac{\delta}{\delta \Phi^A(x)} (\dots) = 0$ ,

$$\begin{aligned} \int d\mu e^{iS} i \frac{\delta S}{\delta \mathcal{L}^B(x)} \delta \mathcal{A}_\nu^A(y) &= \int d\mu \frac{\delta}{\delta \mathcal{L}^B(x)} [e^{iS} \delta \mathcal{A}_\nu^A(y)] - \int d\mu e^{iS} \frac{\delta}{\delta \mathcal{L}^B(x)} \delta \mathcal{A}_\nu^A(y) \\ &= - \int d\mu e^{iS} \frac{\delta}{\delta \mathcal{L}^B(x)} \delta \mathcal{A}_\nu^A(y) = - \left\langle \frac{\delta}{\delta \mathcal{L}^B(x)} (\mathcal{D}_\nu \mathcal{L})^A(y) \right\rangle = -\delta^{AB} \partial_\nu \delta^D(x-y). \end{aligned} \quad (35)$$

Similarly,

$$\begin{aligned} \int d\mu e^{iS} \left( g \frac{\delta S}{\delta \mathcal{B}} \times \bar{\mathcal{C}} \right)^B(x) \delta \mathcal{A}_\nu^A(y) &= \int d\mu e^{iS} g f^{BCE} \frac{\delta S}{\delta \mathcal{B}^C(x)} \bar{\mathcal{C}}^E(x) \delta \mathcal{A}_\nu^A(y) \\ &= \int d\mu \frac{\delta}{\delta i \mathcal{B}^C(x)} [e^{iS} g f^{BCE} \bar{\mathcal{C}}^E(x) \delta \mathcal{A}_\nu^A(y)] + i \int d\mu e^{iS} g f^{BCE} \bar{\mathcal{C}}^E(x) \frac{\delta}{\delta \mathcal{B}^C(x)} \delta \mathcal{A}_\nu^A(y) \\ &= -ig \left\langle \left( \bar{\mathcal{C}} \times \frac{\delta}{\delta \mathcal{B}(x)} \delta \mathcal{A}_\nu^A(y) \right)^B \right\rangle = 0. \end{aligned} \quad (36)$$

Moreover

$$- \int d\mu e^{iS} g\alpha(\mathcal{B} \times \bar{\mathcal{C}})^A(x) \delta\Phi(y) = \int d\mu e^{iS} g\alpha\delta(\mathcal{B} \times \bar{\mathcal{C}})^A(x) \Phi(y) = 0 \quad (37)$$

Finally, the first term of (33) is reduced to

$$\partial_\nu^x \langle (\mathcal{D}^\nu \bar{\mathcal{C}})^B(x) \delta\mathcal{A}_\nu^A(y) \rangle = -\delta^{AB} \partial_\nu \delta^D(x-y) \quad (38)$$

Then, (8)

$$\begin{aligned} & -i \int d^D x e^{ip(x-y)} \partial_\mu^x \langle \mathcal{J}_U^\mu(x) \Phi(y) \rangle \\ & = \Omega_\nu(x) + \int d^D x e^{ipx} \tilde{\Omega}^{\mu A}(x) \left[ \frac{\partial_\mu^x \partial_\nu^x}{\partial_x^2} \delta^D(x-y) \right. \\ & \quad \left. - \left( g_{\mu\rho} - \frac{\partial_\mu^x \partial_\rho^x}{\partial_x^2} \right) \langle g(\mathcal{A}^\rho \times \bar{\mathcal{C}})^A(x) (\mathcal{D}_\nu \bar{\mathcal{C}})^A(y) \rangle \right] \end{aligned} \quad (39)$$

In what follows, we demonstrate that the finite gauge transformation  $\delta_U$  and the BRST transformation  $\delta_B$  commute.

The BRST transformations are given by

$$\begin{aligned}
 \delta_B \mathcal{A}_\mu &= \mathcal{D}_\mu[\mathcal{A}_\mu] \mathcal{C} = \partial_\mu \mathcal{C} - ig[\mathcal{A}_\mu, \mathcal{C}] \\
 \delta_B \mathcal{C} &= ig \mathcal{C} \mathcal{C} \\
 \delta_B \bar{\mathcal{C}} &= i \mathcal{B} \\
 \delta_B \mathcal{B} &= 0.
 \end{aligned} \tag{40}$$

The finite gauge transformations are given by

$$\begin{aligned}
 \delta_U \mathcal{A}_\mu &= U \mathcal{A}_\mu U^\dagger + ig^{-1} U \partial_\mu U^\dagger - \mathcal{A}_\mu \\
 \delta_U \mathcal{C} &= U \mathcal{C} U^\dagger - \mathcal{C} \\
 \delta_U \bar{\mathcal{C}} &= U \bar{\mathcal{C}} U^\dagger - \bar{\mathcal{C}} \\
 \delta_U \mathcal{B} &= U \mathcal{B} U^\dagger - \mathcal{B}.
 \end{aligned} \tag{41}$$

Then, we obtain

$$\delta_U \delta_B \mathcal{B} = 0, \quad \delta_B \delta_U \mathcal{B} = U \delta_B \mathcal{B} U^\dagger - \delta_B \mathcal{B} = 0, \quad (42)$$

$$\delta_U \delta_B \bar{\mathcal{C}} = i \delta_U \mathcal{B} = i(U \mathcal{B} U^\dagger - \mathcal{B}), \quad \delta_B \delta_U \bar{\mathcal{C}} = U \delta_B \bar{\mathcal{C}} U^\dagger - \delta_B \bar{\mathcal{C}} = i U \mathcal{B} U^\dagger - i \mathcal{B} \quad (43)$$

$$\delta_U \delta_B \mathcal{C} = i g \delta_U (\mathcal{C} \mathcal{C}) = i g (U \mathcal{C} \mathcal{C} U^\dagger - \mathcal{C} \mathcal{C}),$$

$$\delta_B \delta_U \mathcal{C} = U \delta_B \mathcal{C} U^\dagger - \delta_B \mathcal{C} = i g U \mathcal{C} \mathcal{C} U^\dagger - i g \mathcal{C} \mathcal{C}, \quad (44)$$

$$\delta_U \delta_B \mathcal{A}_\mu = \delta_U (\mathcal{D}_\mu [\mathcal{A}] \mathcal{C}) = U \mathcal{D}_\mu [\mathcal{A}] \mathcal{C} U^\dagger - \mathcal{D}_\mu [\mathcal{A}] \mathcal{C},$$

$$\delta_B \delta_U \mathcal{A}_\mu = \delta_B (U \mathcal{A}_\mu U^\dagger + i g U \partial_\mu U^\dagger - \mathcal{A}_\mu),$$

$$= U \delta_B \mathcal{A}_\mu U^\dagger - \delta_B \mathcal{A}_\mu = U \mathcal{D}_\mu [\mathcal{A}] \mathcal{C} U^\dagger - \mathcal{D}_\mu [\mathcal{A}] \mathcal{C}, \quad (45)$$

where we have used

$$\begin{aligned} \delta_U (\mathcal{D}_\mu [\mathcal{A}] \mathcal{C}) &= \delta_U (\partial_\mu \mathcal{C} - i g [\mathcal{A}_\mu, \mathcal{C}]) \\ &= \partial_\mu (U \mathcal{C} U^\dagger) - i g [U \mathcal{A}_\mu U^\dagger + i g^{-1} U \partial_\mu U^\dagger, U \mathcal{C} U^\dagger] - (\partial_\mu \mathcal{C} - i g [\mathcal{A}_\mu, \mathcal{C}]) \\ &= U \partial_\mu \mathcal{C} U^\dagger - i g U [\mathcal{A}_\mu, \mathcal{C}] U^\dagger + [U \partial_\mu U^\dagger, U \mathcal{C} U^\dagger] + \partial_\mu U \mathcal{C} U^\dagger + U \mathcal{C} \partial_\mu U^\dagger \\ &= U \partial_\mu \mathcal{C} U^\dagger - i g U [\mathcal{A}_\mu, \mathcal{C}] U^\dagger - \partial_\mu U \mathcal{C} U^\dagger - U \mathcal{C} \partial_\mu U^\dagger + \partial_\mu U \mathcal{C} U^\dagger + U \mathcal{C} \partial_\mu U^\dagger \\ &= U \partial_\mu \mathcal{C} U^\dagger - i g U [\mathcal{A}_\mu, \mathcal{C}] U^\dagger = U \mathcal{D}_\mu [\mathcal{A}] \mathcal{C} U^\dagger, \end{aligned} \quad (46)$$

following

$$U \partial_\mu U^\dagger = -\partial_\mu U U^\dagger. \quad (47)$$

Thus, the finite gauge transformation  $\delta_U$  and the BRST transformation  $\delta_B$  commute.

$$\begin{aligned}
 0 &= U^\dagger [\partial^\mu \Omega_\mu + \partial^\mu (U \mathcal{A}_\mu U^\dagger)] U \\
 &= ig^{-1} U^\dagger \partial^\mu U \partial_\mu U^\dagger U + ig^{-1} U^\dagger U \partial^\mu \partial_\mu U^\dagger U + U^\dagger \partial^\mu U \mathcal{A}_\mu + \mathcal{A}_\mu \partial_\mu U^\dagger U \\
 &= ig^{-1} \partial^\mu U^\dagger U U^\dagger \partial_\mu U + ig^{-1} \partial^\mu \partial_\mu U^\dagger U + ig \tilde{\Omega}^\mu \mathcal{A}_\mu - \mathcal{A}_\mu U^\dagger \partial^\mu U \\
 &= ig^{-1} \partial_\mu (\partial^\mu U^\dagger U) + ig \tilde{\Omega}^\mu \mathcal{A}_\mu - ig \mathcal{A}_\mu \tilde{\Omega}^\mu \\
 &= -ig^{-1} \partial_\mu (U^\dagger \partial^\mu U - ig [\mathcal{A}_\mu, \tilde{\Omega}^\mu]) \\
 &= \partial_\mu \tilde{\Omega}^\mu - ig [\mathcal{A}_\mu, \tilde{\Omega}^\mu] =: \mathcal{D}_\mu \tilde{\Omega}^\mu
 \end{aligned} \tag{48}$$

$$\mathcal{A}_\mu(x) = \frac{\sigma^A}{2} \bar{\eta}_{\mu\nu}^A x_\nu F(x^2) \quad (49)$$

is a typical example that satisfies the Lorenz condition. As a residual symmetry, we consider

$$\tilde{\Omega}_\mu(x) = \frac{\sigma^A}{2} \bar{\eta}_{\mu\nu}^A x_\nu H(x^2) \quad (50)$$

in this case,

$$\partial_\mu \tilde{\Omega}_\mu = 0, \quad (51)$$

$$\begin{aligned} [\mathcal{A}_\mu, \tilde{\Omega}_\mu] &= \left[ \frac{\sigma^A}{2} \bar{\eta}_{\mu\nu}^A x_\nu F(x^2), \frac{\sigma^B}{2} \bar{\eta}_{\mu\rho}^B x_\rho H(x^2) \right] = \left[ \frac{\sigma^A}{2}, \frac{\sigma^B}{2} \right] \bar{\eta}_{\mu\nu}^A \bar{\eta}_{\mu\rho}^B x_\nu x_\rho F(x^2) H(x^2) \\ &= i\varepsilon^{ABC} \frac{\sigma^C}{2} \bar{\eta}_{\mu\nu}^A \bar{\eta}_{\mu\rho}^B x_\nu x_\rho F(x^2) H(x^2) = i \frac{\sigma^C}{2} 2\bar{\eta}_{C\nu\rho} x_\nu x_\rho F(x^2) H(x^2) = 0, \end{aligned} \quad (52)$$

where we have used

$$\begin{aligned} \varepsilon_{CAB} \bar{\eta}_{A\mu\nu} \bar{\eta}_{B\mu\rho} &= \delta_{\mu\mu} \bar{\eta}_{C\nu\rho} - \delta_{\mu\rho} \bar{\eta}_{C\nu\mu} - \delta_{\nu\mu} \bar{\eta}_{C\mu\rho} + \delta_{\nu\rho} \bar{\eta}_{C\mu\mu} \\ &= 4\bar{\eta}_{C\nu\rho} - \bar{\eta}_{C\nu\rho} - \bar{\eta}_{C\nu\rho} \\ &= 2\bar{\eta}_{C\nu\rho}. \end{aligned} \quad (53)$$



Now, gauge fixing condition is

$$n^\mu(x)\mathcal{A}_\mu(x) = 0. \quad (54)$$

Lagrangian is

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{GF+FP}} + \mathcal{L}_J, \\ \mathcal{L}_{\text{YM}} &= -\frac{1}{4}(\mathcal{F}_{\mu\nu})^2, \\ \mathcal{L}_{\text{GF+FP}} &= -i\delta_B \left( \bar{\mathcal{C}}^A \left( n^\mu \mathcal{A}_\mu^A + \frac{\alpha}{2} \mathcal{B}^A \right) \right) = i\bar{\mathcal{C}}^A n^\mu (\mathcal{D}_\mu \mathcal{C})^A + \mathcal{B}^A n^\mu \mathcal{A}_\mu^A + \frac{\alpha}{2} \mathcal{B}^A \mathcal{B}^A, \\ \mathcal{L}_J &= \mathcal{A}_\mu^A J^{\mu A} + \mathcal{B}^A K^A. \end{aligned} \quad (55)$$

To derive the propagator, we consider up to the second order of the field

$$\frac{\partial \mathcal{L}}{\partial \mathcal{A}_\mu^A} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\nu \mathcal{A}_\mu^A} = (g^{\mu\nu} \square - \partial^\mu \partial^\nu) \mathcal{A}_\nu + \mathcal{B}^A n^\mu + J^{\mu A} = 0. \quad (56)$$

If we take the divergence, we get

$$-\partial_\mu J^{\mu A} = n^\mu \partial_\mu \mathcal{B}^A + \partial_\mu n^\mu \mathcal{B}^A \Rightarrow B^A = -\frac{1}{\partial_\rho n^\rho + n^\rho \partial_\rho} \partial_\mu J^{\mu A}. \quad (57)$$

Then we obtain

$$\langle \mathcal{B}^A(x) \mathcal{A}_\mu^B(y) \rangle = \frac{\delta \mathcal{B}^A(x)}{\delta J^{\mu B}(y)} = -\frac{1}{\partial_\rho n^\rho + n^\rho \partial_\rho} \partial_\mu \delta^D(x-y) \delta^{AB} \quad (58)$$

$$i \frac{2QC_D}{D} \left( \frac{D}{2} + \delta \right) \lim_{p \rightarrow 0} \bar{\eta}_{\nu\lambda}^j p^\lambda \int_0^1 d\alpha \int \frac{d^D \ell}{(2\pi)^D} \frac{\alpha^{\frac{D}{2} + \delta - 1} \ell^2}{\{\ell^2 + \alpha(1 - \alpha)p^2\}^{\frac{D}{2} + \delta + 1}} \quad (59)$$

If we estimate the contribution of the UV region  $k > k_R \gg 1$  to  $I_\nu$ , we can take the limit  $p \rightarrow 0$  of the integrand, which is equivalent to the replacement  $\ell_\mu = k_\mu$ :

$$\begin{aligned} &\rightarrow i \frac{2C_D}{D} \left( \frac{D}{2} + \delta \right) \lim_{p \rightarrow 0} 2\varepsilon_{4A\nu\lambda} p^\lambda \int_0^1 d\alpha \int d\Omega_D \int_{k_R}^\infty \frac{dk k^{D-1}}{(2\pi)^D} \frac{\alpha^{\frac{D}{2} + \delta - 1}}{k^{D+2\delta}} \\ &= i \frac{2QC_D}{(4\pi)^{\frac{D}{2}} \Gamma\left(\frac{D}{2} + 1\right)} \left( \frac{D}{2} + \delta \right) \lim_{p \rightarrow 0} 2\varepsilon_{4A\nu\lambda} p^\lambda \int_0^1 d\alpha \int_{k_R}^\infty dk \frac{\alpha^{\frac{D}{2} + \delta - 1}}{k^{2\delta+1}} \\ &= i \frac{2C_D}{(4\pi)^{\frac{D}{2}} \Gamma\left(\frac{D}{2} + 1\right)} \lim_{p \rightarrow 0} 2\varepsilon_{4A\nu\lambda} p^\lambda [\alpha^{\frac{D}{2} + \delta}]_0^1 \left[ -\frac{1}{2\delta} \frac{1}{k^{2\delta}} \right]_{k_R}^\infty \\ &= i \frac{2C_D}{2\delta(4\pi)^{\frac{D}{2}} \Gamma\left(\frac{D}{2} + 1\right)} \lim_{p \rightarrow 0} 2\varepsilon_{4A\nu\lambda} p^\lambda \frac{1}{k_R^{2\delta}} = 0 \quad (\delta > 1/4), \end{aligned} \quad (60)$$

If we think

$$\mathcal{A}_\mu(x) = \sigma^A \bar{\eta}_{\mu\nu}^A \frac{\tilde{x}_\nu \lambda^2}{\tilde{x}^2 (\tilde{x}^2 + \lambda^2)}$$

$$U = U_1 U_0, \quad U_0 = \frac{x_4 I + i \sigma_j x_j}{\sqrt{x^2}}, \quad U_1(x) = \exp \left[ i \sigma^A \frac{x_A}{\sqrt{x^2 + \lambda^2}} \left( \arctan \frac{x_4}{\sqrt{x^2 + \lambda^2}} + \frac{\pi}{2} \right) \right] \quad (61)$$

, gauge fixing condition is satisfied.

$$\mathcal{A}_\mu'' = \mathcal{A}_\mu + \delta_U \mathcal{A}_\mu = 0 \quad (|x| \rightarrow 0) \quad (62)$$

If we introduce the polar coordinate  $(r, \theta_1, \theta_2, \theta_3)$  as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} r \sin \theta_1 \sin \theta_2 \sin \theta_3 \\ r \sin \theta_1 \sin \theta_2 \cos \theta_3 \\ r \sin \theta_1 \cos \theta_2 \\ r \cos \theta_1 \end{pmatrix} \quad (63)$$

Then,

$$U_0 = \cos \theta_1 I + \sin \theta_1 \sin \theta_2 \sin \theta_3 \sigma_1 + \sin \theta_1 \sin \theta_2 \cos \theta_3 \sigma_2 + \sin \theta_1 \cos \theta_2 \sigma_3 \quad (64)$$

$$U_1 = \exp \left[ ir \frac{\sin \theta_1 \sin \theta_2 \sin \theta_3 \sigma_1 + \sin \theta_1 \sin \theta_2 \cos \theta_3 \sigma_2 + \sin \theta_1 \cos \theta_2 \sigma_3}{\sqrt{r^2 \sin^2 \theta_1 + \lambda^2}} \left( \arctan \frac{r \cos \theta_1}{\sqrt{r^2 \sin^2 \theta_1 + \lambda^2}} + \frac{\pi}{2} \right) \right]$$

$$= \exp \left[ i \frac{\sin \theta_1 \sin \theta_2 \sin \theta_3 \sigma_1 + \sin \theta_1 \sin \theta_2 \cos \theta_3 \sigma_2 + \sin \theta_1 \cos \theta_2 \sigma_3}{\sqrt{\sin^2 \theta_1 + (\lambda/r)^2}} \left( \arctan \frac{\cos \theta_1}{\sqrt{\sin^2 \theta_1 + (\lambda/r)^2}} + \frac{\pi}{2} \right) \right] \quad (65)$$

Therefore, if we rescale  $r \rightarrow rp$ ,  $U$  is 0 order of  $p$ , where since  $\lambda$  is unphysical parameter, we can take  $\lambda \rightarrow 0$  as  $p \rightarrow 0$