30 years of non relativistic effective field theories for quarkonium

Antonio Vairo

Technische Universität München

NRQCD for annihilation and production

(1) G. Bodwin, E. Braaten and P. Lepage

Rigorous QCD analysis of inclusive annihilation and production of heavy quarkonium

Phys. Rev. D 51 (1995) 1125-1171 `hep-ph/9407339`

(2) G. Bodwin, E. Braaten and P. Lepage

Rigorous QCD predictions for decays of P-wave quarkonia

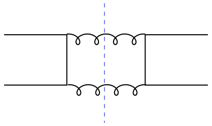
Phys. Rev. D 46 (1992) R1914-R1918 `hep-lat/9205006`

The IR problem in the singlet model

If quarkonium, $\mathcal{Q}$, was just made of a color singlet $Q\bar{Q}$, then

$$|\mathcal{Q}\rangle = \int \frac{d^3k}{(2\pi)^3} \Phi^{ij}_{uv}(\mathbf{k})\, |Q(\mathbf{k})^{iu} Q(-\mathbf{k})^{jv}\rangle \qquad\qquad \Phi^{ij}_{uv}(\mathbf{x}) \sim (\cdots) R(x)$$

On the example of P-wave quarkonium the decay width is (LH = light hadrons)

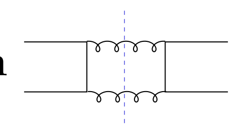
$$\Gamma(\chi_0 \to \mathrm{LH}) = \sum_X \Gamma(\chi_0 \to X) \approx \Gamma(\chi_0 \to gg) = \langle \chi_0 | 2\ \mathrm{Im}$$  $$|\chi_0\rangle$$

The IR problem in the singlet model

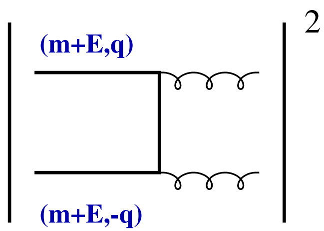
If quarkonium, $\mathcal{Q}$, was just made of a color singlet $Q\bar{Q}$, then

$$|\mathcal{Q}\rangle = \int \frac{d^3k}{(2\pi)^3}\Phi^{ij}_{uv}(\mathbf{k})\; |Q(\mathbf{k})^{iu}Q(-\mathbf{k})^{jv}\rangle \qquad \Phi^{ij}_{uv}(\mathbf{x}) \sim (\cdots)R(x)$$

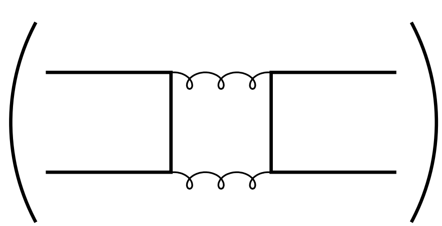
On the example of P-wave quarkonium the decay width is (LH = light hadrons)

$$\Gamma(\chi_0 \to \mathrm{LH}) = \sum_X \Gamma(\chi_0 \to X) \approx \Gamma(\chi_0 \to gg) = \langle \chi_0 | 2\,\mathrm{Im}$$

$$|\chi_0\rangle$$

@ tree level



= 2 Im



$\xrightarrow[q \sim mv \ll m]{}$

$$\frac{4\pi\alpha_s^2}{3}\,\frac{\mathbf{S}\cdot\boldsymbol{\nabla}\delta(\mathbf{r})\mathbf{S}\cdot\boldsymbol{\nabla}}{m^4}$$

$$\Gamma(\chi_0 \to \mathrm{LH}) = \langle \chi_0 | \ldots | \chi_0 \rangle = 9\left(\frac{C_F\pi}{2}\alpha_s^2\right)\frac{\left|R'(0)\right|^2}{\pi m^4}$$

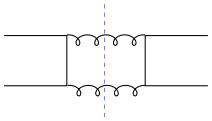
The IR problem in the singlet model

If quarkonium, $\mathcal{Q}$, was just made of a color singlet $Q\bar{Q}$, then

$$|\mathcal{Q}\rangle = \int \frac{d^3k}{(2\pi)^3} \Phi^{ij}_{uv}(\mathbf{k})\, |Q(\mathbf{k})^{iu} Q(-\mathbf{k})^{jv}\rangle \qquad \Phi^{ij}_{uv}(\mathbf{x}) \sim (\cdots) R(x)$$

On the example of P-wave quarkonium the decay width is (LH = light hadrons)

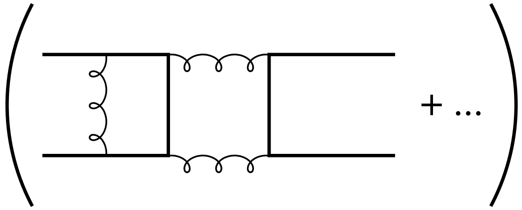
$$\Gamma(\chi_0 \to \mathrm{LH}) = \sum_X \Gamma(\chi_0 \to X) \approx \Gamma(\chi_0 \to gg) = \langle \chi_0 | 2\,\mathrm{Im}$$



$$|\chi_0\rangle$$

@ one loop

2 Im



$$\longrightarrow \left((\ldots)\alpha_s^2 + (\ldots)\alpha_s^3 \ln\frac{\mu}{m}\right) \frac{\mathbf{S}\cdot\boldsymbol{\nabla}\delta(\mathbf{r})\mathbf{S}\cdot\boldsymbol{\nabla}}{m^4}$$

- Barbieri Caffo Gatto Remiddi PLB 95 (1980) 93, NPB 192 (1981) 61

The NRQCD solution to the IR problem

- Quarkonium is a nonrelativistic bound state: $v \ll 1$.
- It has v-suppressed Fock space components made of $Q\bar{Q}$ pairs in a color octet state:

$$|\mathcal{Q}\rangle = \left(|(Q\bar{Q})_1\rangle + |(Q\bar{Q})_8 g\rangle + \cdots\right) \otimes |nljs\rangle$$

$$O(1) \qquad\qquad O(v)$$

- The NRQCD Lagrangian follows from integrating out m from QCD.

Operators are counted in powers of v (or quark momenta). It contains 4-fermion operators that overlap with $Q\bar{Q}$ in a color singlet or octet configuration and whose matching coeffcients have an imaginary part encoding annihilation.

For P-wave states,

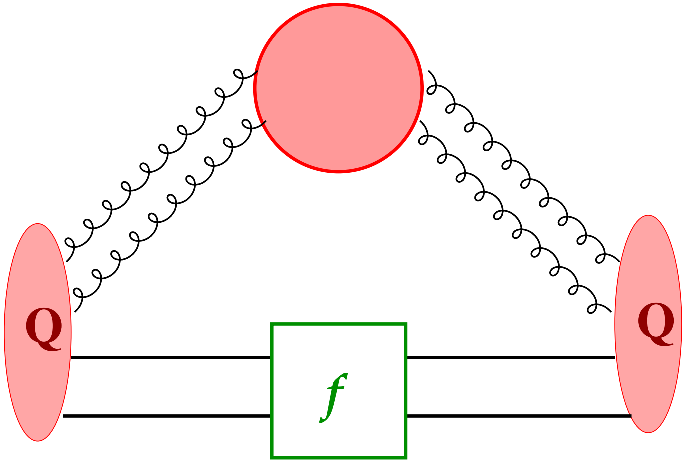
$$\mathcal{L}_{\rm NRQCD} \supset \psi^\dagger \mathcal{K}_N \chi \chi^\dagger \mathcal{K}'_N \psi = \begin{cases} \mathcal{O}_1({}^{2S+1}L_J) \\ \mathcal{O}_8({}^{2S+1}L_J) \end{cases}$$

$$\psi^\dagger T^a \chi \chi^\dagger T^a \psi = \mathcal{O}_8({}^1S_0) = O(1), \quad \psi^\dagger \mathbf{D} \chi \chi^\dagger \mathbf{D} \psi = \mathcal{O}_1({}^1P_1) = O(v^2), \quad \ldots$$

Annihilation widths in NRQCD

$$\Gamma(\mathcal{Q} \to \mathrm{LH}) = \sum_N \frac{2\,\mathrm{Im}\, f^{(N)}}{m^{d_N-4}} \langle \mathcal{Q}|\psi^\dagger \mathcal{K}_N \chi \chi^\dagger \mathcal{K}'_N \psi|\mathcal{Q}\rangle$$

$$\Gamma(\mathcal{Q} \to \mathrm{EM}) = \sum_N \frac{2\,\mathrm{Im}\, f^{(N)}_{\mathrm{em}}}{m^{d_N-4}} \langle \mathcal{Q}|\psi^\dagger \mathcal{K}_N \chi|\Omega\rangle\langle\Omega|\chi^\dagger \mathcal{K}'_N \psi|\mathcal{Q}\rangle$$



P-wave annihilation widths in NRQCD

$$\Gamma(\chi_J \to \mathrm{LH}) \;=\; 9\;\mathrm{Im}\, f_1 \, \frac{\left|R'(0)\right|^2}{\pi m^4} \;+\; \frac{2\;\mathrm{Im}\, f_8}{m^2}\, \langle \chi | \mathcal{O}_8(^1S_0) | \chi \rangle$$

$$\Gamma(\chi_J \to \gamma\gamma) \;=\; 9\;\mathrm{Im}\, f_{\gamma\gamma} \, \frac{\left|R'(0)\right|^2}{\pi m^4} \qquad\qquad J = 0, 2$$

- Octet and singlet contribute to the same order.
- The IR divergence of $\mathrm{Im}\, f_1$ cancels against the octet matrix element $\langle \chi | \mathcal{O}_8(^1S_0) | \chi \rangle$.
- Bottomonium and charmonium (below threshold) P-wave decay widths depend on 6 nonperturbative parameters: 3 wavefunctions $+$ 3 octet matrix elements.

Inclusive production cross sections in NRQCD

In NRQCD, the production cross sections for a quarkonium $\mathcal{Q}$ factorize

- in short distance coefficients, $\sigma_{Q\bar{Q}(N)}$, encoding contributions from energy scales of order m or larger,
- and in long distance matrix elements (LDMEs), $\langle\Omega|\mathcal{O}^{\mathcal{Q}}(N)|\Omega\rangle$, encoding contributions of order mv, mv^2 and $\Lambda_{\rm QCD}$,

so that we can write:

$$\sigma_{\mathcal{Q}+X} = \sum_N \sigma_{Q\bar{Q}(N)} \langle\Omega|\mathcal{O}^{\mathcal{Q}}(N)|\Omega\rangle.$$

The same pattern of IR divergence cancellations between singlet matching coefficients and octet LDMEs happens also for the NRQCD cross sections leading to physical results.

pNRQCD for annihilation

(1) N. Brambilla, H. S. Chung, D. Müller and A. Vairo
Decay and electromagnetic production of strongly coupled quarkonia in pNRQCD
JHEP 04 (2020) 095 `arXiv:2002.07462`

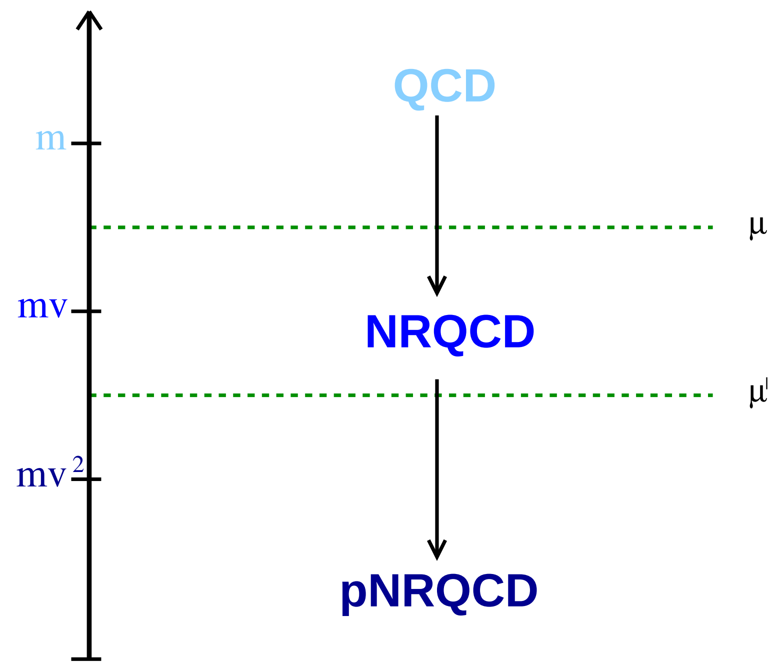
(2) N. Brambilla, D. Eiras, A. Pineda, J. Soto and A. Vairo
Inclusive decays of heavy quarkonium to light particles
Phys. Rev. D 67 (2003) 034018 `hep-ph/0208019`

(3) N. Brambilla, D. Eiras, A. Pineda, J. Soto and A. Vairo
New predictions for inclusive heavy quarkonium P-wave decays
Phys. Rev. Lett. 88 (2002) 012003 `hep-ph/0109130`

Scales and non relativistic EFTs

Quarkonium physics may be described through nonrelativistic effective field theories beyond NRQCD, owing to the hierarchy of scales of nonrelativistic bound states:

$$m \gg mv \gg mv^2$$



- Brambilla Pineda Soto Vairo RMP 77 (2005) 1423

The NRQCD energy eigenstates

The spectral decomposition of $H_{\rm NRQCD}$ in the $Q\bar{Q}$ sector of the Hilbert space reads

$$H_{\rm NRQCD}\big|_{Q\bar{Q}} = \sum_n \int d^3x_1\, d^3x_2\, |\underline{\rm n}; \boldsymbol{x}_1, \boldsymbol{x}_2\rangle\, E_n(\boldsymbol{x}_1, \boldsymbol{x}_2; \boldsymbol{\nabla}_1, \boldsymbol{\nabla}_2)\, \langle \underline{\rm n}; \boldsymbol{x}_1, \boldsymbol{x}_2|$$

$|\underline{\rm n}; \boldsymbol{x}_1, \boldsymbol{x}_2\rangle = \psi^\dagger(\boldsymbol{x}_1)\chi(\boldsymbol{x}_2)|n; \boldsymbol{x}_1, \boldsymbol{x}_2\rangle$ are orthonormal states made of a heavy quark, ψ, a heavy antiquark, χ, and some light d.o.f. labeled by n.
E_n are operators in the coordinate, momentum and spin of the $Q\bar{Q}$.
In the static limit $E_n = E_n^{(0)}$ are the different energy excitations of a static $Q\bar{Q}$.
They may be computed in lattice QCD as a function of the $Q\bar{Q}$ distance.

The eigenstates of the NRQCD Hamiltonian in the $Q\bar{Q}$ sector are

$$|\mathcal{Q}(n, \boldsymbol{P})\rangle = \int d^3x_1 d^3x_2\, \phi_{\mathcal{Q}(n,\boldsymbol{P})}(\boldsymbol{x}_1, \boldsymbol{x}_2)\, |\underline{\rm n}; \boldsymbol{x}_1, \boldsymbol{x}_2\rangle$$

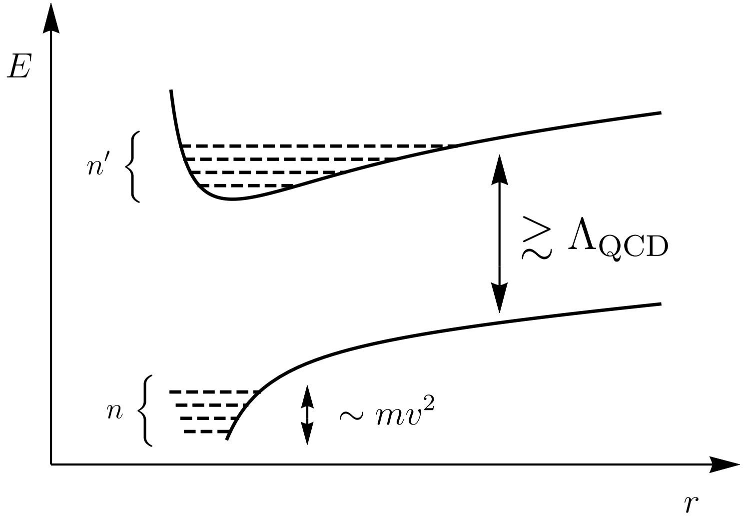
The functions $\phi_{\mathcal{Q}(n,\boldsymbol{P})}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ are eigenfunctions of $E_n(\boldsymbol{x}_1, \boldsymbol{x}_2; \boldsymbol{\nabla}_1, \boldsymbol{\nabla}_2)$;
$\boldsymbol{P}$ is the center of mass momentum of the $Q\bar{Q}$ pair.

The NRQCD energy levels

We expect the different $E_n^{(0)}$ to develop an energy gap of order $\Lambda_{\rm QCD}$ much larger than the energies of the eigenstates of each single E_n, which are of order mv^2:

$$mv^2 \ll \Lambda_{\rm QCD}$$

This qualifies the strong coupling regime.



Strong coupling is suited to describe excited (non Coulombic) quarkonium states.

○ Bali et al PRD 62 (2000) 054503

Matching the potential

In strongly coupled pNRQCD, the non interacting part of the Hamiltonian is given by

$$H_{\rm pNRQCD} \supset \int d^3x_1\, d^3x_2\; S_n^\dagger\, h_n(\boldsymbol{x}_1, \boldsymbol{x}_2; \boldsymbol{\nabla}_1, \boldsymbol{\nabla}_2)\, S_n$$

S_n is a color singlet field containing a $Q\bar{Q}$;
h_n is obtained by matching the NRQCD energy E_n.

The matching may be performed order by order in $1/m$ by expanding the NRQCD Hamiltonian and the states $|\underline{\rm n}; \boldsymbol{x}_1, \boldsymbol{x}_2\rangle$ using quantum mechanical perturbation theory. At leading order in v we have

$$h_n(\boldsymbol{x}_1, \boldsymbol{x}_2; \boldsymbol{\nabla}_1, \boldsymbol{\nabla}_2) = -\frac{\boldsymbol{\nabla}_1^2}{2m} - \frac{\boldsymbol{\nabla}_2^2}{2m} + V^{(0;n)}(\boldsymbol{x}_1, \boldsymbol{x}_2)$$

The matching fixes the static potential $V^{(0;n)}$ to be the static energy $E_n^{(0)}$ of $H^{(0)}_{\rm NRQCD}$.
As a consequence of the matching, the functions $\phi_{\mathcal{Q}(n,\boldsymbol{P})}$ are also eigenfunctions of h_n.

- Brambilla Pineda Soto Vairo PRD 63 (2001) 014023
  Pineda Vairo PRD 63 (2001) 054007

Matching the quarkonium annihilation matrix elements

The pNRQCD factorization formula for the quarkonium annihilation matrix elements reads

$$\langle \mathcal{Q}|\mathcal{O}(N)|\mathcal{Q}\rangle = \frac{1}{\langle \boldsymbol{P}=\boldsymbol{0}|\boldsymbol{P}=\boldsymbol{0}\rangle} \int d^3x_1 d^3x_2 d^3x_1' d^3x_2' \, \phi_{\mathcal{Q}}(\boldsymbol{x}_1 - \boldsymbol{x}_2)$$
$$\left[-V_{\mathcal{O}(N)}(\boldsymbol{x}_1, \boldsymbol{x}_2; \boldsymbol{\nabla}_1, \boldsymbol{\nabla}_2)\delta^{(3)}(\boldsymbol{x}_1 - \boldsymbol{x}_1')\delta^{(3)}(\boldsymbol{x}_2 - \boldsymbol{x}_2')\right] \phi^*_{\mathcal{Q}}(\boldsymbol{x}_1' - \boldsymbol{x}_2')$$

$\phi_{\mathcal{Q}}$ is the quarkonium wavefunction in pNRQCD (corresponding to the state $n=0$),
$V_{\mathcal{O}(N)}$ has to be determined by matching NRQCD with pNRQCD order by order in $1/m$ via quantum mechamical perturbation theory.

Matching P-wave annihilation matrix elements

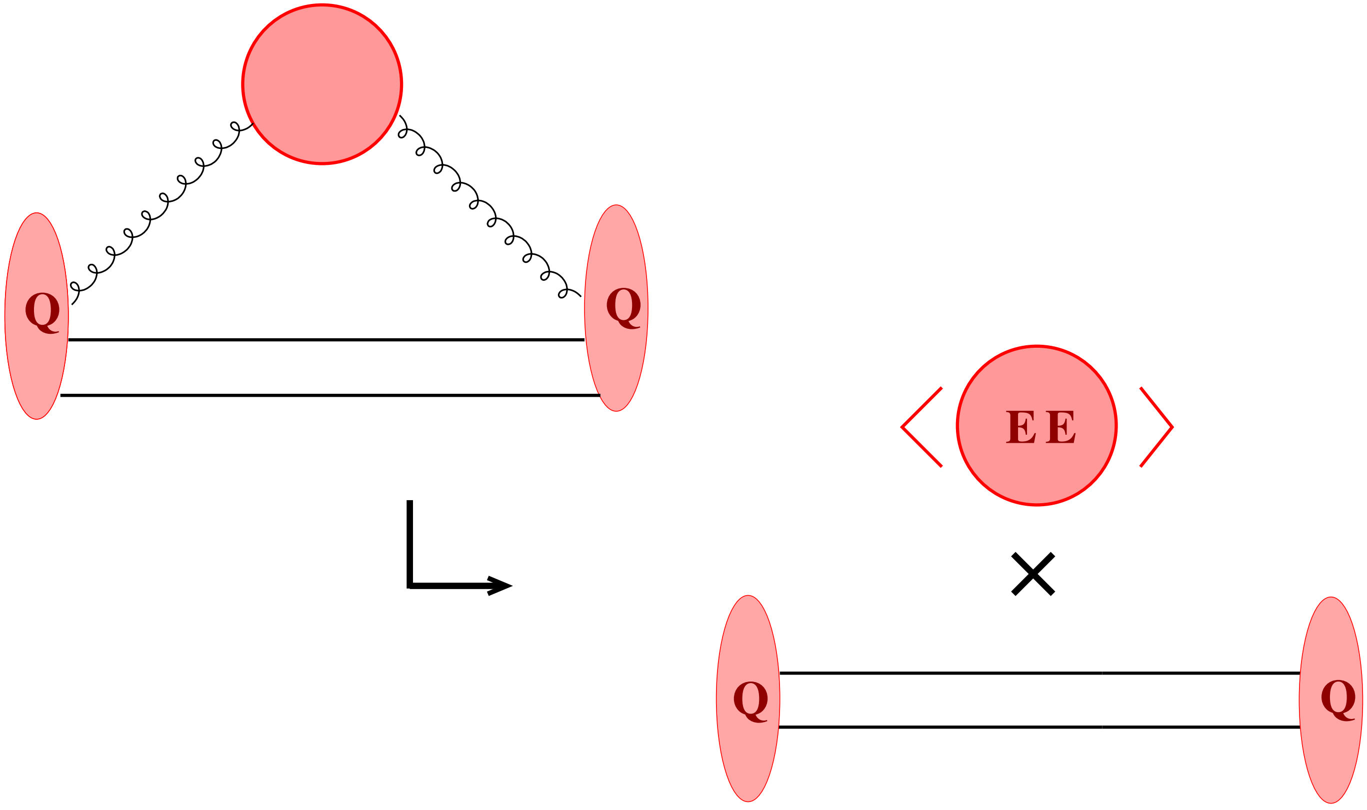
The matching of the octet matrix element appearing at LO in P-wave quarkonium annihilation leads to

$$\langle \chi_{QJ}(nP)|\mathcal{O}_8(^1S_0)|\chi_{QJ}(nP)\rangle = \frac{2T_F}{9N_c m^2}\,\frac{3N_c}{2\pi}|R'(0)|^2\mathcal{E}_3$$

where

$$\mathcal{E}_3 = \frac{T_F}{N_c}\int_0^\infty dt\, t^3 \langle\Omega|gE^{i,a}(t,\mathbf{0})\Phi_{ab}(t,0)gE^{i,b}(0,\mathbf{0})|\Omega\rangle$$

is a universal chromoelectric correlator (to be computed in lattice QCD ...).



Q
Q
E E
Q
Q

P-wave annihilation widths in pNRQCD

$$\Gamma(\chi_J \to \mathrm{LH}) = \frac{\left|R'(0)\right|^2}{\pi m^4} \left[9 \,\mathrm{Im}\, f_1 \; + \; \frac{\mathrm{Im}\, f_8}{3} \; \mathcal{E}_3 \right]$$

$$\Gamma(\chi_J \to \gamma\gamma) \; = \; 9 \,\mathrm{Im}\, f_{\gamma\gamma} \, \frac{\left|R'(0)\right|^2}{\pi m^4} \qquad J = 0, 2$$

- The quarkonium state dependence factorizes.
- The IR divergence of $\mathrm{Im}\, f_1$ cancels against the chromoelectric correlator $\mathcal{E}_3$.
- Bottomonium and charmonium (below threshold) P-wave decay widths depend on 4 nonperturbative parameters: 3 wavefunctions $+$ 1 universal correlator.

Correlator and octet matrix element

$\mathcal{E}_3(\Lambda)$ can be obtained from a least squares fit to the ratios of decay rates $\Gamma(\chi_{c0}(1P) \to \mathrm{LH})/\Gamma(\chi_{c1}(1P) \to \mathrm{LH})$, $\Gamma(\chi_{c1}(1P) \to \mathrm{LH})/\Gamma(\chi_{c2}(1P) \to \mathrm{LH})$, $\Gamma(\chi_{c0}(1P) \to \mathrm{LH})/\,\Gamma(\chi_{c0}(1P) \to \gamma\gamma)$, and $\Gamma(\chi_{c2}(1P) \to \mathrm{LH})/\Gamma(\chi_{c2}(1P) \to \gamma\gamma)$ (from PDG) at leading order in v. In the $\overline{\mathrm{MS}}$ scheme, we obtain

$$\mathcal{E}_3(1\ \text{GeV}) = 2.05^{+0.94}_{-0.65}.$$

$\mathcal{E}_3(\Lambda)$ at different scales follows from the one loop renormalization group improved expression ($\beta_0 = 11N_c/3 - 4T_F n_f/3$)

$$\mathcal{E}_3(\Lambda) = \mathcal{E}_3(\Lambda') + \frac{24C_F}{\beta_0} \log \frac{\alpha_s(\Lambda')}{\alpha_s(\Lambda)},$$

The octet matrix element on charmonium state is

$$\langle \chi_{cJ}(1P)|\mathcal{O}_8({}^1S_0)|\chi_{cJ}(1P)\rangle = (3.53^{+1.05}_{-1.15}{}^{+1.62}_{-1.12}) \times 10^{-3}\ \text{GeV}^3$$

computing the wavefunction with several potential models.

P-wave charmonium annihilation widths

From the ratio of widths and the experimental value of the electromagnetic width, we get

$$\Gamma(\chi_{c0}(1P) \to \mathrm{LH}) = 8.3^{+3.0}_{-3.1} \text{ MeV} \qquad [10.6 \pm 0.6 \text{ MeV from PDG}]$$

$$\Gamma(\chi_{c1}(1P) \to \mathrm{LH}) = 0.42^{+0.06+0.28}_{-0.06-0.22} \text{ MeV} \qquad [0.552 \pm 0.041 \text{ MeV from PDG}]$$

$$\Gamma(\chi_{c2}(1P) \to \mathrm{LH}) = 1.4^{+0.6}_{-0.6} \text{ MeV} \qquad [1.60 \pm 0.09 \text{ MeV from PDG}]$$

P-wave bottomonium annihilation widths

From the ratio of widths and the experimental value of the electromagnetic width, we get

$$\Gamma(\chi_{b0}(nP) \to \mathrm{LH}) = 1.07^{+0.33}_{-0.37} \text{ MeV}$$
$$\Gamma(\chi_{b1}(nP) \to \mathrm{LH}) = 0.14 \pm 0.06 \text{ MeV}$$
$$\Gamma(\chi_{b2}(nP) \to \mathrm{LH}) = 0.28^{+0.09}_{-0.10} \text{ MeV}$$

which are almost independent of the principal quantum number $n = 1, 2, 3$.
These are proper predictions that exploit the universality of the chromoelectric correlator.

pNRQCD for hadroproduction

(1) N. Brambilla, H. S. Chung, A. Vairo and X. P. Wang
Inclusive production of J/ψ, $\psi(2S)$ and Υ states in pNRQCD
JHEP 03 (2023) 242 `arXiv:2210.17345`

(2) N. Brambilla, H. S. Chung, A. Vairo and X. P. Wang
Production and polarization of S-wave quarkonia in potential Nonrelativistic QCD
Phys. Rev. D 105 (2022) 11, L111503 `arXiv:2203.07778`

(3) N. Brambilla, H. S. Chung and A. Vairo
Inclusive production of heavy quarkonia in pNRQCD
JHEP 09 (2021) 032 `arXiv:2106.09417`

(4) N. Brambilla, H. S. Chung and A. Vairo
Inclusive hadroproduction of P-wave heavy quarkonia in potential Nonrelativistic QCD
Phys. Rev. Lett. 126 (2021) 8 `arXiv:2007.07613`

Hadroproduction LDMEs

Color singlet and octet operators for hadroproduction of quarkonia have the form

$$\mathcal{O}^{\mathcal{Q}}(N_{\text{color singlet}}) = \chi^\dagger \mathcal{K}_N \psi \mathcal{P}_{\mathcal{Q}(\boldsymbol{P}=\boldsymbol{0})} \psi^\dagger \mathcal{K}'_N \chi$$

$$\mathcal{O}^{\mathcal{Q}}(N_{\text{color octet}}) = \chi^\dagger \mathcal{K}_N T^a \psi \Phi_\ell^{\dagger ab}(0) \mathcal{P}_{\mathcal{Q}(\boldsymbol{P}=\boldsymbol{0})} \Phi_\ell^{bc}(0) \psi^\dagger \mathcal{K}'_N T^c \chi$$

$\Phi_\ell(x)$ is a Wilson line along the direction ℓ in the adjoint representation required to ensure the gauge invariance of the color octet LDME.

- Nayak Qiu Sterman PLB 613 (2005) 45

$\mathcal{P}_{\mathcal{Q}(\boldsymbol{P})}$ projects onto a state containing a heavy quarkonium $\mathcal{Q}$ with momentum $\boldsymbol{P}$.
$\mathcal{P}_{\mathcal{Q}(\boldsymbol{P})}$ commutes with the NRQCD Hamiltonian (the number of quarkonia is conserved) and is diagonalized by the same eigenstates of the NRQCD Hamiltonian:

$$\mathcal{P}_{\mathcal{Q}(\boldsymbol{P})} = \sum_{n \in \mathbb{S}} |\mathcal{Q}(n, \boldsymbol{P})\rangle \langle \mathcal{Q}(n, \boldsymbol{P})|$$

The sum extends over $\mathbb{S}$, which are all states where the $Q\bar{Q}$ is in a color singlet at the origin in the static limit. This is a necessary condition to produce a quarkonium.

Matching the wavefunctions $\phi_{\mathcal{Q}(n,\boldsymbol{P})}$

The projector $\mathcal{P}_{\mathcal{Q}(\boldsymbol{P})}$ depends on the wavefunction $\phi_{\mathcal{Q}(n,\boldsymbol{P})}$ with $n \in \mathbb{S}$.
$\phi_{\mathcal{Q}(n,\boldsymbol{P})}$ is a solution of the Schrödinger equation with static potential $V^{(0;n)}$.
$V^{(0;n)}$ is the energy of a static Wilson loop in the presence of disconnected gluon fields.

Lattice QCD determinations of $V^{(0;n)}$ for $n \in \mathbb{S}$ and $n \neq 0$ are not available yet.
One expects, however, disconnected gluon fields to produce mainly a constant shift to the potentials, e.g. in the form of a glueball mass. This is supported by the large N_c limit: the vacuum expectation value of a Wilson loop with additional disconnected gluon fields factorizes into the vacuum expectation value of the Wilson loop times the vacuum expectation value of the additional gluon fields up to corrections of order $1/N_c^2$.

If the slopes of the static potentials are the same for all $n \in \mathbb{S}$, then

$$\phi_{\mathcal{Q}(n,\boldsymbol{P})}(\boldsymbol{x}_1, \boldsymbol{x}_2) \approx e^{i\boldsymbol{P}\cdot(\boldsymbol{x}_1+\boldsymbol{x}_2)/2}\phi^{(0)}_{\mathcal{Q}}(\boldsymbol{x}_1 - \boldsymbol{x}_2)$$

$\phi^{(0)}_{\mathcal{Q}}$ is the leading order quarkonium wavefunction in the center of mass frame.

Matching the hadroproduction LDMEs

The pNRQCD factorization formula for the LDMEs reads at LO

$$\langle\Omega|\mathcal{O}^{\mathcal{Q}}(N)|\Omega\rangle = \frac{1}{\langle \boldsymbol{P}=\boldsymbol{0}|\boldsymbol{P}=\boldsymbol{0}\rangle}\int d^3x_1 d^3x_2 d^3x_1' d^3x_2'\,\phi_{\mathcal{Q}}^{(0)}(\boldsymbol{x}_1-\boldsymbol{x}_2)$$
$$\times\left[-V_{\mathcal{O}^{\mathcal{Q}}(N)}(\boldsymbol{x}_1,\boldsymbol{x}_2;\boldsymbol{\nabla}_1,\boldsymbol{\nabla}_2)\delta^{(3)}(\boldsymbol{x}_1-\boldsymbol{x}_1')\delta^{(3)}(\boldsymbol{x}_2-\boldsymbol{x}_2')\right]\phi_{\mathcal{Q}}^{(0)\,*}(\boldsymbol{x}_1'-\boldsymbol{x}_2')$$

$V_{\mathcal{O}^{\mathcal{Q}}(N)}$ has to be determined from matching the NRQCD LDMEs to pNRQCD.

The matching is performed by expanding the states $|\underline{\mathrm{n}};\boldsymbol{x}_1,\boldsymbol{x}_2\rangle$ order by order in $1/m$ using quantum mechanical perturbation theory, similarly to what done for the NRQCD matrix elements entering annihilation widths.

$pp \to \chi_Q + X$

We consider

$$pp \to h_Q(nP) + X \qquad \text{and} \qquad pp \to \chi_{QJ}(nP) + X$$

The NRQCD factorization formulas at leading order in v read

$$\sigma_{h_Q+X} = \sigma_{Q\bar{Q}(^1P_1^{[1]})} \langle \Omega | \mathcal{O}^{h_Q}(^1P_1^{[1]}) | \Omega \rangle + \sigma_{Q\bar{Q}(^1S_0^{[8]})} \langle \Omega | \mathcal{O}^{h_Q}(^1S_0^{[8]}) | \Omega \rangle$$

$$\sigma_{\chi_{QJ}+X} = \sigma_{Q\bar{Q}(^3P_J^{[1]})} \langle \Omega | \mathcal{O}^{\chi_{QJ}}(^3P_J^{[1]}) | \Omega \rangle + \sigma_{Q\bar{Q}(^3S_1^{[8]})} \langle \Omega | \mathcal{O}^{\chi_{QJ}}(^3S_1^{[8]}) | \Omega \rangle$$

Matching the contact terms in pNRQCD

After matching with pNRQCD, the contact terms read

$$
\begin{aligned}
V_{\mathcal{O}(^1P_1^{[1]})}(\boldsymbol{r},\boldsymbol{\nabla}_{\boldsymbol{r}}) =& N_c \nabla^i_{\boldsymbol{r}} \delta^{(3)}(\boldsymbol{r}) \nabla^i_{\boldsymbol{r}} \\
V_{\mathcal{O}(^1S_0^{[8]})}(\boldsymbol{r},\boldsymbol{\nabla}_{\boldsymbol{r}}) =& N_c \nabla^i_{\boldsymbol{r}} \delta^{(3)}(\boldsymbol{r}) \nabla^j_{\boldsymbol{r}} \frac{\mathcal{E}^{ij}}{N_c^2 m^2} \\
V_{\mathcal{O}(^3P_J^{[1]})}(\boldsymbol{r},\boldsymbol{\nabla}_{\boldsymbol{r}}) =& T^{ij}_{1J} N_c \nabla^i_{\boldsymbol{r}} \delta^{(3)}(\boldsymbol{r}) \nabla^j_{\boldsymbol{r}} \\
V_{\mathcal{O}(^3S_1^{[8]})}(\boldsymbol{r},\boldsymbol{\nabla}_{\boldsymbol{r}}) =& \sigma^k \otimes \sigma^k N_c \nabla^i_{\boldsymbol{r}} \delta^{(3)}(\boldsymbol{r}) \nabla^j_{\boldsymbol{r}} \frac{\mathcal{E}^{ij}}{N_c^2 m^2}
\end{aligned}
$$

$\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ and T^{ij}_{1J} are spin projectors.

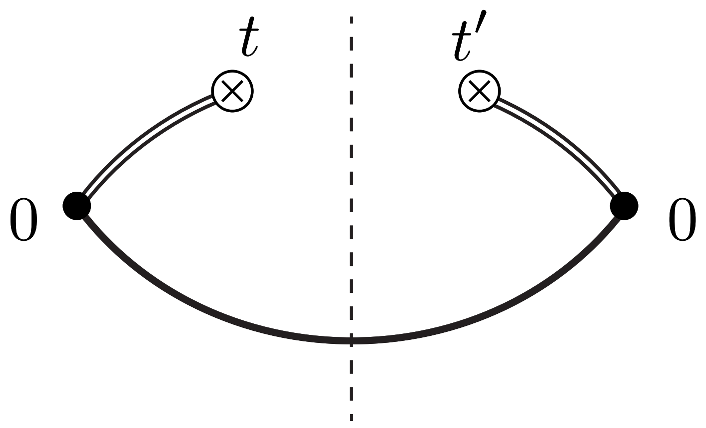
The tensor $\mathcal{E}^{ij}$ is defined as

$$
\mathcal{E}^{ij} = \int_0^\infty dt\, t \int_0^\infty dt'\, t' \, \langle \Omega | \Phi_\ell^{\dagger ab} \Phi^{\dagger ad}(0;t) g E^{d,i}(t) g E^{e,j}(t') \Phi^{ec}(0;t') \Phi_\ell^{bc} | \Omega \rangle
$$

$\Phi^{ab}(0,t)$ is a Wilson line in the adjoint representation connecting $(t,\mathbf{0})$ with $(0,\mathbf{0})$.

The chromoelectric correlators $\mathcal{E}^{ij}$ and $\mathcal{E}$

For a suitable choice of ℓ^0, the fields in $gE^{e,j}(t')\Phi^{ec}(0;t')\Phi_\ell^{bc}$ are time ordered and those in $\Phi_\ell^{\dagger ab}\Phi^{\dagger ad}(0;t)gE^{d,i}(t)$ are anti-time ordered.
Hence the correlator $\mathcal{E}^{ij}$ may be interpreted as a cut diagram:



For polarization-summed cross sections or for production of scalar states only the isotropic part of $\mathcal{E}^{ij}$ is relevant. This is the dimensionless gluonic correlator $\mathcal{E}$:

$$\mathcal{E} = \frac{3}{N_c}\int_0^\infty dt\, t \int_0^\infty dt'\, t'\, \langle\Omega|\Phi_\ell^{\dagger ab}\Phi^{\dagger ad}(0;t)gE^{d,i}(t)gE^{e,i}(t')\Phi^{ec}(0;t')\Phi_\ell^{bc}|\Omega\rangle$$

LDMEs in pNRQCD

The pNRQCD factorization formulas for P-wave quarkonium hadroproduction LDMEs are

$$
\begin{aligned}
\langle\Omega|\mathcal{O}^{h_Q}({}^1P_1^{[1]})|\Omega\rangle &= 3\frac{3N_c}{2\pi}|R'(0)|^2 \\
\langle\Omega|\mathcal{O}^{h_Q}({}^1S_0^{[8]})|\Omega\rangle &= 3\frac{3N_c}{2\pi}|R'(0)|^2\frac{1}{9N_c m^2}\mathcal{E} \\
\langle\Omega|\mathcal{O}^{\chi_{QJ}}({}^3P_J^{[1]})|\Omega\rangle &= (2J+1)\frac{3N_c}{2\pi}|R'(0)|^2 \\
\langle\Omega|\mathcal{O}^{\chi_{QJ}}({}^3S_1^{[8]})|\Omega\rangle &= (2J+1)\frac{3N_c}{2\pi}|R'(0)|^2\frac{1}{9N_c m^2}\mathcal{E}
\end{aligned}
$$

LDMEs are polarization summed in the case of χ_{QJ} states.

The above expressions imply (at leading order in v) the universality of the ratios

$$
\frac{m^2\langle\Omega|\mathcal{O}^{\chi_{QJ}}({}^3S_1^{[8]})|\Omega\rangle}{\langle\Omega|\mathcal{O}^{\chi_{QJ}}({}^3P_J^{[1]})|\Omega\rangle} = \frac{m^2\langle\Omega|\mathcal{O}^{h_Q}({}^1S_0^{[8]})|\Omega\rangle}{\langle\Omega|\mathcal{O}^{h_Q}({}^1P_1^{[1]})|\Omega\rangle} = \frac{\mathcal{E}}{9N_c}
$$

Infrared divergences in NRQCD

For the pNRQCD expressions of the LDMEs to be consistent with perturbative QCD, they must reproduce the same infrared divergences. At two loop accuracy and at the lowest order in the relative momentum q of the Q and $\bar{Q}$, the infrared diverges in the NRQCD LDMEs can be cast in the infrared factor

$$\mathcal{I}_2(p,q) = \sum_N \int_0^\infty d\lambda'\, \lambda' \langle\Omega|\bar{\mathcal{T}}\left\{\Phi_\ell^{\dagger c'b}\Phi_p^{\dagger a'c'}(\lambda')[p^\mu q^\nu F^{a'}_{\nu\mu}(\lambda' p)]\right\}|N\rangle$$
$$\times \int_0^\infty d\lambda\, \lambda \langle N|\mathcal{T}\left\{\Phi_\ell^{bc}[p^\mu q^\nu F^{a}_{\nu\mu}(\lambda p)]\Phi_p^{ac}(\lambda)\right\}|\Omega\rangle$$

The sum over N contains all possible intermediate states, p is half the center-of-mass momentum of the $Q\bar{Q}$, and

$$\Phi_p(\lambda) = \mathcal{P}\exp\left[-ig\int_0^\lambda d\lambda'\, p\cdot A^{\rm adj}(\lambda' p)\right]$$

is an adjoint Wilson line along p.

- Nayak Qiu Sterman PLB 613 (2005) 45, PRD 72 (2005) 114012
  Nayak Qiu Sterman PRD 74 (2006) 074007

Consistency of pNRQCD with the NRQCD factorization

- Since in $\mathcal{I}_2(p,q)$ a momentum q comes from each side of the cut, the infrared factor contributes to the production of a color-singlet P-wave state.
- In the rest frame of the $Q\bar{Q}$: $\boldsymbol{p}=0$, $q^0=0$, $\Phi_p(\lambda)=\Phi(0,t)$ with $t=\sqrt{p^2}\lambda$, $p^\mu q^\nu F^a_{\nu\mu}(\lambda p)=-\sqrt{p^2}q^i E^{a\,i}(t)$ and $\mathcal{I}_2(p,q)$ can be written as

$$\mathcal{E}^{ij}\,\frac{q^i q^j}{p^2}$$

  Since this expression is proportional to the contact terms $V_{\mathcal{O}(^1S_0^{[8]})}$ and $V_{\mathcal{O}(^3S_1^{[8]})}$ in momentum space, the pNRQCD expressions for the color-octet LDMEs reproduce the same infrared divergences cast in the NRQCD infrared factor.
- The one-loop running of $\mathcal{E}$ ($C_F=(N_c^2-1)/(2N_c)$):

$$\frac{d}{d\log\Lambda}\mathcal{E}(\Lambda)=12C_F\frac{\alpha_{\mathrm{s}}}{\pi}$$

  implies $\frac{d}{d\log\Lambda}\langle\mathcal{O}^{\chi_{QJ}}(^3S_1^{[8]})\rangle=\frac{4C_F\alpha_{\mathrm{s}}}{3N_c\pi m^2}\langle\mathcal{O}^{\chi_{QJ}}(^3P_J^{[1]})\rangle$.
  This agrees with the one-loop evolution equation derived in perturbative NRQCD.

Chromoelectric correlator for hadroproduction

The correlator $\mathcal{E}$ can be fitted from the ratio

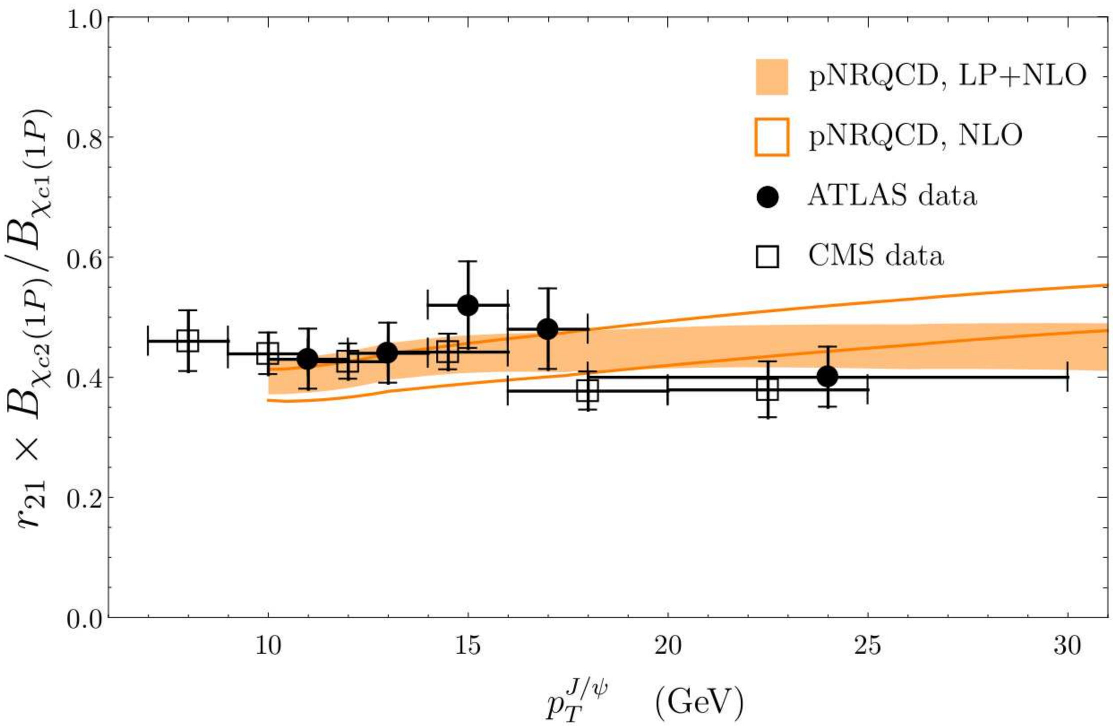
$$r_{21} = \frac{d\sigma_{\chi_{c2}(1P)}/dp_T}{d\sigma_{\chi_{c1}(1P)}/dp_T}$$

which does not depend (at leading order in v) on the wavefunction. One obtains

$$\mathcal{E}(\Lambda = 1.5 \text{ GeV}) = 2.8 \pm 1.7$$

The correlator is universal: it does not depend neither on the flavor of the heavy quark nor on the quarkonium state. The universal nature of the correlator allows to use it to compute cross sections for quarkonia with different principal quantum number and for bottomonia (once accounted for the running) without having to fit new octet LDMEs.

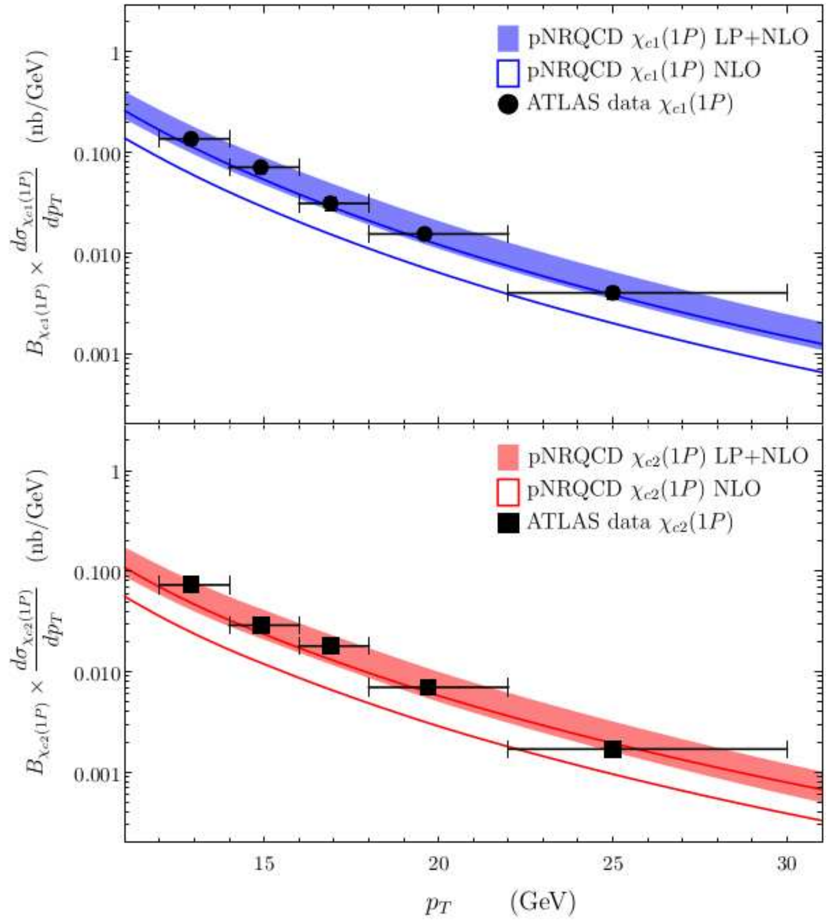
$(d\sigma_{\chi_{c2(1P)}}/dp_T)/(d\sigma_{\chi_{c1(1P)}}/dp_T)$



@ center of mass energy $\sqrt{s} = 7$ TeV and rapidity range $|y| <$0.75.

- CMS coll EPJC 72 (2012) 2251
  ATLAS coll JHEP 07 (2014) 154

$\sigma(pp \to \chi_{cJ}(1P) + X)$

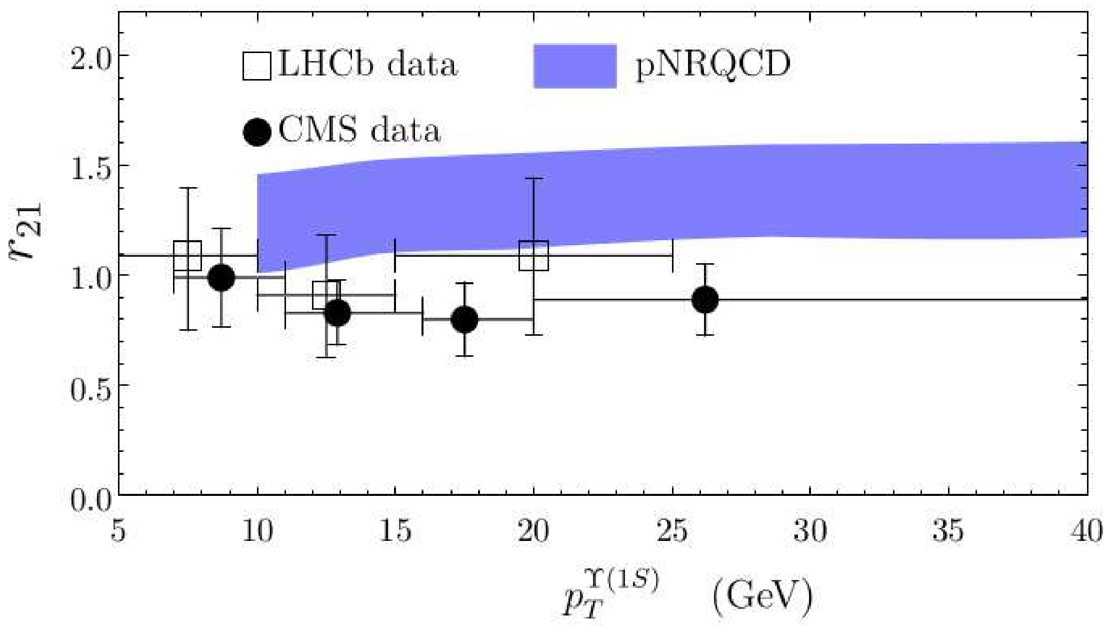


@ center of mass energy $\sqrt{s} = 7$ TeV and rapidity range $|y| < 0.75$.
Wavefunctions at the origin (at leading order in v) determined from $\Gamma(\chi_{c0,2}(1P) \to \gamma\gamma)$.

- ATLAS coll JHEP 07 (2014) 154

$(d\sigma_{\chi_{b2}(1P)}/dp_T)/(d\sigma_{\chi_{b1}(1P)}/dp_T)$

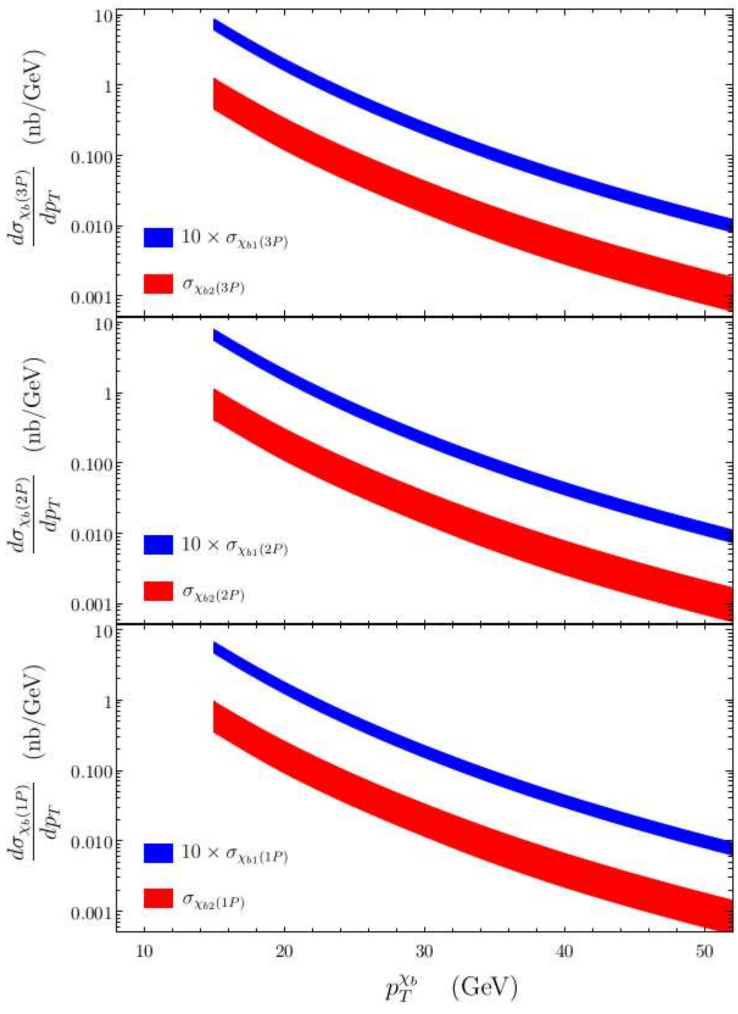
A test of the universality of the pNRQCD factorization is provided by the ratio $(d\sigma_{\chi_{b2}(1P)}/dp_T)/(d\sigma_{\chi_{b1}(1P)}/dp_T)$ that depends only on $\mathcal{E}$ (at the scale of the b mass) and therefore is expected to be the same also for $2P$ and $3P$ bottomonium states.



@ center of mass energy $\sqrt{s} = 7$ TeV and rapidity range $2 < y < 4.5$.

- LHCb coll EPJC 74 (2014) 3092
  CMS coll PLB 743 (2015) 383

$\sigma(pp \to \chi_{bJ}(1P) + X)$

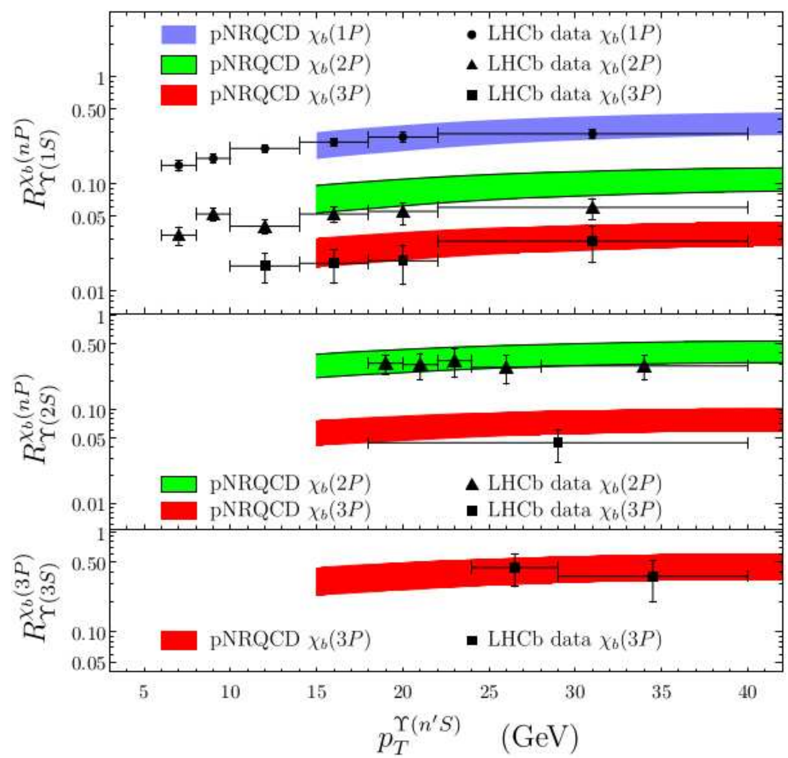


@ center of mass energy $\sqrt{s} = 7$ TeV and rapidity range $2 < y < 4.5$.
Wavefunctions at the origin (at leading order in v) determined from models.

$\chi_{bJ}(nP)$ feeddown fractions

Feeddown fractions, $R^{\chi_b(nP)}_{\Upsilon(n'S)} = \dfrac{\sum_{J=1,2} \mathrm{Br}(\chi_{bJ}(nP) \to \Upsilon(n'S) + \gamma)\, \sigma_{\chi_{bJ}(nP)}}{\sigma_{\Upsilon(n'S)}}$,

are model dependent in the χ_{bJ} wavefunctions and in some Br.



@ center of mass energy $\sqrt{s} = 7$ TeV and rapidity range $2 < y < 4.5$.

- LHCb coll EPJC 74 (2014) 3092

Conclusions

After 30 years ...

... NRQCD and the nonrelativistic EFTs that have originated from it (pNRQCD, BOEFT, and others) have become the framework to address quarkonium physics in all its aspects:

- precision determinations of masses, splittings, widths, transitions for the charmonium and bottomonium lowest states;
- determinations of the long range potentials for quarkonium, hybrids, tetraquarks;
- computation of splitting and transitions for hybrids and other quarkonium exotics;
- computation of charmonium and borttomonium production cross sections;
- determination of quarkonium suppression observables in heavy-ion collisions;
- ...

Outlook on quarkonium annihilation and production

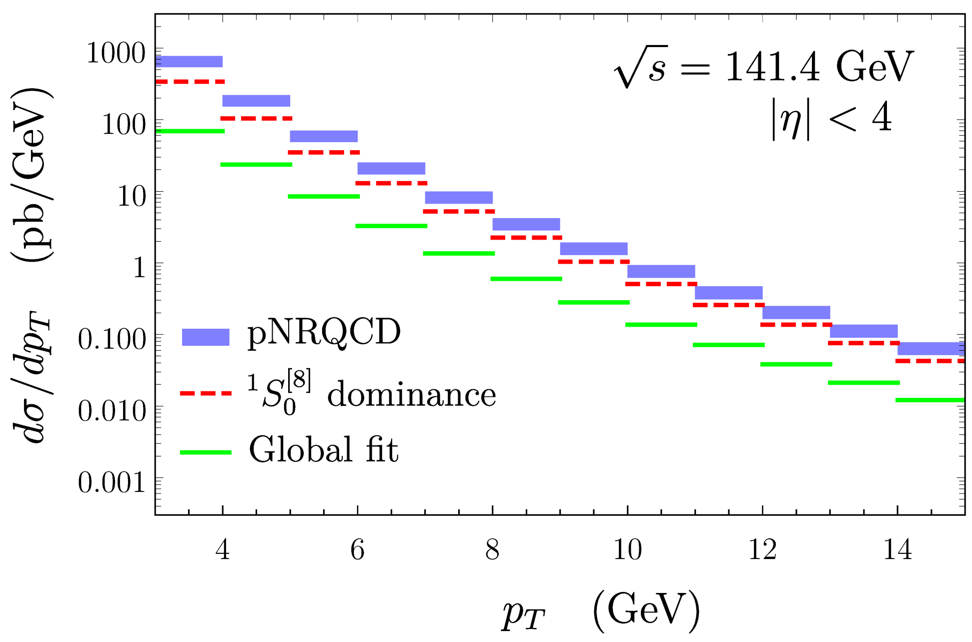
Theoretical predictions could significantly benefit from

- computation of (integrals of) gauge field correlators in lattice QCD,
- model independent determinations of the bottomonium wavefunctions at the origin. This may require new data, e.g. for P-wave bottomonium electromagnetic decays.

Possible developments include:

- Computation of higher order corrections in the velocity expansion. They come from higher dimensional operators in the NRQCD factorization formula, from higher order corrections to the pNRQCD expansion of the NRQCD long-distance matrix elements, and from higher order corrections to the wavefunctions originating from higher order corrections to the pNRQCD potential.
- Extension of the formalism to quarkonium exotica (hybrids, tetraquarks) and to quarkonium production in electron-ion and heavy ion colliders.

$\sigma(eh \to J/\psi + X)$



@ center of mass energy $\sqrt{s} = 7$ TeV.

- Feng Gong Chang Wang PRD 99 (2019) 014044 [$1S_0^{[8]}$ dominance]
  Butenschoen Kniehl PRD 84 (2011) 051501 [Global fit]