Factorial growth of perturbation theory, power corrections, and extraction of quark masses and α_s

Andreas S. Kronfeld
Fermilab & IAS TU München

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INSPIRE

Prototypical α_s Determination

- Consider an “effective charge” with a single hard scale:

$$\mathscr{R}(Q) = R(Q) + C_p \frac{\Lambda^p}{Q^p}$$

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"perturbative part"

perturbative series in
mass-independent scheme

$$R(Q) = \sum_{l=0} r_l(\mu/Q)\alpha_s(\mu)^{l+1}$$

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power $p \leftrightarrow$ factorial growth

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power p ↔ factorial growth

perturbative series in
mass-independent scheme

$$R(Q) = \sum_{l=0} r_l(\mu/Q)\,\alpha_s(\mu)^{l+1}$$

- Perturbative part and power correction inseparable.

Factorial Growth

- Even in quantum mechanics, high orders of perturbation theory grow factorially [*e.g.*, Bender & Wu 1971, 1973].

- Also in QFT [*e.g.*, Gross & Neveu 1974, Lautrup 1977].

- In pQCD, r_l grow factorially (known for a long time):

$$r_l \sim R_0^{(p)} \left(\frac{2\beta_0}{p}\right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)} \equiv R_l^{(p)}$$

for $l \gg 1$. Here $b = \beta_1/2\beta_0^2 \overset{n_f=3}{=} 32/81 \approx 0.4$.

- Does $r_l = \{1, 1.38, 5.46, 26.7\}$ start growing by $l = 3$?

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Examples

- Static energy = energy between two static sources, $p = 1$:
  - its Fourier transform, $p > 1$;
  - its derivative, the "static force" ($p \geq 9$);
- Bjorken sum rule, $p \in \{2, 4, 6, \ldots\}$.
- Quark mass, $p \in \{1, 2, 3, \ldots\}$.
- Adler function, $p \in \{4, 6, 8, \ldots\}$.

Outline

- Introduction
- Power Corrections and Factorial Growth
- New Approximation for Perturbative Series
- Borel Summation
- Worked Example: Static Energy
- Two or More Power Corrections
- Conclusions & Outlook

Power Corrections and Factorial Growth

Summary of Math in arXiv:2310.151137 [in JHEP]

- Use some simple steps and the RGE (which connects μ independence of $R(Q)$ to Q dependence of $R(Q)$—
  - obtain a more slowly growing set of coefficients, $f_k^{(p)}$.
- Invert an infinite matrix (lower triangular).
- Simplify and clarify "minimal renormalon subtraction (MRS)" of arXiv:1701.00347 and arXiv:1712.04983 [Komijani].

Main New Result

- Exact result (“=” not “~”):

$$r_l = \left(\frac{2\beta_0}{p}\right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)} \sum_{k=0}^{l-1} (k+1) \frac{\Gamma(1+pb)}{\Gamma(k+2+pb)} \left(\frac{p}{2\beta_0}\right)^k f_k^{(p)} + f_l^{(p)}$$

- In some problems, the $f_k^{(p)}$ grow, but more slowly (i.e., same formula with $p'>p$).

- Another result is generalization to cascade of powers.

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$$r_l = \underbrace{\left(\frac{2\beta_0}{p}\right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)}}_{\text{well-known growth}} \sum_{k=0}^{l-1} (k+1) \frac{\Gamma(1+pb)}{\Gamma(k+2+pb)} \left(\frac{p}{2\beta_0}\right)^k f_k^{(p)} + f_l^{(p)}$$

- In some problems, the $f_k^{(p)}$ grow, but more slowly (i.e., same formula with $p' > p$).

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well-known growth | Komijani R_0 (with finite # of terms) | extra

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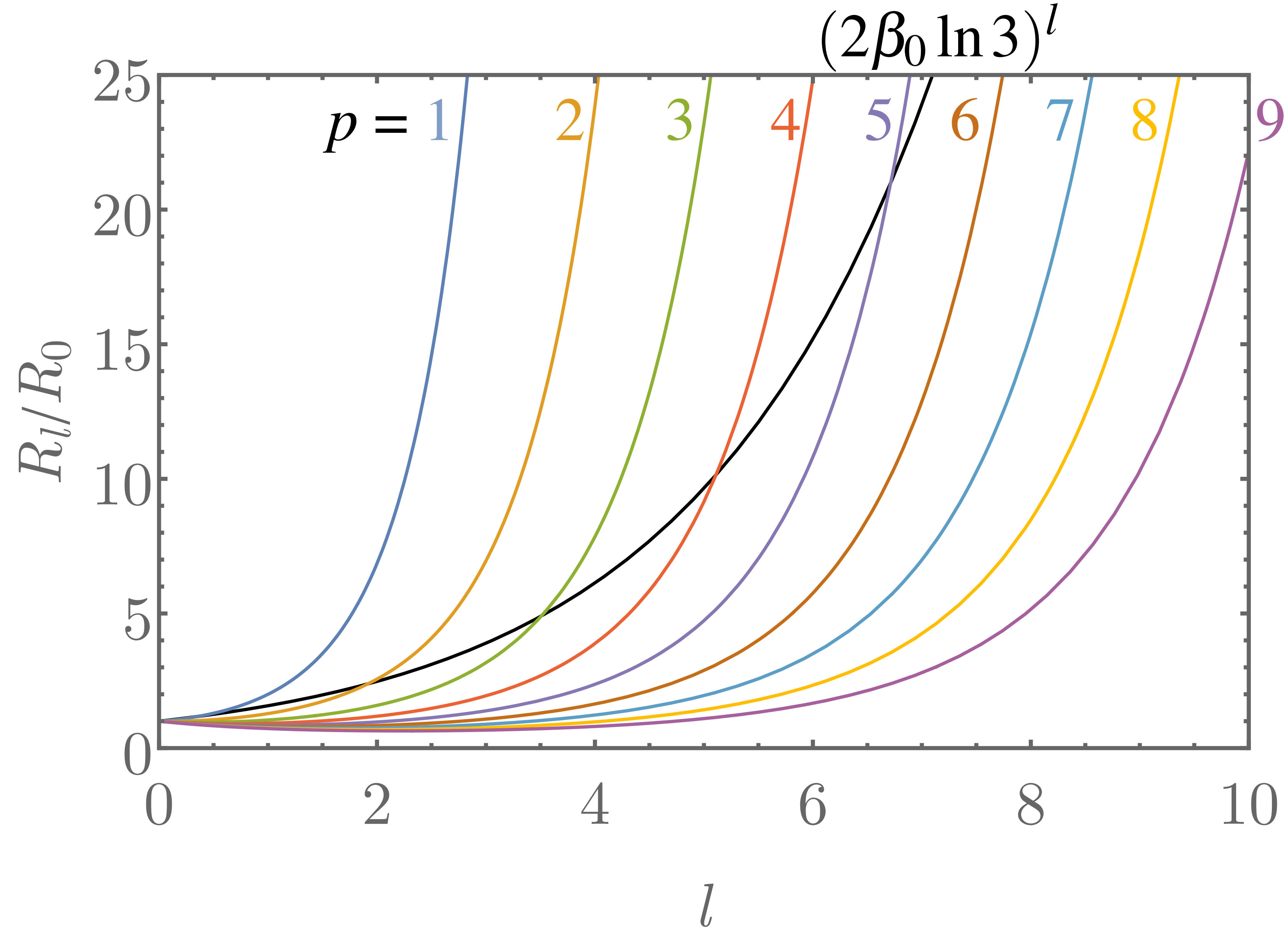
We will use this form for a resummation.

- In some problems, the $f_k^{(p)}$ grow, but more slowly (i.e., same formula with $p'>p$).
- Another result is generalization to cascade of powers.

Growth ↔ Power

- Larger p ⇒ growth takes over at larger l.



New Approximation for Perturbative Series

Perturbative Series

- In practice, the r_l are in the literature for $l < L$.
- The f_l, $l < L$, are obtained from them, and the formula returns these r_l (as it must).
- For $l \geq L$, the formula tells us (formally) the largest part:

$$r_l = \underbrace{\left(\frac{2\beta_0}{p}\right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)}}_{\text{well-known growth}} \underbrace{\sum_{k=0}^{l-1} (k+1) \frac{\Gamma(1+pb)}{\Gamma(k+2+pb)} \left(\frac{p}{2\beta_0}\right)^k f_k^{(p)}}_{\text{Komijani } R_0 \text{ (truncated)}} \underbrace{+ f_l^{(p)}}_{\text{drop}}$$

use the approximate formula for the uncalculated terms.

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- For $l \geq L$, the formula tells us (formally) the largest part:

$$r_l \approx \underbrace{\left(\frac{2\beta_0}{p}\right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)}}_{\text{well-known growth}} \underbrace{\sum_{k=0}^{L-1} (k+1) \frac{\Gamma(1+pb)}{\Gamma(k+2+pb)} \left(\frac{p}{2\beta_0}\right)^k f_k^{(p)}}_{\text{Komijani } R_0 \text{ (truncated)}}$$

use the approximate formula for the uncalculated terms.

Recap & Compendium

- That means $\sum_{l=0}^{\infty} r_l \alpha_s^{l+1} \to \sum_{l=0}^{L-1} r_l \alpha_s^{l+1} + \sum_{l=L}^{\infty} R_l^{(p)} \alpha_s^{l+1}$

  with

$$R_l^{(p)} \equiv R_0^{(p)} \left(\frac{2\beta_0}{p}\right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)}$$

$$R_0^{(p)} \equiv \sum_{k=0}^{L-1} (k+1) \frac{\Gamma(1+pb)}{\Gamma(k+2+pb)} \left(\frac{p}{2\beta_0}\right)^k f_k^{(p)}$$

- Justified because the retained terms are formally larger than the ones omitted.

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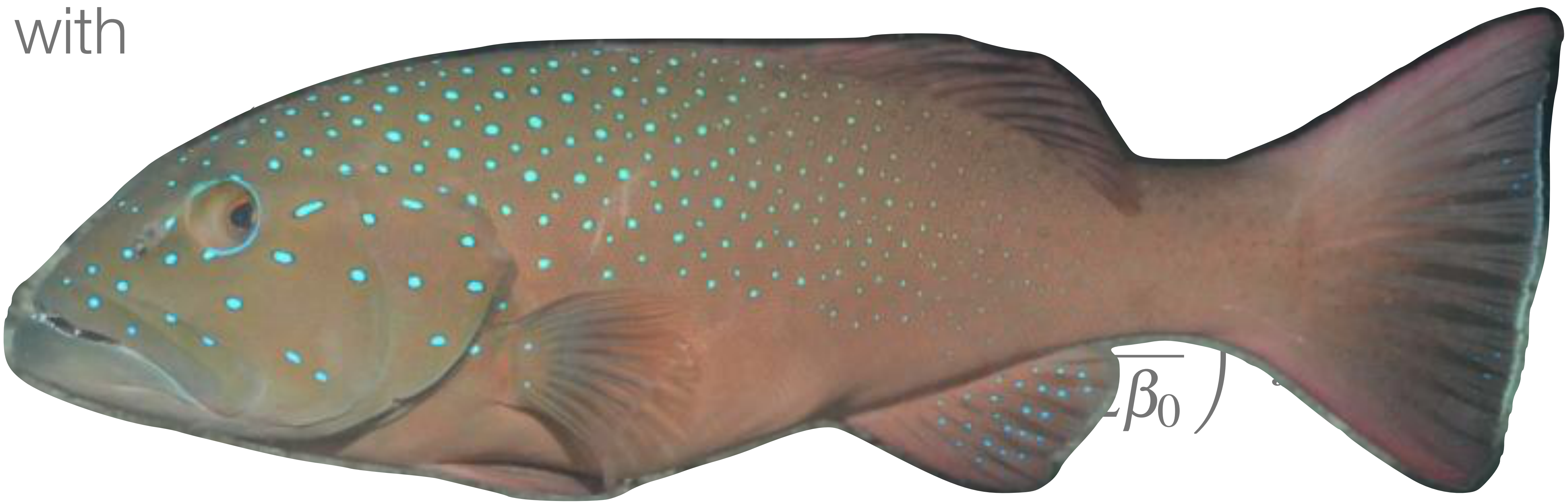

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Borel Summation

Rearrange and React

- We have

$$R(Q) = \sum_{l=0}^{\infty} r_l \alpha_s^{l+1} \to \sum_{l=0}^{L-1} r_l \alpha_s^{l+1} + \sum_{l=L}^{\infty} R_l^{(p)} \alpha_s^{l+1}$$

$$= \underbrace{\sum_{l=0}^{L-1} \left(r_l - R_l^{(p)} \right) \alpha_s^{l+1}}_{R_{\mathrm{RS}}^{(p)}(Q)} + \underbrace{\sum_{l=0}^{\infty} R_l^{(p)} \alpha_s^{l+1}}_{R_{\mathrm{B}}^{(p)}(Q)}$$

- The “renormalon subtracted” part and the “Borel” part.

- The R_l from above yield divergent sum for R_{B}, but we’re not done yet: use Borel summation to assign meaning.

Assignment

- Thus, we now define

$$R_{\mathrm{B}}^{(p)}(Q) = R_0^{(p)} \frac{p}{2\beta_0} \mathscr{J}(pb, p/2\beta_0 \alpha_{\mathrm{g}}(Q))$$

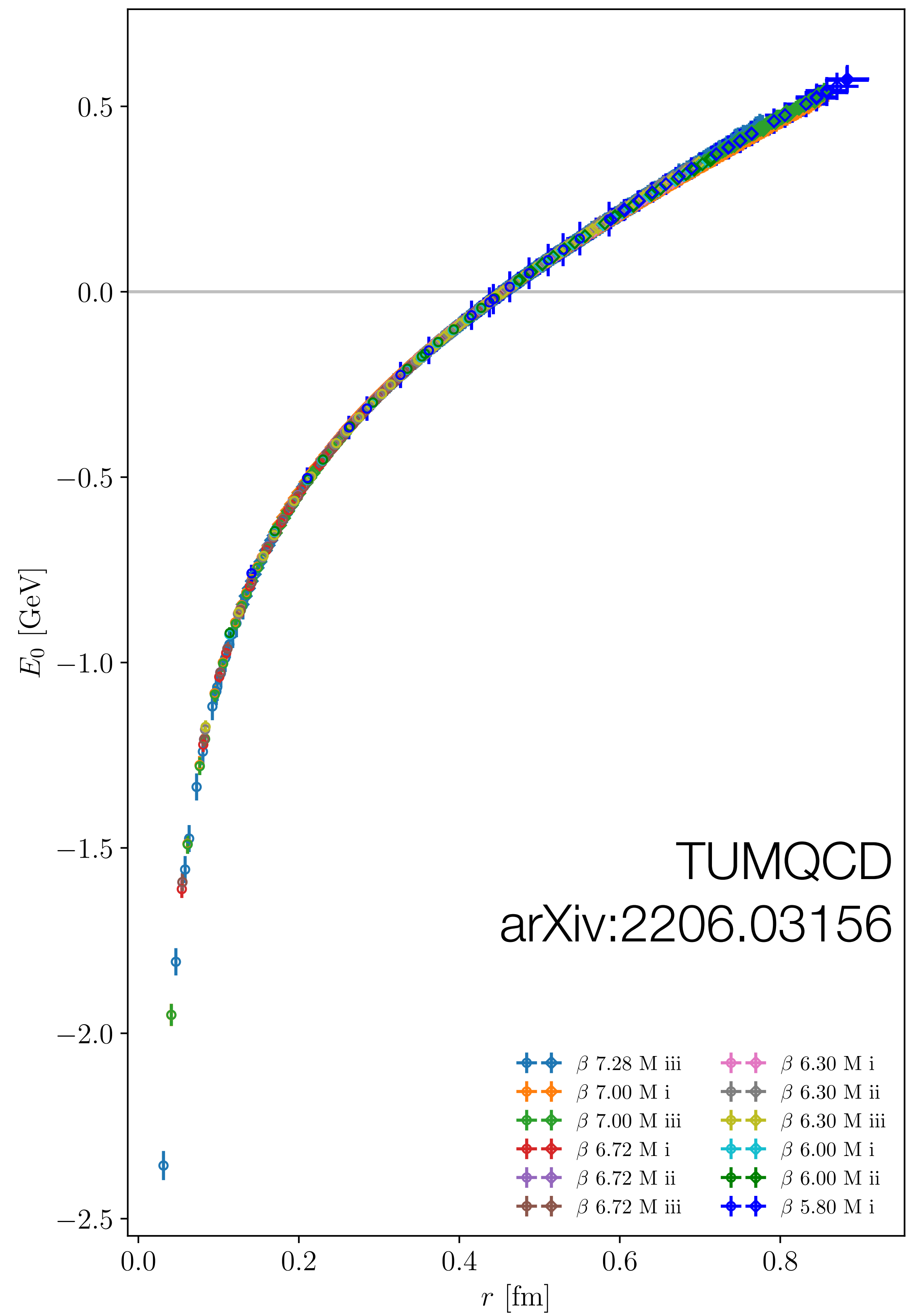
$$\mathscr{J}(c,y) = \mathrm{e}^{-y}\Gamma(-c)\gamma^\star(-c,-y)$$

where $\gamma^\star(a,x)$ is an analytic function of both a and x:

limiting function of the incomplete gamma function

- convergent expansion in $x = -1/2\beta_0\alpha_{\mathrm{g}}$;
- asymptotic expansion in α_{g} regenerates the starting point; the dropped term is $\mathrm{O}(\mathrm{e}^{-p/2\beta_0\alpha_{\mathrm{g}}})$.

Static Energy



Static Energy

- Quantity extracted from oblong Wilson loops:
  - perturbative potential has IR divergences starting at 3 loops [Appelquist, Dine, Muzinich 1978];
  - compensated by multipole (retardation) term [Brambilla, Pineda, Soto, Vairo 1999, 2000].
- Perturbative series:

$$E_0(r) = -\frac{C_F}{r}\sum_{l=0} v_l(\mu r)\alpha_s(\mu)^{l+1} + \Lambda_0$$

- In notation used above, $Q \to 1/r$, $\mathscr{R}(1/r) = -rE_0(r)/C_F$.

Related Quantities

- Perturbation theory carried out in momentum space:

$$\tilde{R}(q) = \sum_{l=0} a_l(\mu/q)\alpha_s(\mu)^{l+1}$$

- Leading power/factorial comes from Fourier transform, so $\tilde{R}(q)$ has $p > 1$.

- The "static force"

$$\mathfrak{F}(r) = -\frac{\mathrm{d}E_0}{\mathrm{d}r} \qquad \mathscr{F}(r) = F^{(1)}(1/r) = -r^2\mathfrak{F}(r)/C_F$$

has no power corrections (until instantons at $p \geq 9$).

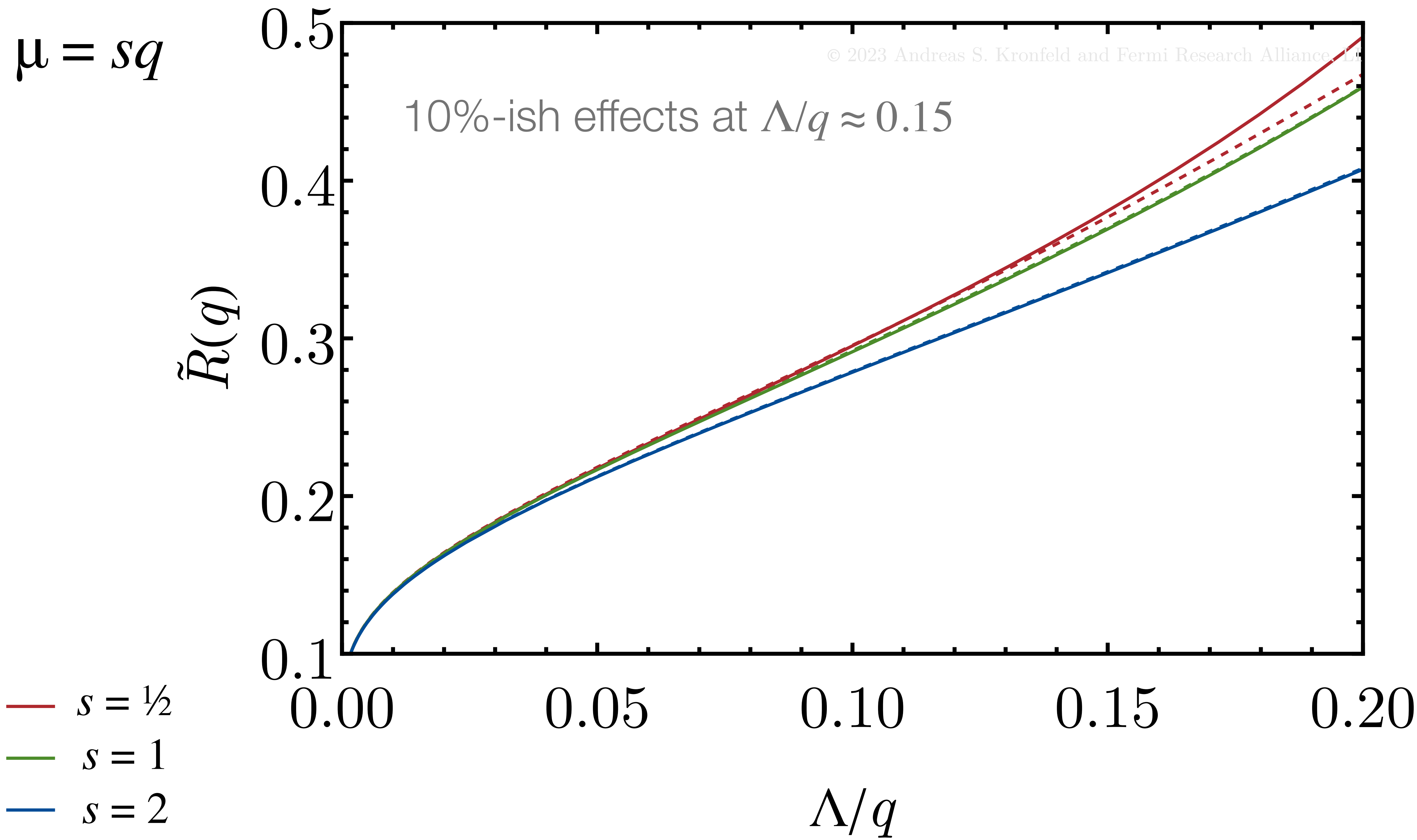
Coefficients at $\mu = 1/r$ or $\mu = q$

l	$\overline{\text{MS}}$ $a_l(1)$	$\overline{\text{MS}}$ $f_l(1)$	geometric $a_l(1)$	geometric $f_l(1)$	α_2 $a_l(1)$	α_2 $f_l(1)$
0	1	1	1	1	1	1
1	0.557042	−0.048552	0.557042	−0.048552	0.557042	−0.048552
2	1.70218	0.687291	1.83497	0.820079	1.83497	0.820079
3	2.43687	0.323257	2.83268	0.558242	3.01389	0.739452

l	$\overline{\text{MS}}$ $v_l(1)$	$\overline{\text{MS}}$ $v_l(1) - V_l(1)$	geometric $v_l(1)$	geometric $v_l(1) - V_l(1)$	α_2 $v_l(1)$	α_2 $v_l(1) - V_l(1)$
0	1	0.206061	1	0.182531	1	0.177584
1	1.38384	−0.202668	1.38384	−0.249689	1.38384	−0.259574
2	5.46228	0.019479	5.59507	−0.009046	5.59507	−0.042959
3	26.6880	0.219262	27.3034	0.050179	27.4846	0.066468

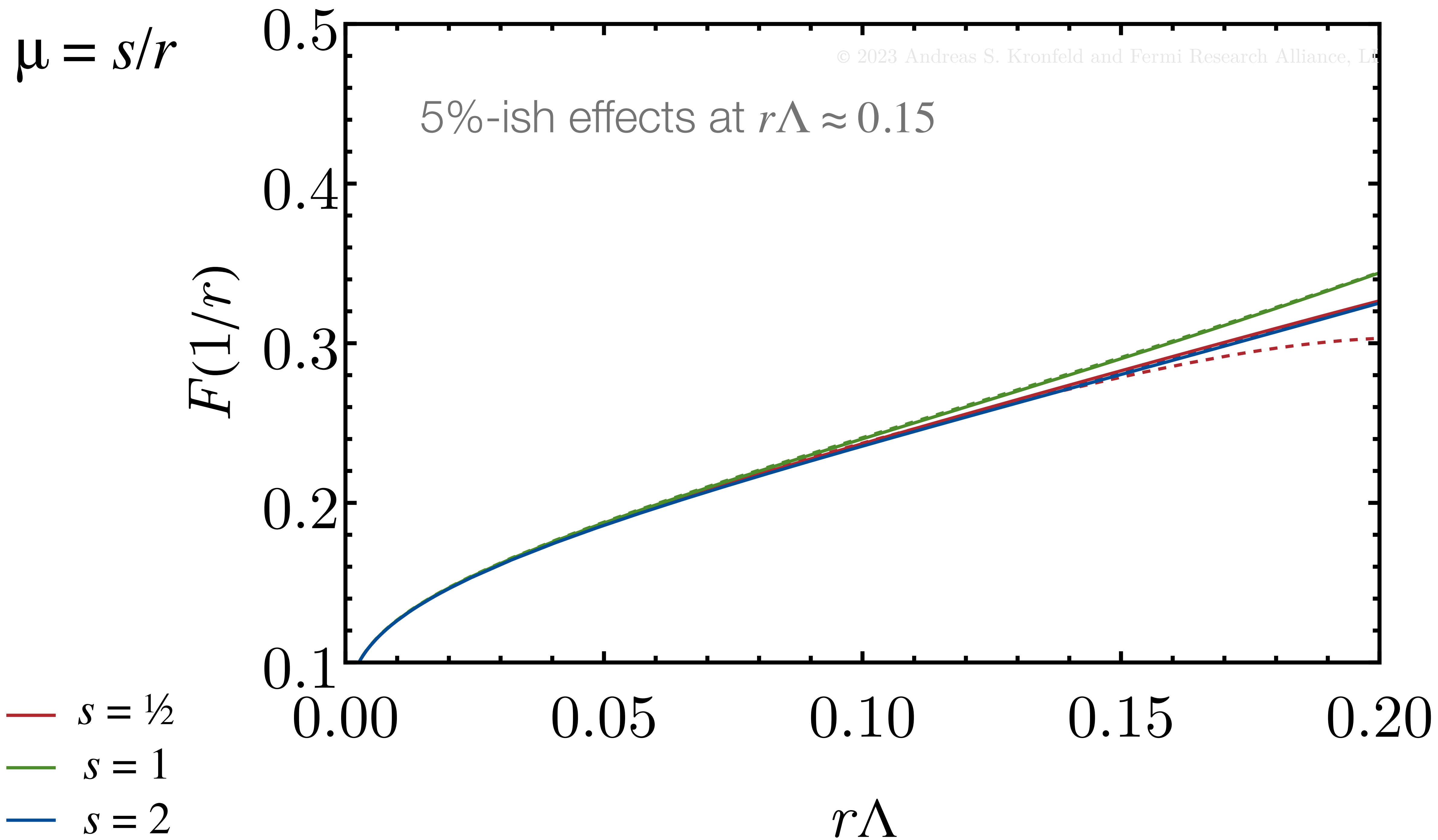
Good Series (at most $p > 1$ growth)

$\mu = sq$



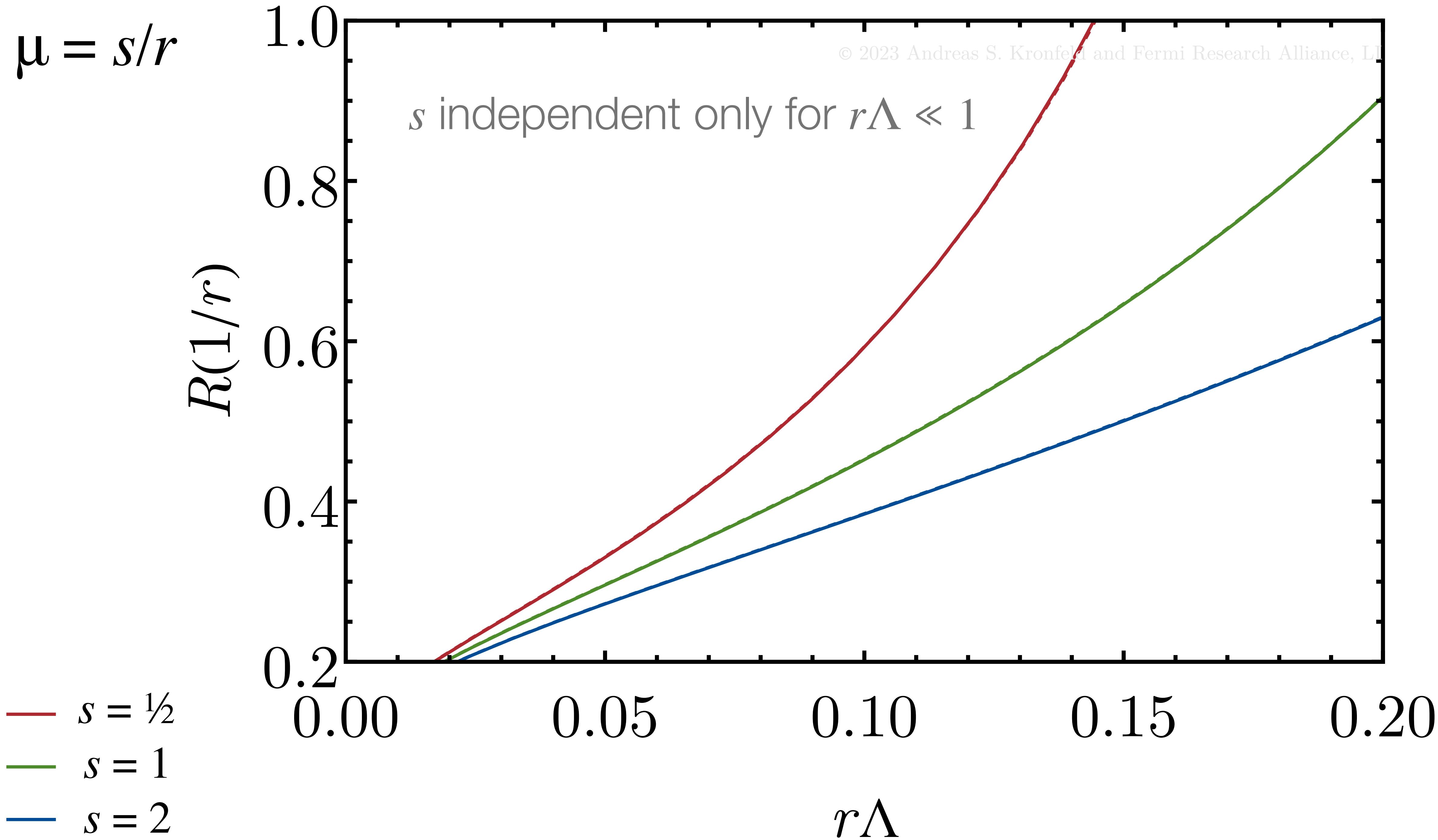
Great Series (instanton power $p \geq 9$)

$\mu = s/r$



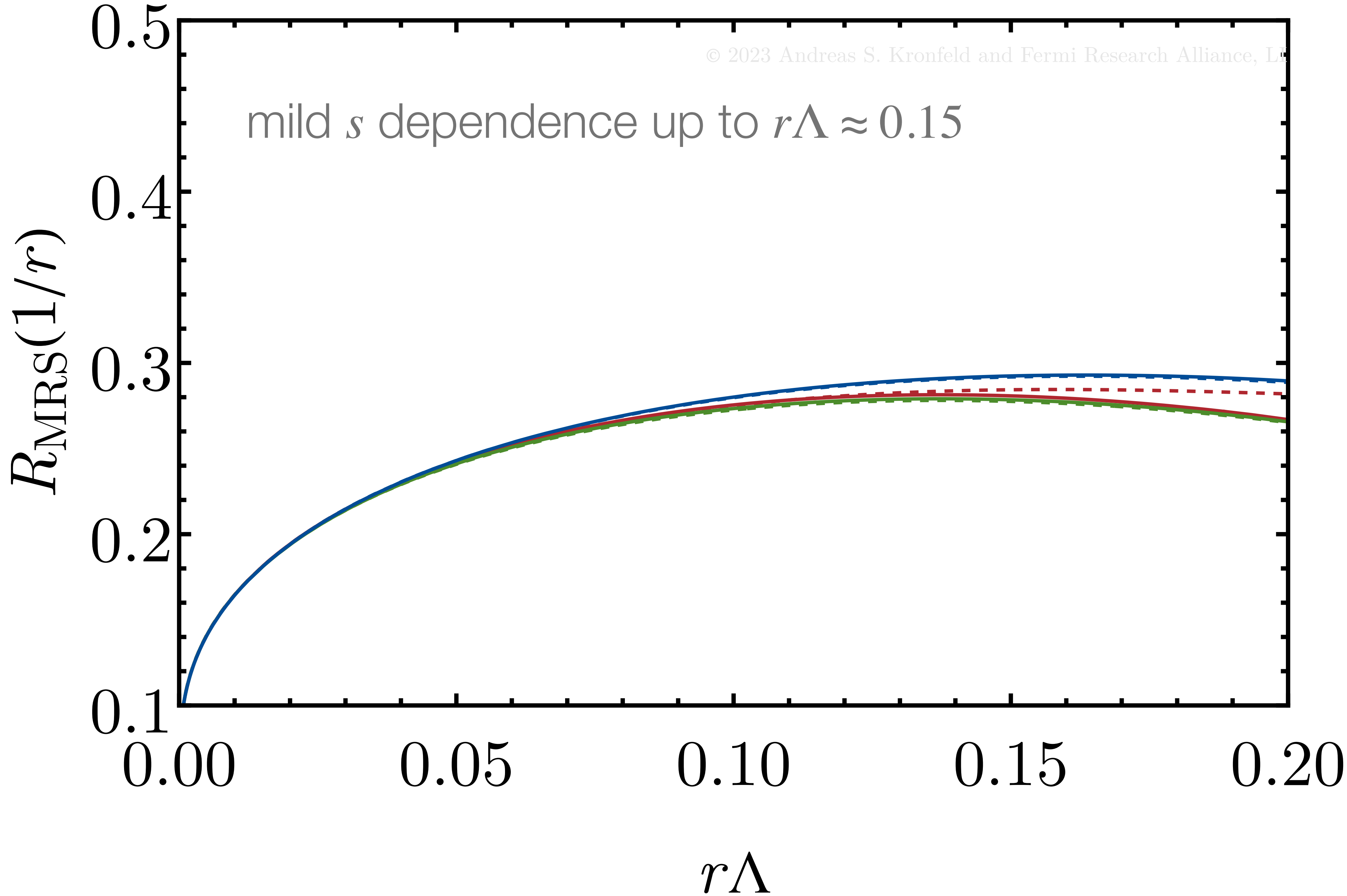
Horrible Series ($p = 1$)

$\mu = s/r$



MRS Series

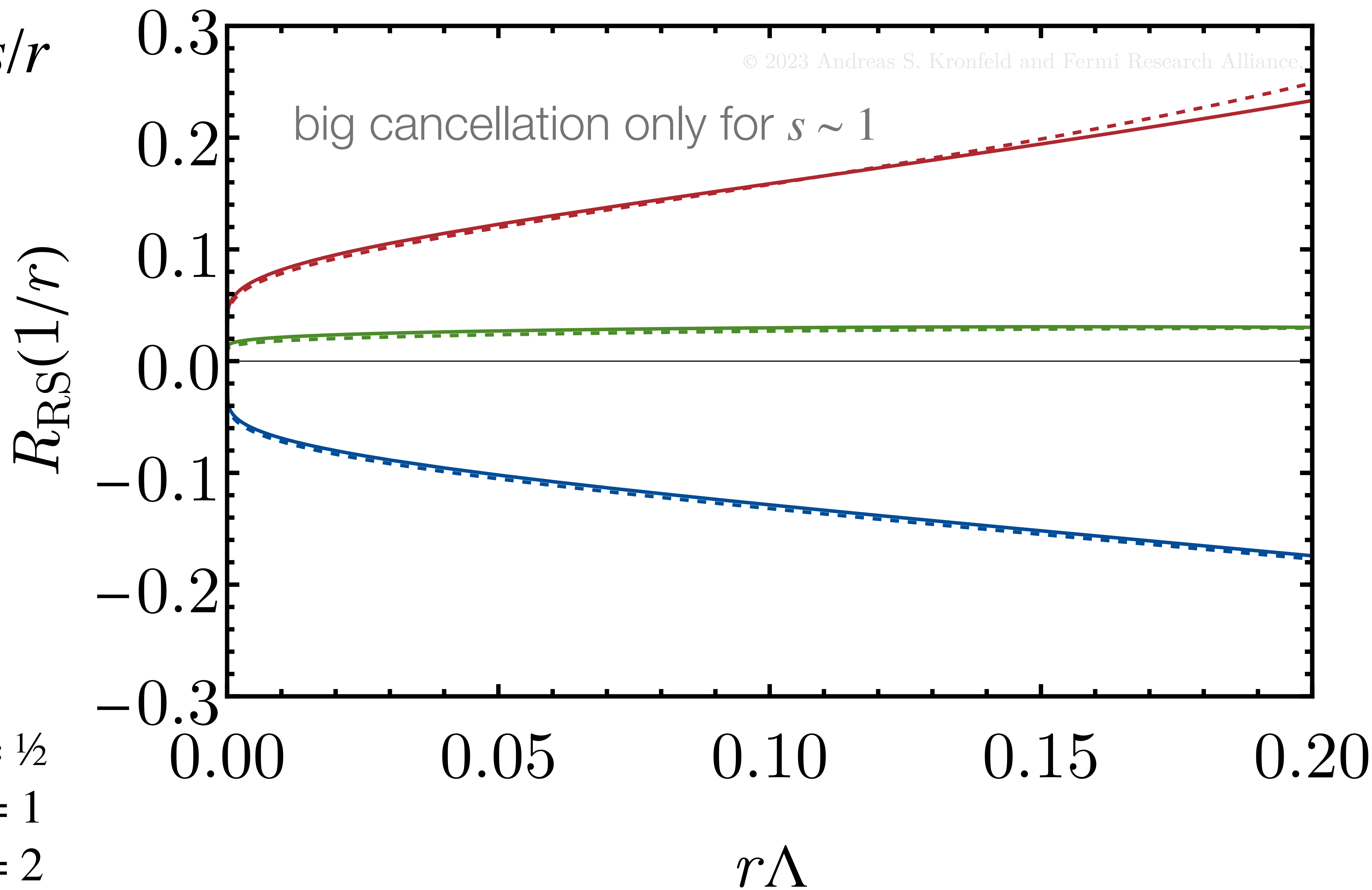
$\mu = s/r$



$s = ½$
$s = 1$
$s = 2$

Renormalon Subtracted Series

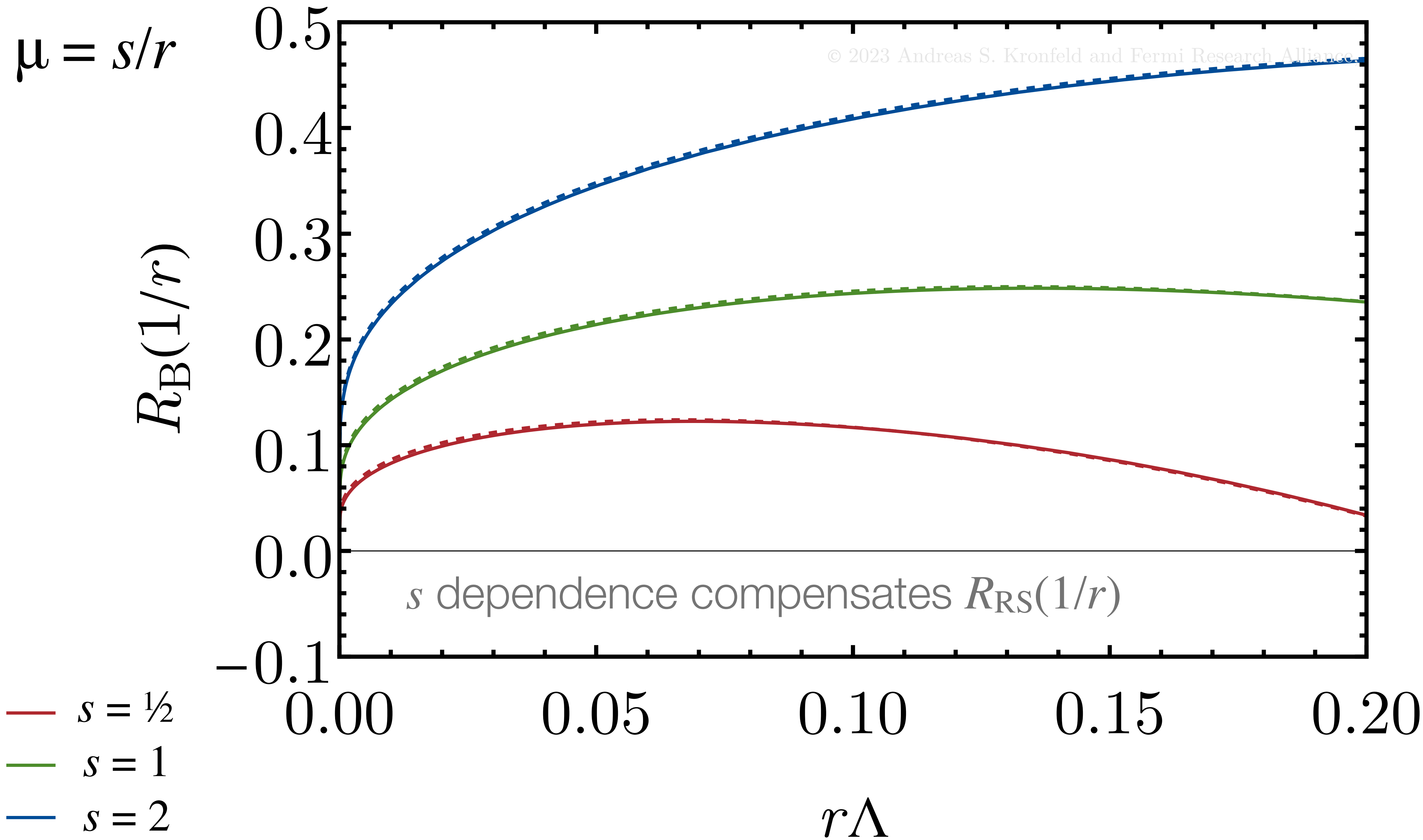
$\mu = s/r$



$s = ½$
$s = 1$
$s = 2$

The part that is a convergent series in $1/\alpha_s$

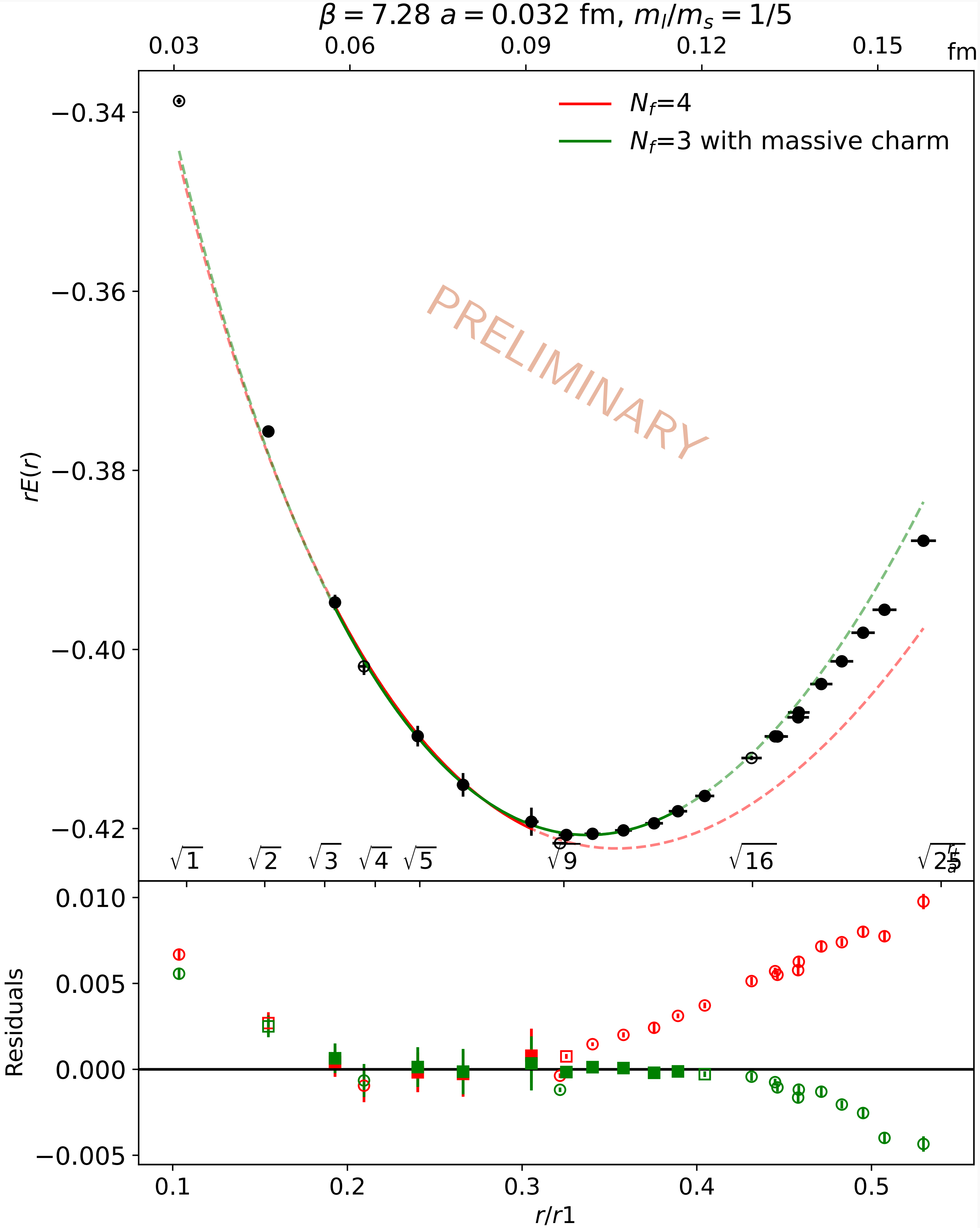
$\mu = s/r$



Fitting with Power Corrections

- The Λ on the horizontal axis is $\Lambda_{\overline{\text{MS}}}$—
  - fits to data will have this as free parameter, i.e., optimization will stretch/shrink the curves to fit.
- Let's go back to the plots and get a feel for adding small amounts of order $(\Lambda/q)^{2 \text{ or } 3 \text{ or } 4}$, $(\Lambda r)^9$, or Λr.
- Disentangling power-law and logarithmic dependence seems hard for $R(1/r)$, but not for $R_{\text{MRS}}(1/r)$.

★



- Start fits from $r/a = \sqrt{3}$
- From TUMQCD2019
  PT works up to $\sim 0.13 fm$
- Charm effects noticeable already at $r > 0.1 fm$
- Charm effects:
  limit to 2-loop accuracy
- Drop on-axis points due to large discretization effects
- Model average (AIC) over valid fit ranges
- Correlated fits,
  blocked jackknife

← Example: Finest ensemble, 2-loops no us-resum., MRS

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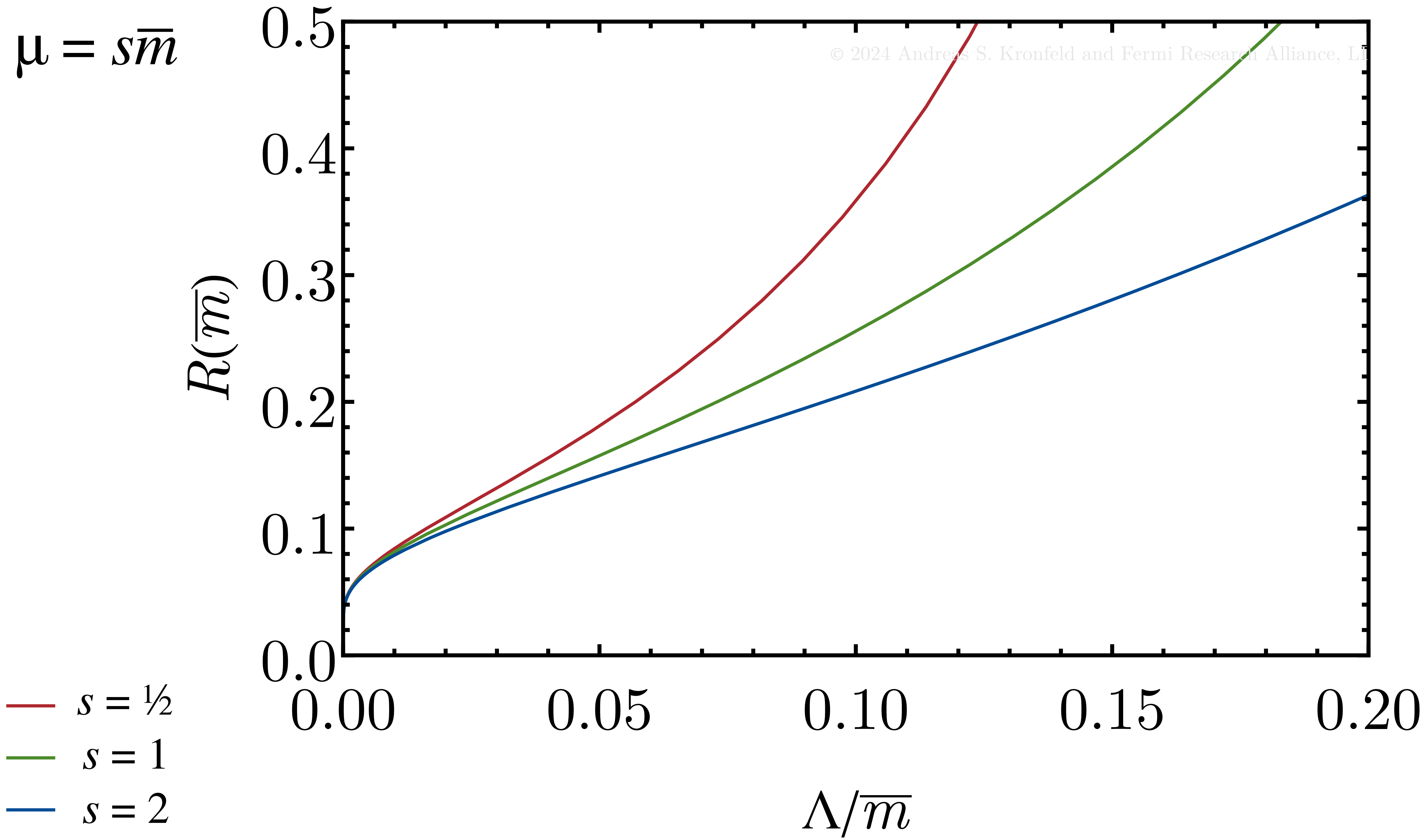
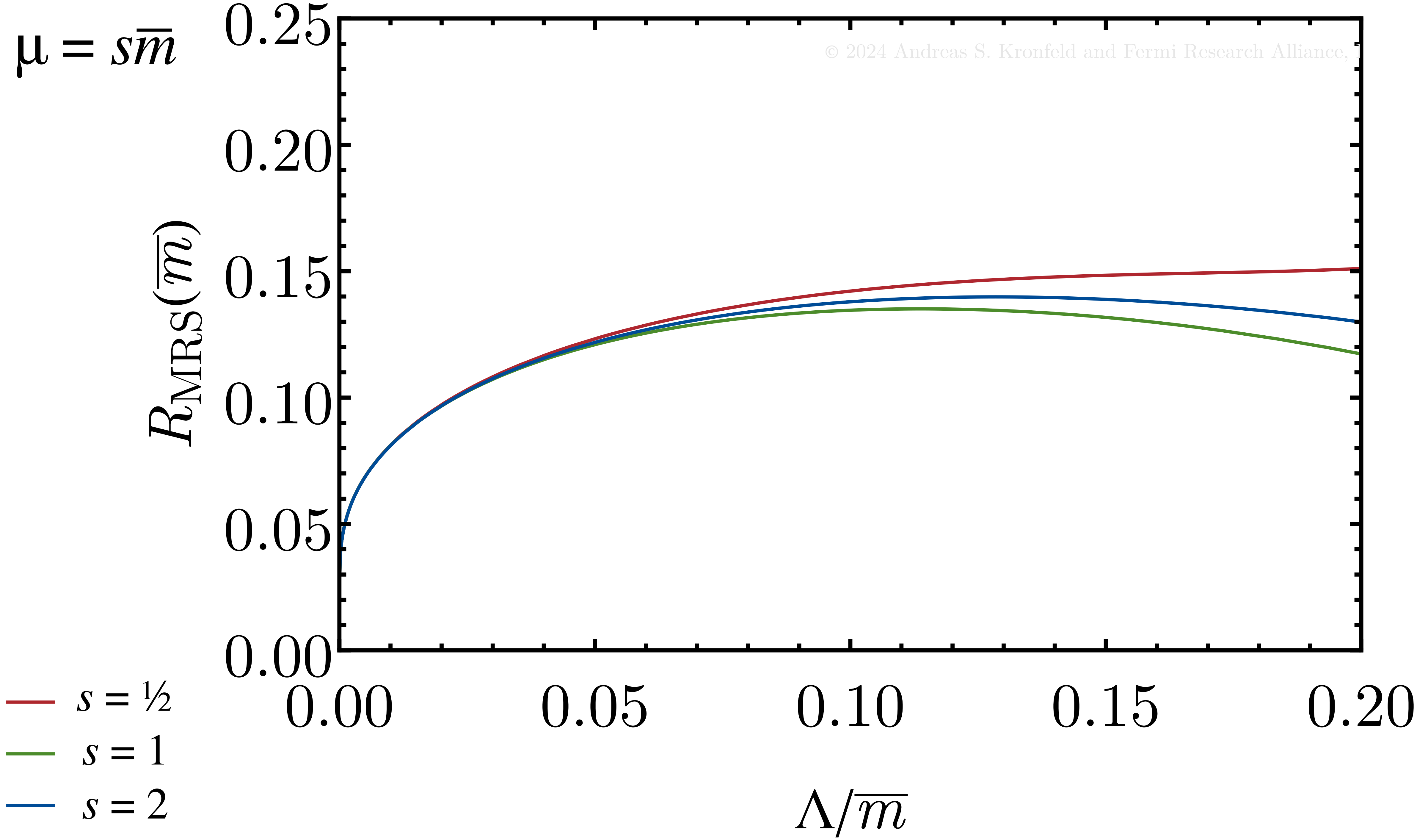
Pole Mass's Horrible Series ($p = 1, 2, 3, \ldots$)

$\mu = s\overline{m}$



Pole Mass's MRS Series



Quark Mass Results

arXiv:1802.04248

- Masses in numerical form:

$$m_{l,\overline{\mathrm{MS}}}(2\ \mathrm{GeV}) = 3.402(15)_{\mathrm{stat}}(05)_{\mathrm{syst}}(19)_{\alpha_s}(04)_{f_\pi,\mathrm{PDG}}\ \mathrm{MeV}$$

$$m_{u,\overline{\mathrm{MS}}}(2\ \mathrm{GeV}) = 2.130(18)_{\mathrm{stat}}(35)_{\mathrm{syst}}(12)_{\alpha_s}(03)_{f_\pi,\mathrm{PDG}}\ \mathrm{MeV}$$

$$m_{d,\overline{\mathrm{MS}}}(2\ \mathrm{GeV}) = 4.675(30)_{\mathrm{stat}}(39)_{\mathrm{syst}}(26)_{\alpha_s}(06)_{f_\pi,\mathrm{PDG}}\ \mathrm{MeV}$$

$$m_{s,\overline{\mathrm{MS}}}(2\ \mathrm{GeV}) = 92.47(39)_{\mathrm{stat}}(18)_{\mathrm{syst}}(52)_{\alpha_s}(11)_{f_\pi,\mathrm{PDG}}\ \mathrm{MeV}$$

$$m_{c,\overline{\mathrm{MS}}}(3\ \mathrm{GeV}) = 983.7(4.3)_{\mathrm{stat}}(1.4)_{\mathrm{syst}}(3.3)_{\alpha_s}(0.5)_{f_\pi,\mathrm{PDG}}\ \mathrm{MeV}$$

$$m_{b,\overline{\mathrm{MS}}}(m_{b,\overline{\mathrm{MS}}}) = 4201(12)_{\mathrm{stat}}(1)_{\mathrm{syst}}(8)_{\alpha_s}(1)_{f_\pi,\mathrm{PDG}}\ \mathrm{MeV}$$

- Mass ratios:

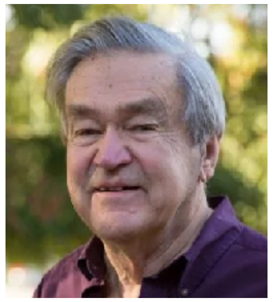
$$m_c/m_s = 11.783(11)_{\mathrm{stat}}(21)_{\mathrm{syst}}(00)_{\alpha_s}(08)_{f_\pi,\mathrm{PDG}}$$

$$m_b/m_s = 53.94(6)_{\mathrm{stat}}(10)_{\mathrm{syst}}(1)_{\alpha_s}(5)_{f_\pi,\mathrm{PDG}}$$

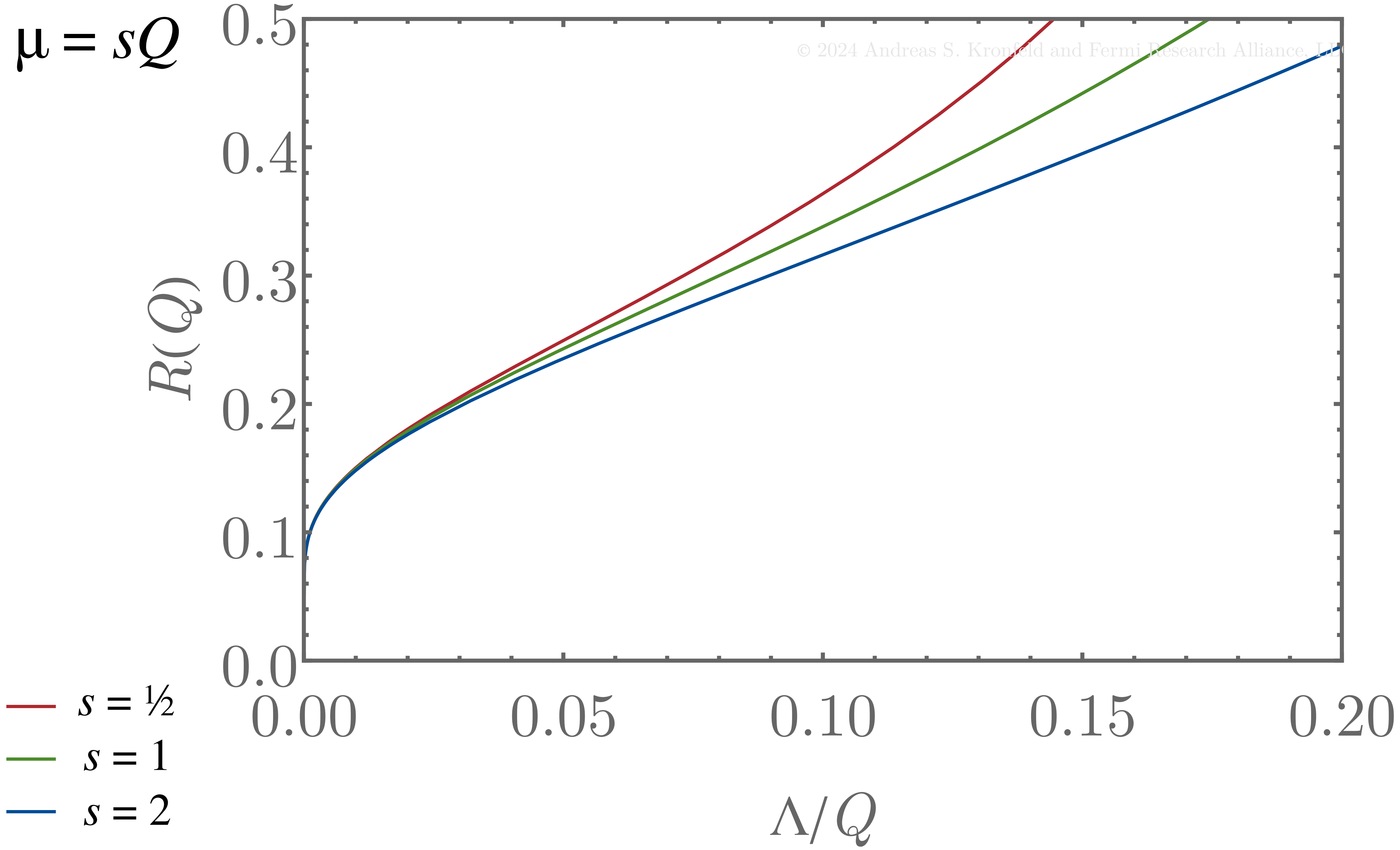
$$m_b/m_c = 4.578(5)_{\mathrm{stat}}(6)_{\mathrm{syst}}(0)_{\alpha_s}(1)_{f_\pi,\mathrm{PDG}}$$

Two or More Power Corrections

Bjorken Sum Rule's Horrible Series ($p = 2$)

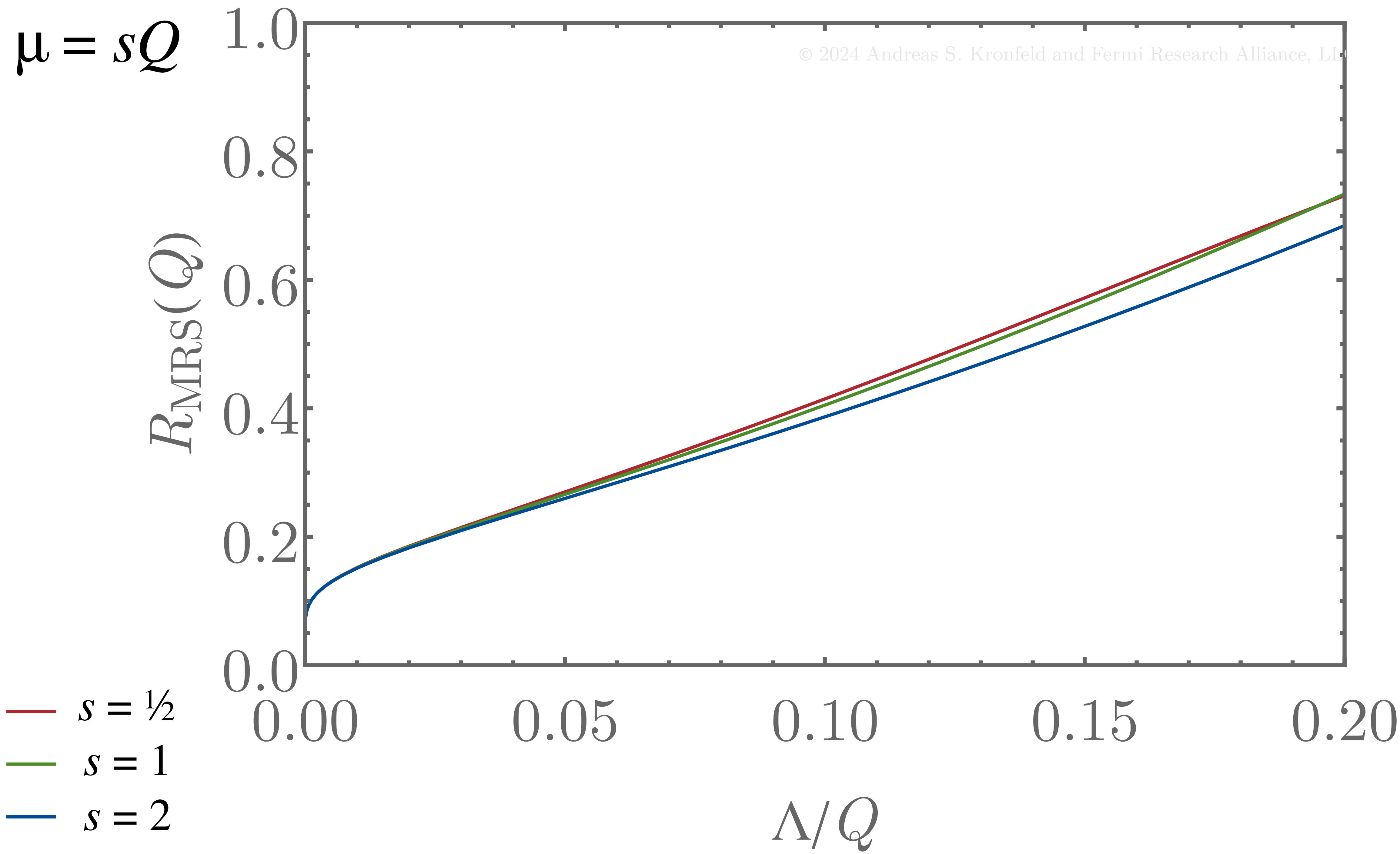


$\mu = sQ$



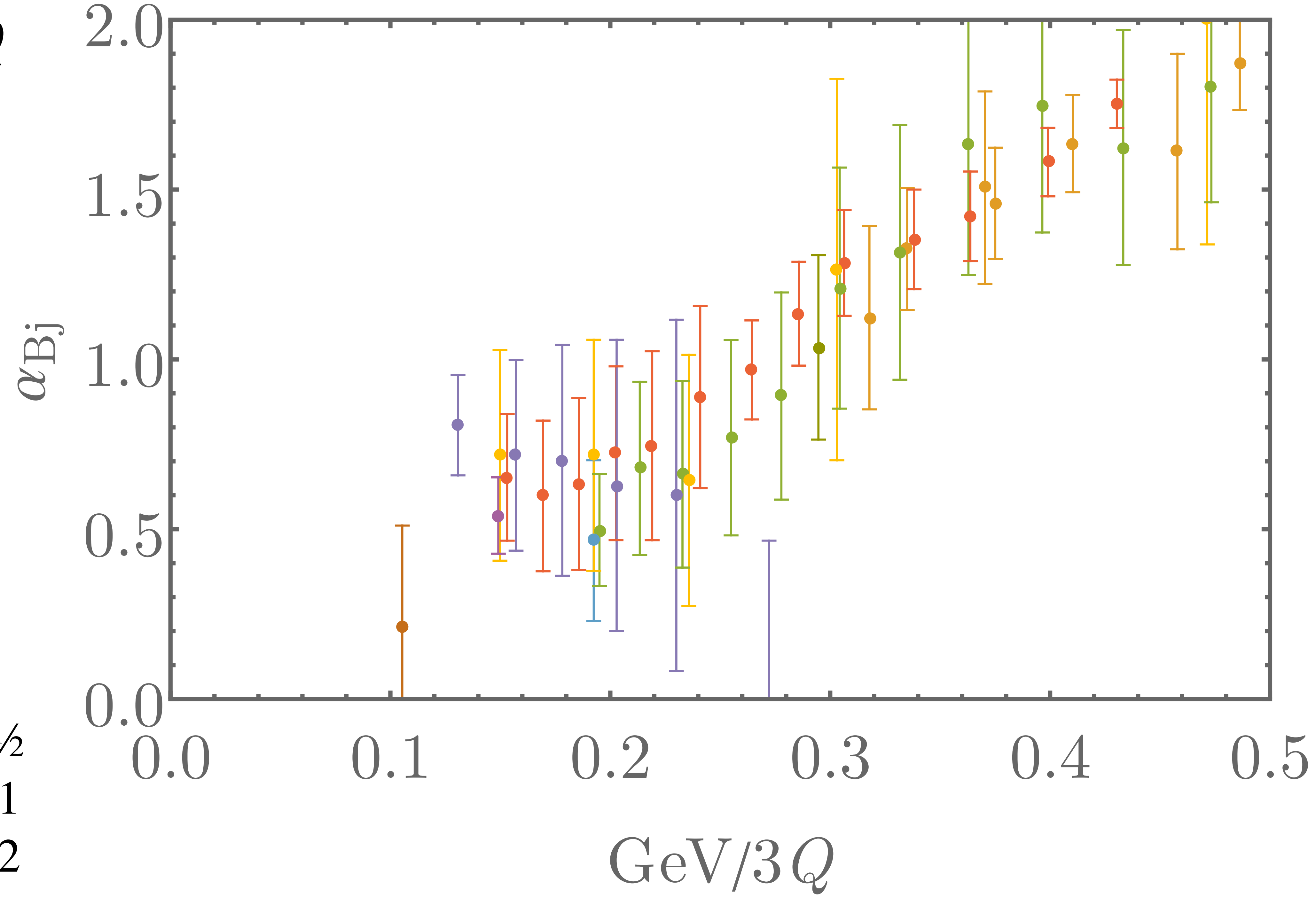
Bjorken Sum Rule's MRS Series

$\mu = sQ$



Bjorken Sum Rule Experimental Data

μ = Q



s = ½
s = 1
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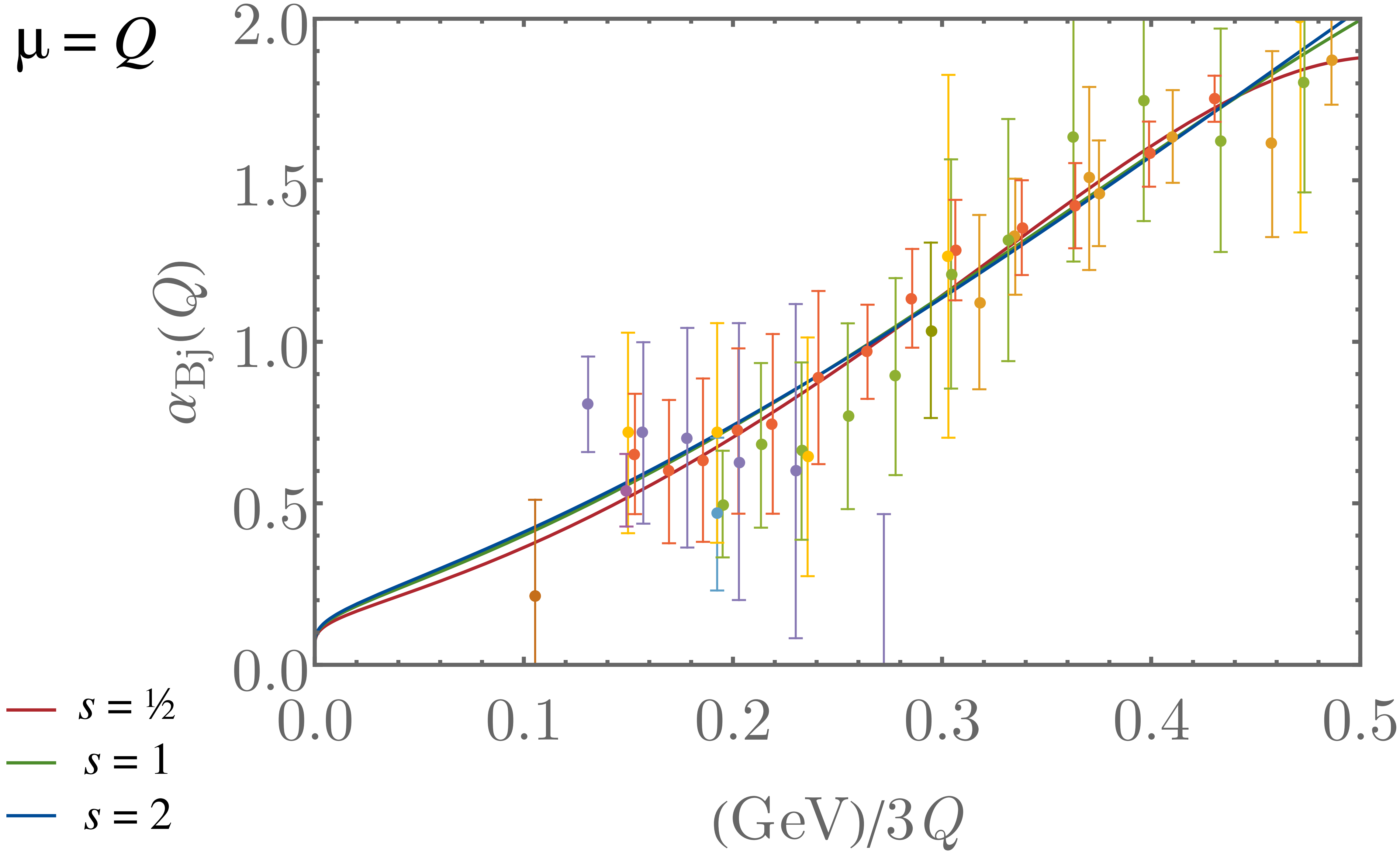
Bjorken Sum Rule Two-Parameter Fits

μ = Q



Bjorken Sum Rule Two-Parameter Fits

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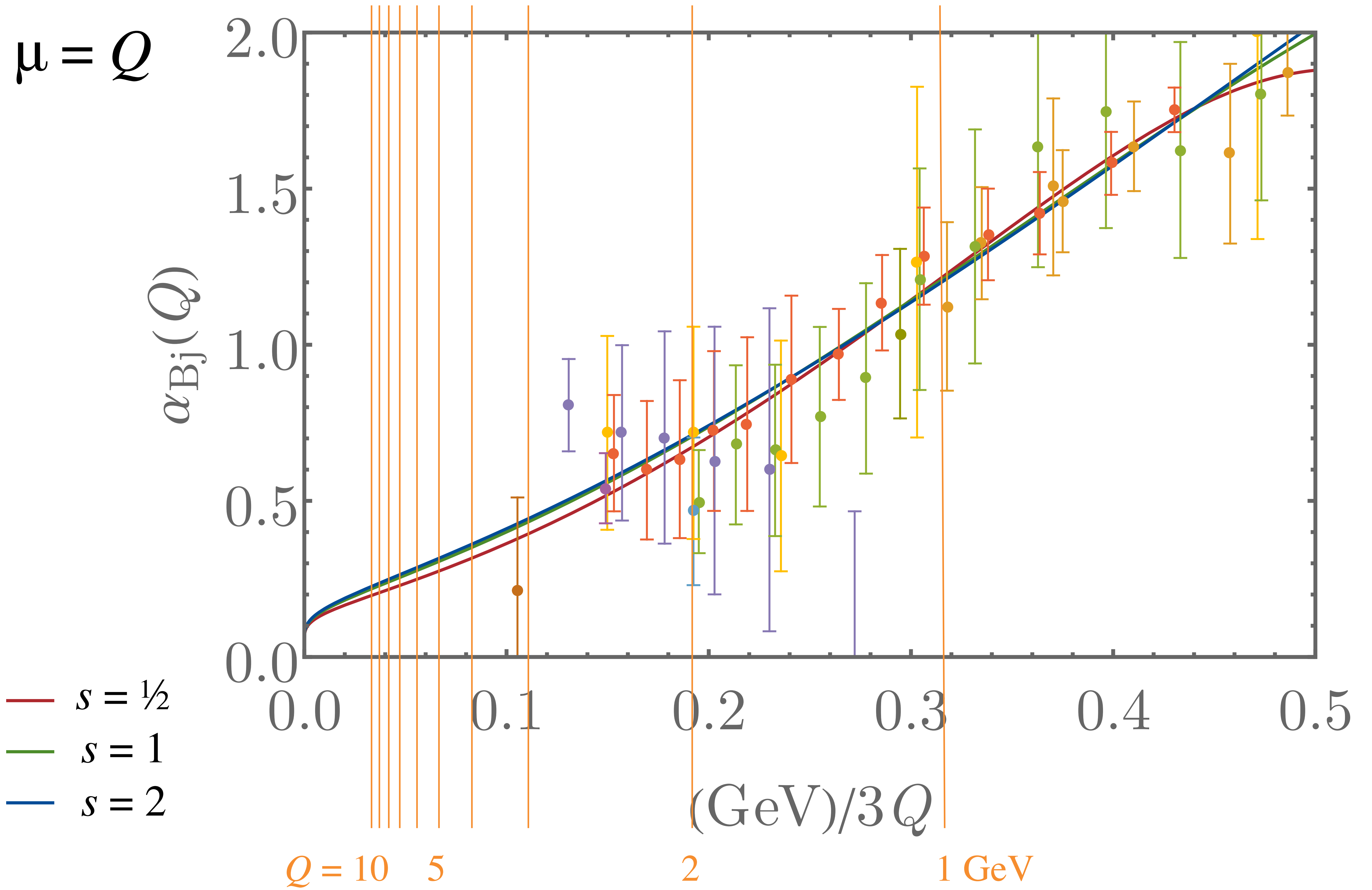

Summary

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- MRS revisited for any sequence of power corrections ↔ dominant, subdominant, sub-subdominant, … growth.
- Formulas for growth and normalization both follow from RGE and hold exactly at low orders.
- Standard to sum logarithms; let's sum factorials too—
    - reduction or elimination of truncation uncertainty!
- Better name needed: "renormalons" are not subtracted, but a class of (now) known contributions is summed.

Thank you for your attention

Questions?