# Heavy Quarks and the QCD Coupling

Mattia Dalla Brida, Roman Höllwieser, Francesco Knechtli,
Tomasz Korzec, Alberto Ramos, Stefan Sint, Rainer Sommer

ALPHA
Collaboration

# Motivation

- The strong coupling $\alpha_s$
  - The magnitude of the strong coupling between quarks and gluons is essential for our quantitative understanding of all high energy processes
  - Traditionally $\alpha_s$ is extracted from various high energy experiments
- Determination from low energy experiments
  e.g. hadron masses and decay constants
  - $\overline{\text{MS}}$ scheme does not exist beyond perturbation theory
  - Low energy inputs $\rightarrow \mu_{\text{had}}$ will be small in $\alpha(\mu_{\text{had}})$
  - Lattice QCD works at all energies
    $\rightarrow$ use this tool to evolve $\alpha(\mu_{\text{had}}) \rightarrow \alpha(\mu_{\text{PT}})$ non-perturbatively

## Outline

- Non-perturbative $\beta$-functions from the lattice
- New method exploiting the decoupling of heavy quarks
- Recent improvements

# Lambda Parameter

- Renormalized couplings $\alpha(\mu) \equiv \frac{\overline{g}^2(\mu)}{4\pi}$ depend on renormalization scale $\mu$
- Dependence is described by $\beta$ function
  $\mu \frac{\partial \overline{g}}{\partial \mu} = \beta(\overline{g})$
- In perturbation theory
  $\beta(g) \sim -g^3(b_0 + b_1 g^2 + b_2 g^4 + \ldots)$
- Integration of RG equation introduces the dimensionful $\Lambda$ parameter

$$\Lambda/\mu = \underbrace{(b_0\overline{g}^2)^{-b_1/(2{b_0}^2)}\; e^{-1/(2b_0\overline{g}^2)}\; \exp\left\{-\int_0^{\overline{g}} \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{{b_0}^2 x}\right] dx\right\}}_{\equiv \varphi(\overline{g})}$$

- Two values of the coupling, $\overline{g}_1$, $\overline{g}_2$ correspond to scale ratio
  $\frac{\mu_1}{\mu_2} = \frac{\varphi(\overline{g}_2)}{\varphi(\overline{g}_1)}$
- Special case: scale ratio of $\mu_1/\mu_2 = 2 \Rightarrow$ step-scaling function
  $\sigma(\overline{g}_1^2) = \overline{g}_2^2$
  Contains the same information as $\beta(g)$, but better suited for numerical methods

# Gradient Flow Coupling

- We need: Precise, renormalized, dimensionless quantity, accessible to lattice QCD
- Traditionally: Couplings based on the static potential or SF coupling

More precise, especially at low $\mu$: Gradient flow couplings

## Gradient Flow

Gradient flow $\sim$ (covariant) diffusion in "flow time" $t$

[M.F. Atiyah, R. Bott, Phil.Trans.Roy.Soc.Lond. A308 (1982)]

$$\begin{aligned} \partial_t B_\mu(t,x) &= D_\nu G_{\nu\mu}(t,x), \qquad B_\mu(0,x) = A_\mu(x) \\ D_\mu &= \partial_\mu + [B_\mu, \cdot] \\ G_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] \end{aligned}$$

- Correlators of $B$ at $t > 0$ need no renormalization

  [M. Lüscher, JHEP 1008 (2010)]

  [M. Lüscher and P. Weisz, JHEP 1102 (2011)]

# Gradient Flow Coupling

It has been quickly realized, that the “flowed” action density can serve as a renormalized coupling with $\mu = (8t)^{-1/2}$

[M. Lüscher JHEP 08 (2010) 071]

[R.Harlander, T. Neumann, JHEP 06 (2016)]

Finite size couplings are obtained, when $\mu$ is tied to $L$

[Z. Fodor, K. Holland, J. Kuti, D. Nogradi, and C. H. Wong, JHEP 1211 (2012)]

[P. Fritzsch, A. Ramos, JHEP 1310 (2013)]

Schrödinger Functional (SF) boundary conditions,

$$\bar{g}^2_{\mathrm{GF}}(\mu) = \mathcal{N}^{-1} \sum_{k,l=1}^{3} \left. \frac{t^2 \langle \mathrm{tr}\,\{G_{kl}(t,x) G_{kl}(t,x)\}\,\delta_{Q,0}\rangle}{\langle \delta_{Q,0}\rangle} \right|^{x_0=T/2,\; c=\sqrt{8t}/L}_{\mu=1/L, T=L,\, M=0}$$

- $G_{\mu\nu}$ field strength tensor at finite flow time $t$
- Different $c$ (e.g. 0.3) $\leftrightarrow$ different scheme
- Topological charge $Q$ restricted to $Q=0$ sector
- $\mathcal{N}$ known normalization factor
- $z = ML$ mass of the three degenerate quarks

# Gradient Flow Coupling

It has been quickly realized, that the "flowed" action density can serve as a renormalized coupling with $\mu = (8t)^{-1/2}$

[M. Lüscher JHEP 08 (2010) 071]

[R.Harlander, T. Neumann, JHEP 06 (2016)]

Finite size couplings are obtained, when $\mu$ is tied to $L$

[Z. Fodor, K. Holland, J. Kuti, D. Nogradi, and C. H. Wong, JHEP 1211 (2012)]

[P. Fritzsch, A. Ramos, JHEP 1310 (2013)]

Schrödinger Functional (SF) boundary conditions,

$$\bar{g}^2_{\mathrm{GFT}}(\mu, M) = \mathcal{N}^{-1} \sum_{k,l=1}^{3} \frac{t^2 \langle \mathrm{tr}\{G_{kl}(t,x) G_{kl}(t,x)\}\, \delta_{Q,0}\rangle}{\langle \delta_{Q,0}\rangle}\Bigg|^{x_0 = T/2,\, c=\sqrt{8t}/L}_{\mu=1/L, T=2L, M=z/L}$$

- $G_{\mu\nu}$ field strength tensor at finite flow time $t$
- Different $c$ (e.g. 0.3) $\leftrightarrow$ different scheme
- Topological charge $Q$ restricted to $Q = 0$ sector
- $\mathcal{N}$ known normalization factor
- $z = ML$ mass of the three degenerate quarks

# Strategy 2017

[ALPHA, Phys.Rev.Lett. 119 (2017) 10, Phys.Rev.Lett. 117 (2016) 18, Phys.Rev.D 95 (2017) 1, Eur.Phys.J.C 78 (2018) 5]

$$\Lambda^{(3)}_{\overline{\mathrm{MS}}} = \frac{1}{\sqrt{t_0^\star}} \times \sqrt{t_0^\star}\mu_{\mathrm{had}} \times \frac{\mu_0}{\mu_{\mathrm{had}}} \times \frac{\mu_{\mathrm{PT}}}{\mu_0} \times \varphi_{\mathrm{SF}}(\bar{g}_{\mathrm{SF}}(\mu_{\mathrm{PT}})) \times \frac{\Lambda^{(3)}_{\overline{\mathrm{MS}}}}{\Lambda^{(3)}_{\mathrm{SF}}}$$

- $\sqrt{8t_0^\star} = 0.413(5)(2)$ fm

  [M. Bruno, T.K., S. Schaefer, Phys.Rev.D 95 (2017) 7]

  "Scale setting", involves CLS large volume simulations to relate the technical scale $\sqrt{t_0^\star}$ to physical hadron masses and decay constants
- $\mu_{\mathrm{had}} \approx 197$ MeV

  Low energy scale, implicitly defined by $\bar{g}^2_{\mathrm{GF}}(\mu_{\mathrm{had}}) = 11.31$
- $\mu_0 \approx 4$ GeV

  Intermediate scale, implicitly defined by $\bar{g}^2_{\mathrm{SF}}(\mu_0) = 2.012$

  Scheme switch from GF to SF, $\bar{g}^2_{\mathrm{GF}}(\mu_0/2) = 2.6723(64)$
- $\mu_{\mathrm{PT}} = O(100\ \mathrm{GeV})$

  High energy scale at which perturbative $\varphi$ is reliable.
- $\varphi_{\mathrm{SF}}(\bar{g}_{\mathrm{SF}}(\mu_{\mathrm{PT}}))$

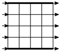
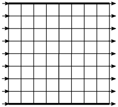
  Analytical 3-loop expression

  [A. Bode, P. Weisz, U. Wolff Nucl.Phys.B 576 (2000)]
- $\Lambda^{(3)}_{\mathrm{SF}}/\Lambda^{(3)}_{\overline{\mathrm{MS}}} = 0.38286(2)$

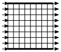
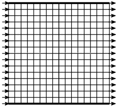
  M. Lüscher, R. Sommer, P. Weisz, U. Wolff, Nucl. Phys. B413, 481 (1994)]

# Computing Step Scaling Functions

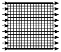
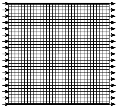
Instead of $\beta(\bar{g})$, compute: $\sigma(u) = \bar{g}^2(\mu/2)|_{u=\bar{g}^2(\mu)}$

$m_0^{(1)}, g_0^{(1)}$:  $\overset{\text{same } a^{(1)}}{\leftrightarrow}$  $= \Sigma(u, \frac{a^{(1)}}{L})$

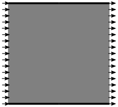
$\updownarrow$ same $L$, $\bar{g}^2(L^{-1})$

$m_0^{(2)}, g_0^{(2)}$:  $\overset{\text{same } a^{(2)}}{\leftrightarrow}$  $= \Sigma(u, \frac{a^{(2)}}{L})$

$\updownarrow$ same $L$, $\bar{g}^2(L^{-1})$

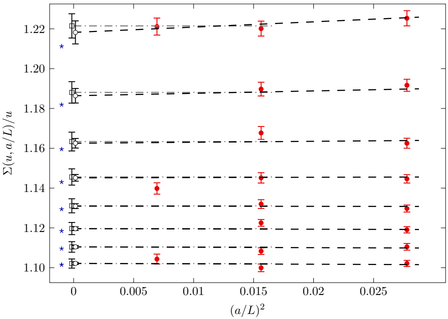
$m_0^{(3)}, g_0^{(3)}$:  $\overset{\text{same } a^{(3)}}{\leftrightarrow}$  $= \Sigma(u, \frac{a^{(3)}}{L})$

$\downarrow$cont. limit

$\bar{g}^2 = u, \ \overline{m} = 0$  $= \sigma(u)$

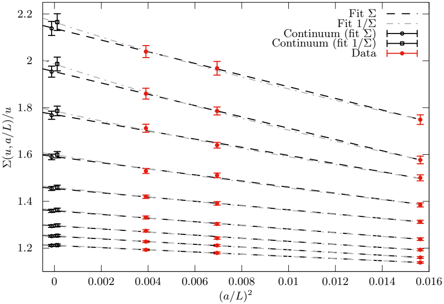
## SF-coupling

[M. Dalla Brida, P. Fritzsch, T. K., A. Ramos, S. Sint, R. Sommer, PRL 117 (2016)]



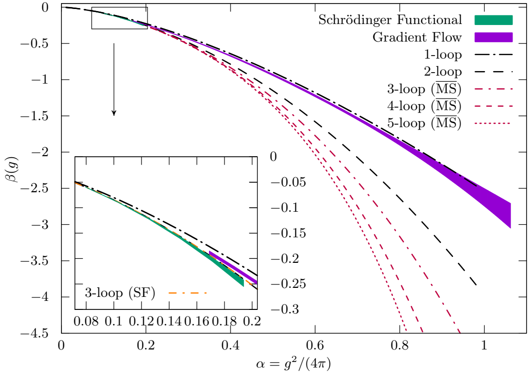

## GF-coupling

[M. Dalla Brida, P. Fritzsch, T. K., A. Ramos, S. Sint, R. Sommer, PRD 95 (2017)]



$\sigma(g)$ and $\beta(g)$ contain the same information

$$\ln(\mu_1/\mu_2) = -\int_{\bar{g}(\mu_1)}^{\bar{g}(\mu_2)} \frac{dg}{\beta(g)}$$



# Result 2017

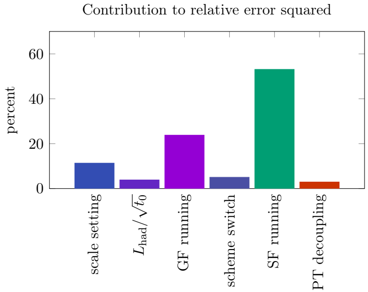
## Final Result

$$\Lambda^{(3)}_{\overline{\mathrm{MS}}} = 341(12)\ \mathrm{MeV}$$

$$\Lambda^{(5)}_{\overline{\mathrm{MS}}} = 215(10)(03)\ \mathrm{MeV} \qquad \text{pert. decoupling}$$

$$\alpha_{\overline{\mathrm{MS}}}(M_Z) = 0.1185(8)(3)$$

$$0.1174(16) \qquad \text{PDG non-lattice 2017}$$



[T.Appelquist, J.Carazzone, PRD11 (1975)], [S.Weinberg, Phys.Lett b91 (1980)]

- Heavy quarks decouple and have a suppressed influence on low-energy observables
- E.g. QCD with $N_f = 3$ heavy quarks of mass $M$
  $\rightarrow$ Effective theory for energies $\ll M$ = pure Yang Mills
- Matching: choose correct $\Lambda^{(0)}$, depending on $M/\Lambda$
- Then low energy quantities are well described by the effective theory

$$\mathcal{S} = \mathcal{S}^{(0)} \qquad + O(M^{-2})$$

# Physical vs Practical Couplings

- Decoupling applies also to "physical couplings", i.e. couplings that depend on the mass

$$\left[\bar{g}^{(0)}(\mu/\Lambda^{(0)})\right]^2 = \bar{g}^2(\mu/\Lambda, M/\Lambda) \qquad + O((\mu/M)^2, (\Lambda/M)^2)$$

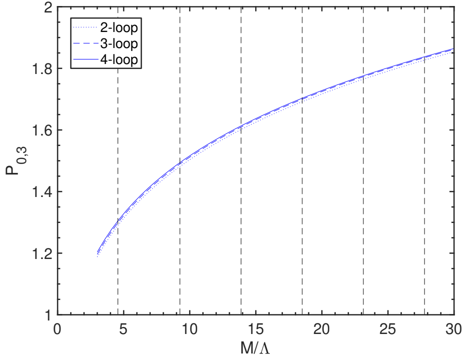
- This does not hold for mass-independent couplings. These have to be matched, e.g.

$$\left[\bar{g}^{(0)}(\mu/\Lambda^{(0)})\right]^2 = \bar{g}^2(\mu/\Lambda) + c_1(\mu/\overline{m}(\mu))\, \bar{g}^4(\mu/\Lambda) + \ldots$$

  - $\overline{m}(\mu)$ renormalized heavy quark mass $\Leftrightarrow$ $M$ RGI mass
  - Convenient choice of scheme and scale:
    $\overline{\mathrm{MS}}$-scheme with $\mu = m_*$ such that $\overline{m}(m_*) = m_*$
    - ⋆ $c_1 = 0$
    - ⋆ $log(\mu/\overline{m})$ vanish $\Rightarrow c_2, \ldots, c_4$ are pure numbers
    - ⋆ $c_2, \ldots, c_4$ known
      [K.Chetyrkin, J.H.Kühn, C.Sturm, Nucl.Phys.B744 (2006)]
      [B.A.Kniehl, A.V.Kotikov, A.I.Onishchenko, O.L.Veretin, PRL 97 (2006)]
      [Y. Schröder and M. Steinhauser, JHEP 01, 051 (2006)]
      [A.G.Grozin, M.Hoeschele, J.Hoff, M.Steinhauser, JHEP 1109 (2011)]
      [M. Gerlach, F. Herren, and M. Steinhauser, JHEP 11, 141 (2018)]

Perturbative matching relations between couplings can be translated into relations between Λ parameters

$$\Lambda^{(0)} = P_{0,3}(M/\Lambda^{(3)})\,\Lambda^{(3)}$$



# Strategy 2022

[ALPHA, Eur.Phys.J.C 82 (2022) 12]

$$\Lambda^{(3)}_{\overline{\mathrm{MS}}} = \underbrace{\frac{1}{\sqrt{t_0^\star}} \times \sqrt{t_0^\star}\,\mu_{\mathrm{had}} \times \frac{\mu_{\mathrm{dec}}}{\mu_{\mathrm{had}}}}_{N_\mathrm{f}=3} \times \left( \lim_{M\to\infty} \underbrace{\frac{\Lambda^{(0)}_{\overline{\mathrm{MS}}}(M/\Lambda)}{\mu_{\mathrm{dec}}}}_{N_\mathrm{f}=0} \times \underbrace{P^{-1}_{3,0}(M/\Lambda)}_{PT} \right)$$

- A short piece of running in $N_\mathrm{f} = 3$ up to $\mu_{\mathrm{dec}} = 789(15)$ MeV, $\bar{g}^2_{\mathrm{GF}}(\mu_{\mathrm{dec}}) = 3.949$
- Turning on heavy masses, determining $\bar{g}_{\mathrm{GFT}}(\mu_{\mathrm{dec}}, M)$ $\rightarrow$ challenging continuum extrapolation
- Assume that decoupling holds, $\bar{g}^{(0)}_{\mathrm{GFT}}(\mu_{\mathrm{dec}}) \overset{!}{=} \bar{g}_{\mathrm{GFT}}(\mu_{\mathrm{dec}}, M)$
- Determine $\frac{\Lambda^{(0)}_{\overline{\mathrm{MS}}}(M/\Lambda)}{\mu_{\mathrm{dec}}}$ in pure gauge theory

# $\mu_{\text{dec}}$

In principle all running starting at $\mu_{\text{had}}$ could be done in $N_f = 0$
Reasons to start at a slightly larger $\mu_{\text{dec}} \approx 800$ MeV

- Difficulty of the $M \to \infty$ limit
  We want to reach large $M$, but for asymptotic scaling, we need
  - $aM \ll 1$

  Practicable lattice sizes are limited
  - $\frac{L}{a} = \frac{1}{a\mu_{\text{dec}}} \leq 48$
    $\to M/\mu_{\text{dec}}$ should not be much larger than 10
- Simulations at smaller $a$ (smaller $g_0^2$)
  $\to$ Lattice PT works better
  - $c_t$, $\tilde{c}_t$
  - $b_g$, $b_m$

# Turning on Heavy Masses

- Known: For $L/a \in \{12, 16, 20, 24, 32, 40, 48\}$
  $(g_0^2, m_0, L/a)$ such that $L = 1/\mu_{\rm dec}, M = 0$
- How to choose
  $(g_0^2, m_0, L/a)$ such that $L = 1/\mu_{\rm dec}, M = z\,\mu_{\rm dec}$?
  $z \in \{2, 4, 6, 8, 10, 12\}$

## Up to $O(a)$

- $g_0^2$ as in the massless case
- $\overline{m} = Z_m m_q, \qquad m_q = m_0 - m_{\rm crit}$

$\rightarrow$ We need

- $Z_m(g_0^2, \mu_{\rm dec})$
- $m_{\rm crit}(g_0^2)$
- Relation $\overline{m} \leftrightarrow M$

[ALPHA, Eur.Phys.J.C 80 (2020) 2]

# Turning on Heavy Masses

- Known: For $L/a \in \{12, 16, 20, 24, 32, 40, 48\}$
  $(g_0^2, m_0, L/a)$ such that $L = 1/\mu_{\text{dec}}, M = 0$
- How to choose
  $(g_0^2, m_0, L/a)$ such that $L = 1/\mu_{\text{dec}}, M = z\,\mu_{\text{dec}}$?
  $z \in \{2, 4, 6, 8, 10, 12\}$

## Up to $O(a^2)$

- $\tilde{g}_0^2$ as in the massless case
  $\tilde{g}_0^2 = g_0^2(1 + b_g a m_q)$
- $\overline{m} = Z_m m_q(1 + b_m a m_q)$

$\rightarrow$ We need

- $Z_m(g_0^2, \mu_{\text{dec}})$
- $m_{\text{crit}}(g_0^2)$
- Relation $\overline{m} \leftrightarrow M$, [ALPHA, Eur.Phys.J.C 80 (2020) 2]
- $b_g(g_0^2) = 0.01200 \times N_{\text{f}}\, g_0^2 \quad + \quad O(g_0^4)$, [S.Sint, R. Sommer Nucl.Phys. B465 (1996)]
- $b_m(g_0^2)$

# Continuum Limits

For a fixed $z$ (mass) and $c$ (scheme), the continuum extrapolation is

$$\bar{g}_z^2(a) = \bar{g}^2_{\mathrm{GFT}}(\mu_{\mathrm{dec}}, M) + p\,\left[\alpha_{\overline{\mathrm{MS}}}(a^{-1})\right]^{\hat{\Gamma}}(a\mu_{\mathrm{dec}})^2$$

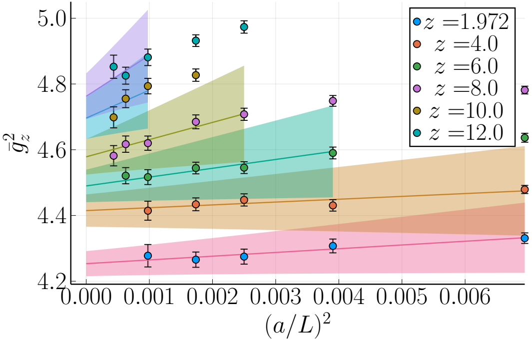
with fit parameters $\bar{g}^2_{\mathrm{GFT}}$ and $p$

Impact of logarithmic corrections assessed by varying $\hat{\Gamma} \in [-1, 1]$.

[N. Husung, P. Marquard, R. Sommer, Eur.Phys.J.C 80 (2020) 3]

Example on the right with

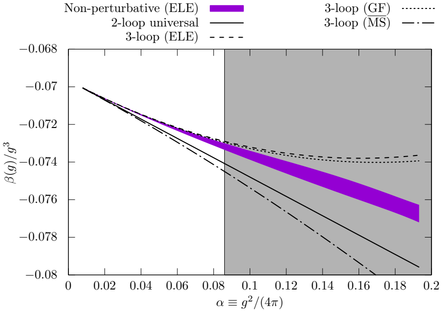
- $\hat{\Gamma} = 0$
- $c = 0.3$
- cut $aM < 0.4$

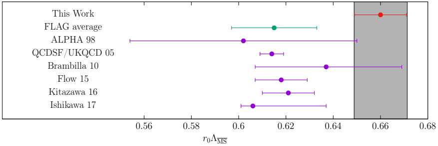


# Running in Pure Gauge Theory

## Result based on finite volume GF schemes:

[M.Dalla Brida, A.Ramos, Eur.Phys.J. C79 (2019)]




- State 2019: tensions!
- Meanwhile confirmed by several precise calculations

[A. Hasenfratz et al, Phys. Rev. D 108, 014502 (2023)]

[C. Wong et al, PoS LATTICE2022, 043 (2023)]

[N. Brambilla et al, Phys.Rev.D 109 (2024) 11]

Given a coupling $\overline{g}^2_{\mathrm{GF}}(\mu_{\mathrm{dec}})$ we can obtain $\Lambda^{(0)}/\mu_{\mathrm{dec}}$

E.g. at $z = 6$: $\overline{g}^2_{\mathrm{GF}}(\mu_{\mathrm{dec}}) = 4.466(37) \Rightarrow \Lambda^{(0)}/\mu_{\mathrm{dec}} = 0.741(12)$

$$\Lambda^{(3)}_{\overline{\mathrm{MS}},\mathrm{eff}}(z) = \Lambda^{(3)}_{\overline{\mathrm{MS}}} + \frac{B}{z^2}\left[\alpha_{\overline{\mathrm{MS}}}(m^\star)\right]^{\hat{\Gamma}_m}$$

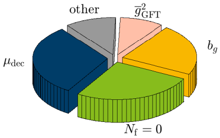
$$\hat{\Gamma}_m \in [0,1]$$

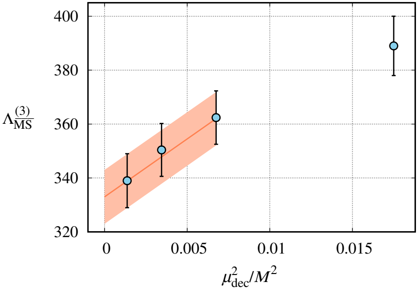
## Results

$$\Lambda^{(3)}_{\overline{\mathrm{MS}}} = 336(12)\ \mathrm{MeV}$$

$$\Lambda^{(5)}_{\overline{\mathrm{MS}}} = 211.3(9.8)\ \mathrm{MeV}$$

$$\alpha_{\overline{\mathrm{MS}}}(M_Z) = 0.11823(84)$$




- $c = 0.36$
- $\hat{\Gamma}_m = 0$

# Recent Improvements

- More precise scale setting
  $\sqrt{t_0^\star} = 0.1434(19)$ fm

  [B. Strassberger et al, PoS LATTICE2021 (2022) 135]

  [RQCD, JHEP 05 (2023) 035, JHEP 05 (2023) 035]

  $\rightarrow \mu_{\mathrm{dec}} = 802(13)$ MeV
- Non-perturbative determination of $b_g$

# Nonperturbative $b_g$

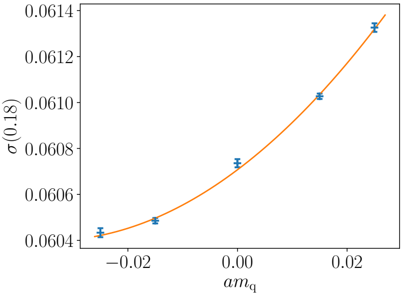
[Alpha Collaboration, JHEP 2024 (2024)]

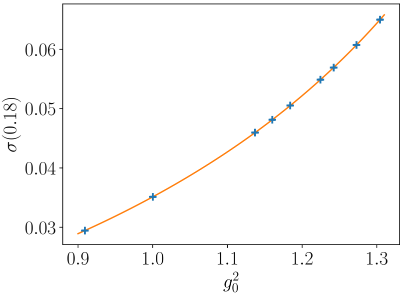
In QCD, gluonic observables are symmetric under $\overline{m} \to -\overline{m}$

## Improvement Condition

$$b_g(g_0^2) = \left.\frac{\partial\langle O_g\rangle}{\partial a m_q}\right|_{g_0^2, m_q=0} \times \left[ g_0^2 \left.\frac{\partial\langle O_g\rangle}{\partial g_0^2}\right|_{m_q=0} \right]^{-1}$$

for some renormalized gluonic observable $\langle O_g \rangle$



# Nonperturbative $b_g$

[Alpha Collaboration, JHEP 2024 (2024)]

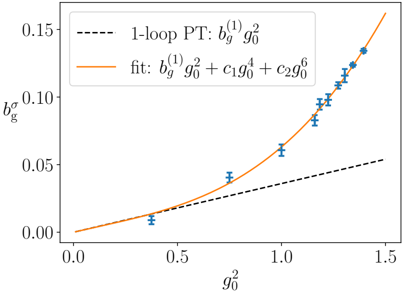
In QCD, gluonic observables are symmetric under $\overline{m} \to -\overline{m}$

## Improvement Condition

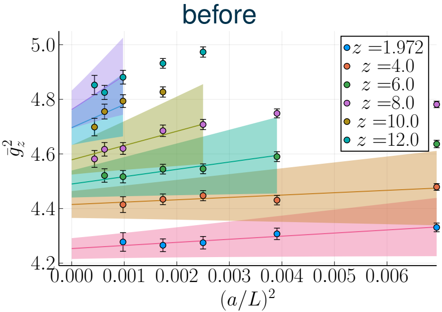
$$b_g(g_0^2) = \left.\frac{\partial \langle O_g \rangle}{\partial a m_q}\right|_{g_0^2, m_q=0} \times \left[ g_0^2 \left.\frac{\partial \langle O_g \rangle}{\partial g_0^2}\right|_{m_q=0} \right]^{-1}$$

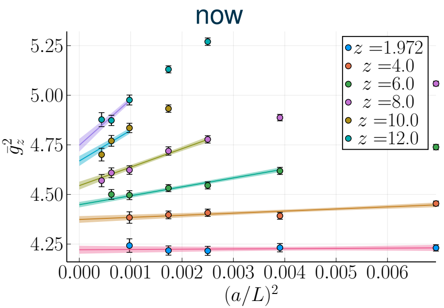
for some renormalized gluonic observable $\langle O_g \rangle$

- box of size $L^4$
- around $m_q = 0$
- anti-periodic b.c.
- $\langle O_g \rangle \sim$ GF coupling



# Updated Result




- Coupling definition with $c = 0.3$
- Data with $aM > 0.4$ neglected
- $\Gamma = 0$

# Result and Outlook

## Preliminary Result

$$\Lambda^{(3)}_{\overline{\mathrm{MS}}} = 342.3(9.6)\ \mathrm{MeV}$$

## Outlook

- Renewed interest in high accuracy $N_f = 0$ running
  - Better algorithms, e.g. multi-level
  - Novel renormalization schemes
- Improved scale setting