# Resummation of Threshold Double Logarithms in Inclusive Production of Heavy Quarkonium

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Based on arXiv:2408.04255 [hep-ph]

XVIth Quark Confinement and the Hadron Spectrum
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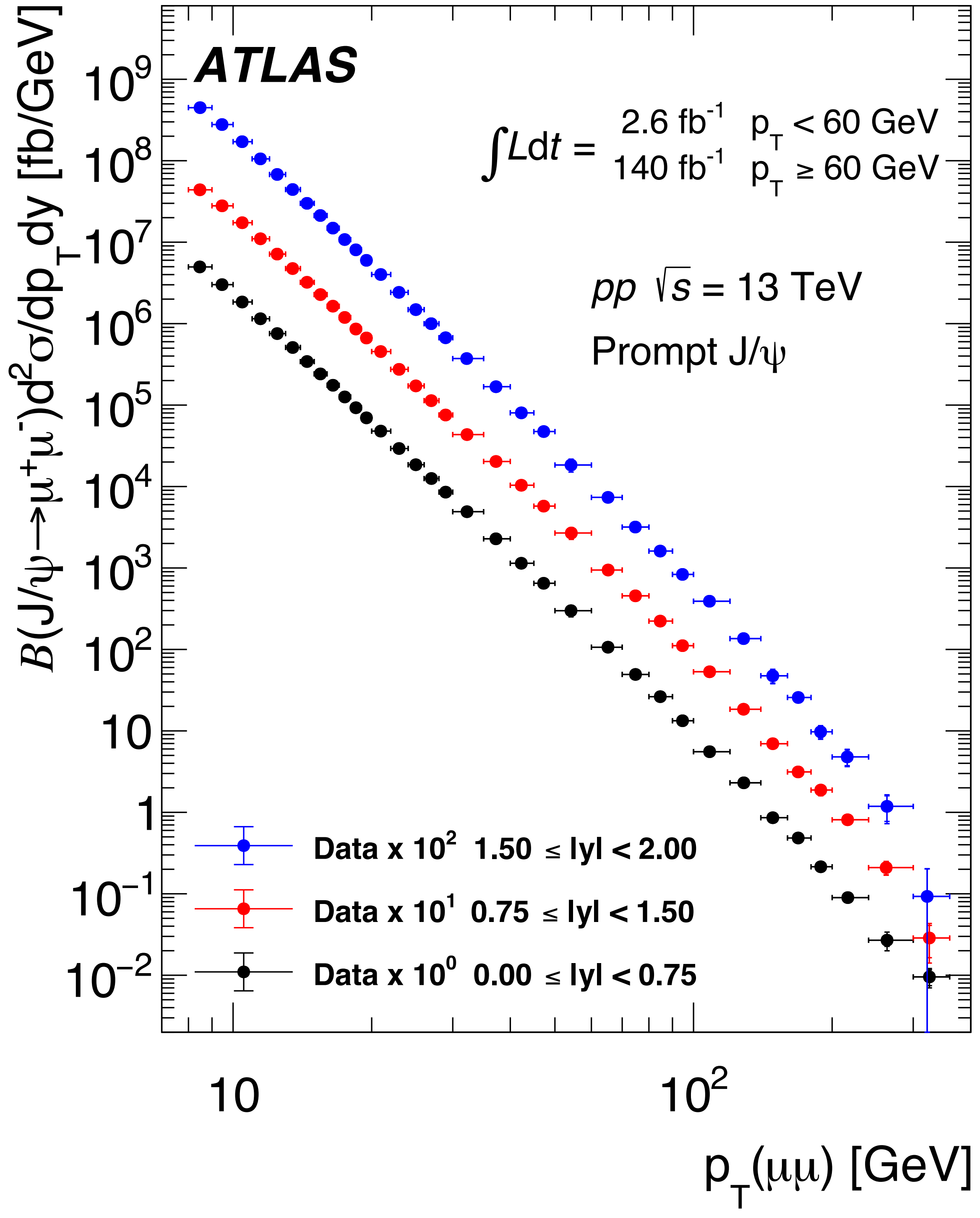
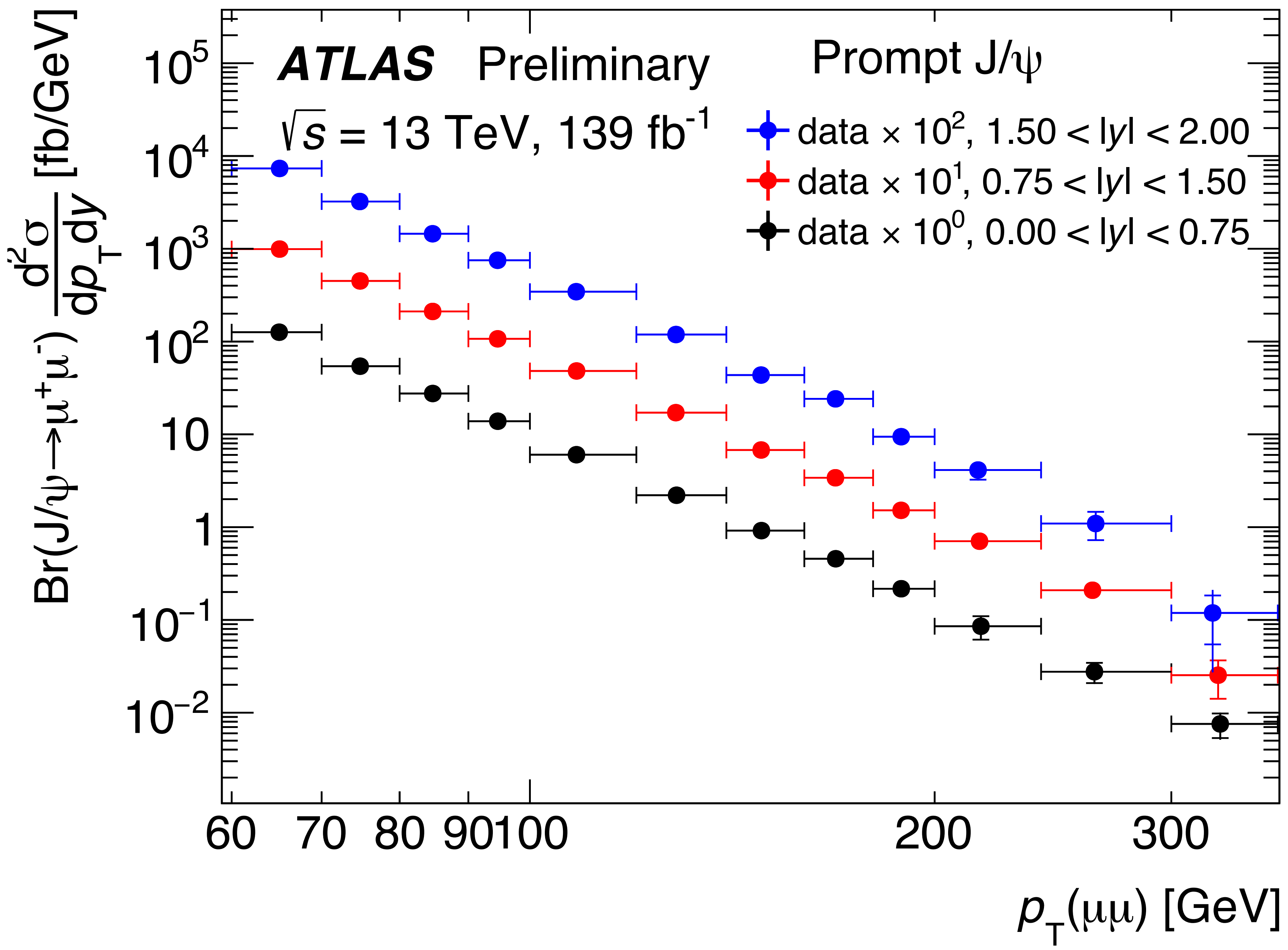
# NRQCD vs. Data at large $p_T$

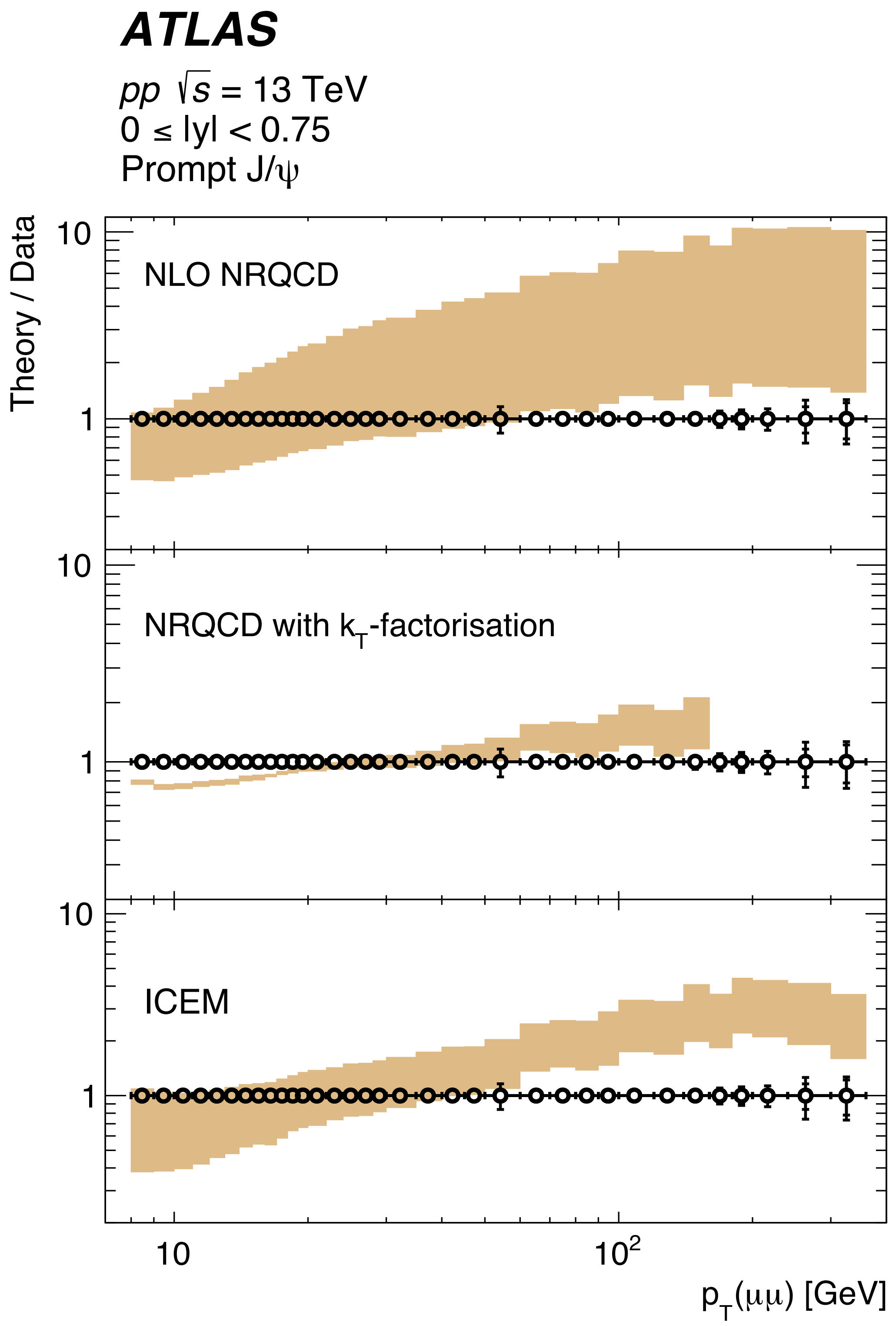
- Prompt $J/\psi$ cross sections have been measured up to $p_T = 360$ GeV at the 13 TeV LHC




**ATLAS EPJC 84, 169 (2024)**




**ATLAS-CONF-2019-047**




**NLO NRQCD calculation from Butenschoen and Kniehl, based on PRD 84, 051501 (2011)**

- Preliminary results have been available since **2019**.
  It seemed that NLO NRQCD has trouble describing data for $p_T \gg 100$ GeV.

# NRQCD factorization

- Inclusive production cross section of a quarkonium $\mathcal{Q}$ is given by

$$\sigma_{\mathcal{Q}} = \sum_{\mathcal{N}} \sigma_{Q\bar{Q}(\mathcal{N})} \langle \mathcal{O}^{\mathcal{Q}}(\mathcal{N}) \rangle$$

- $Q\bar{Q}$ cross sections are computed in perturbative QCD.
  Long-distance matrix elements describe evolution of $Q\bar{Q}$ into quarkonium+$X$.
- For $J/\psi$ or $\psi$(2S), dominant contributions come from $\mathcal{N}$ = ${}^3S_1^{[1]}$, ${}^3S_1^{[8]}$, ${}^1S_0^{[8]}$, ${}^3P_J^{[8]}$.
  For $\chi_{cJ}$, $\mathcal{N}$ = ${}^3P_J^{[1]}$ and ${}^3S_1^{[8]}$ at leading order in the nonrelativistic expansion.
- Color-singlet $Q\bar{Q}$ has direct overlap with quarkonium state, while color-octet $Q\bar{Q}$ must go through transition to evolve into a color-singlet quarkonium.
- $Q\bar{Q}({}^1S_0^{[8]})$ → $J/\psi$+$X$ occurs through $\Delta S$=1 (spin flip), while
  $Q\bar{Q}({}^3P_J^{[8]})$ → $J/\psi$+$X$ needs $\Delta L$=1 with $\Delta S$=0, and $Q\bar{Q}({}^3S_1^{[8]})$ → $J/\psi$+$X$ requires $\Delta L$=$\Delta S$=0.
  $Q\bar{Q}({}^3S_1^{[8]})$ → $\chi_{cJ}$+$X$ occurs through $\Delta L$=1 with $\Delta S$=0. Color-octet MEs are determined from data.
- $P$-wave $Q\bar{Q}$ cross sections are negative at large $p_T$.

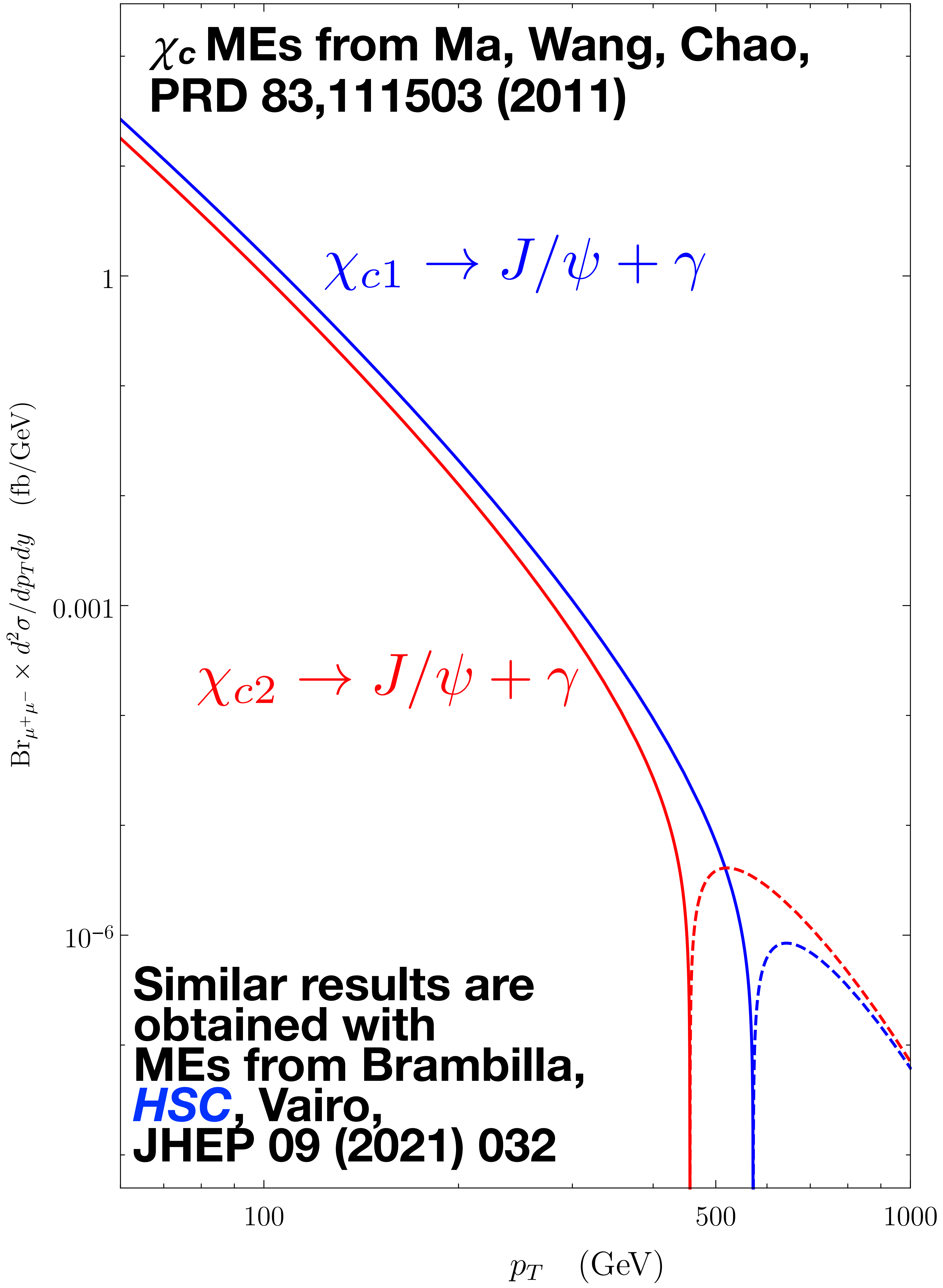
# NLO NRQCD vs. Data at large $p_T$

- Prompt $J/\psi$ data vs. direct $J/\psi$ from NLO NRQCD from various ME sets. $\sqrt{s}$=13 TeV LHC, |y|<0.75. Uncertainties from matrix elements only.

# $\chi_{cJ}$ production in NRQCD at large $p_T$

- $\chi_{cJ}$ cross sections from NLO NRQCD ***always turn negative at large $p_T$***, regardless of choice of the color-octet ME $^3S_1^{[8]}$.

- A significant amount of prompt $J/\psi$ comes from feeddowns from $\chi_{c1,2} \to J/\psi+\gamma$.
  Without solving the ***negative cross section problem***, ***it is IMPOSSIBLE to make any solid prediction of prompt $J/\psi$ production rates.***



# What happens at large $p_T$

- Large-$p_T$ cross sections are dominated by gluon fragmentation.

$$\sigma_{\mathcal{Q}} = \sum_{i=g,q,\bar{q}} \int_0^1 dz\, \hat{\sigma}_i \times D_{i\to\mathcal{Q}}(z)$$

Fragmentation function describes production of hadron $\mathcal{Q}$ from massless parton $i{=}g,\ q,\ \bar{q}$.
$z$ = fraction of $\mathcal{Q}$ momentum compared to parton momentum in the + direction

- NRQCD factorization for fragmentation functions

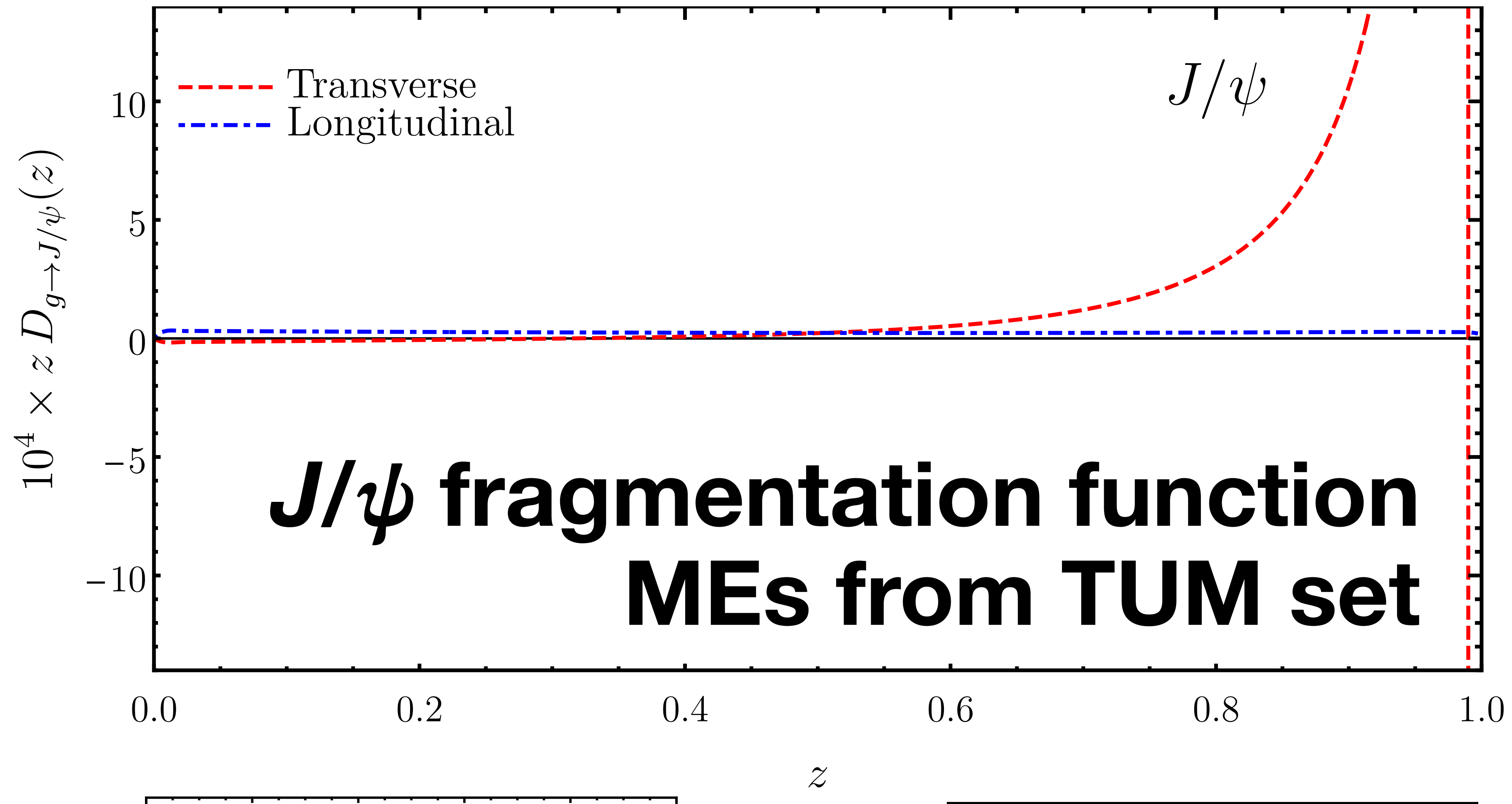
$$D_{g\to\mathcal{Q}}(z) = \sum_{\mathcal{N}} D_{g\to Q\bar{Q}(\mathcal{N})}(z)\langle\mathcal{O}^{\mathcal{Q}}(\mathcal{N})\rangle$$

The perturbative fragmentation functions $D_{g\to Q\bar{Q}(\mathcal{N})}(z)$ are singular distributions at $z=1$ for $\mathcal{N}$ = ${}^3S_1^{[8]}$, ${}^3P_J^{[8]}$, and ${}^3P_J^{[1]}$, because $\Delta S=0$ processes can occur by emitting soft gluons.

- The ${}^3S_1^{[1]}$ and ${}^1S_0^{[8]}$ fragmentation functions are regular functions.

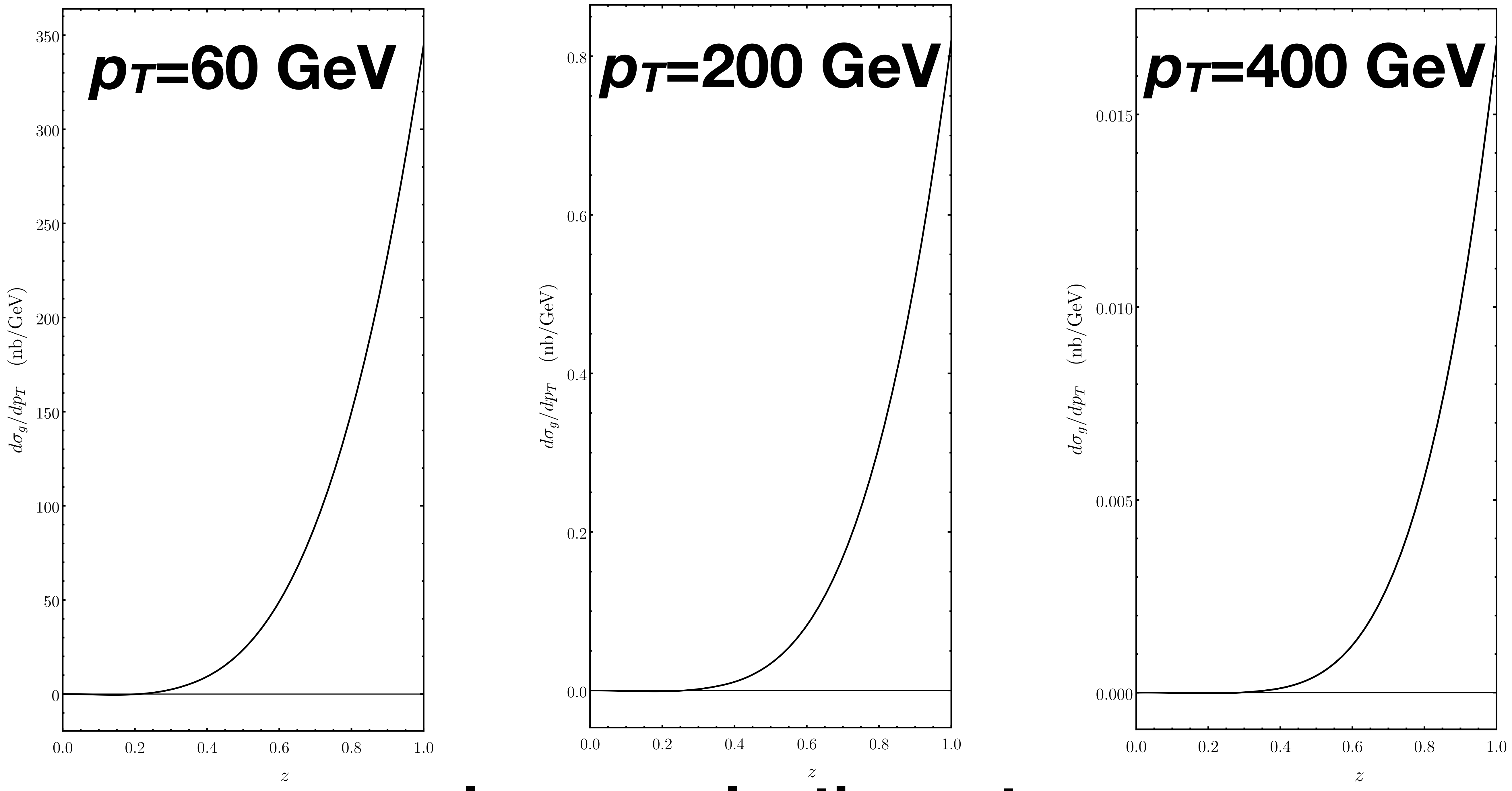
# What happens at large $p_T$

- The $J/\psi$ fragmentation function contains singular distributions $\delta(1-z)$, $1/(1-z)_+$, $[\log(1-z)/(1-z)]_+$, … . These change sign rapidly near $z=1$, so that ***no choice of ME sets will lead to positive definite fragmentation functions.***




- Gluon production rates rise with $z$. Slope at $z=1$ grows steeper with $p_T$.

- Contributions from singular distributions become increasingly more important as $p_T$ increases. As more singular distributions appear at higher orders in $\alpha_s$, this disturbs convergence of perturbation theory.



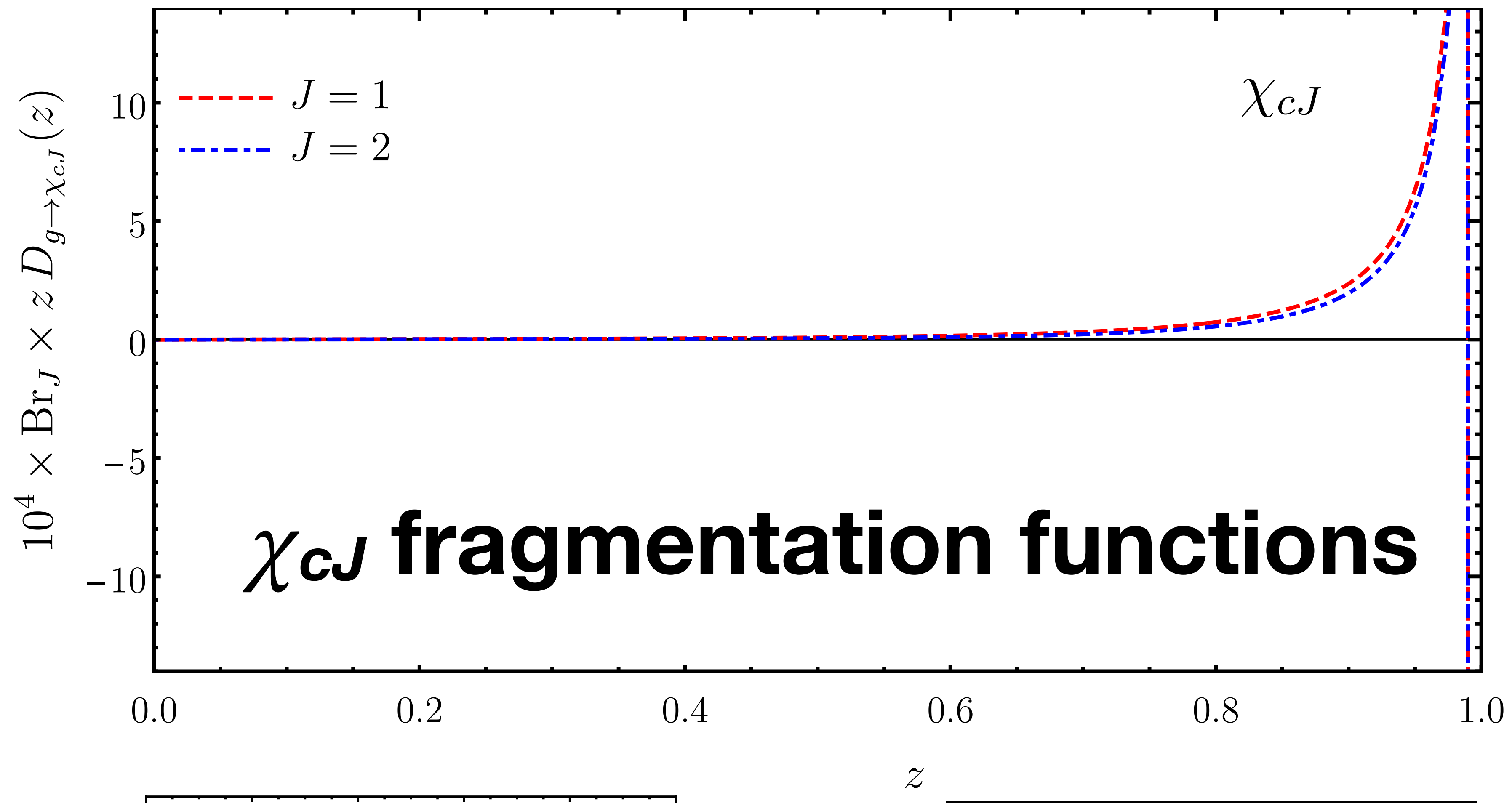

**gluon production rates**

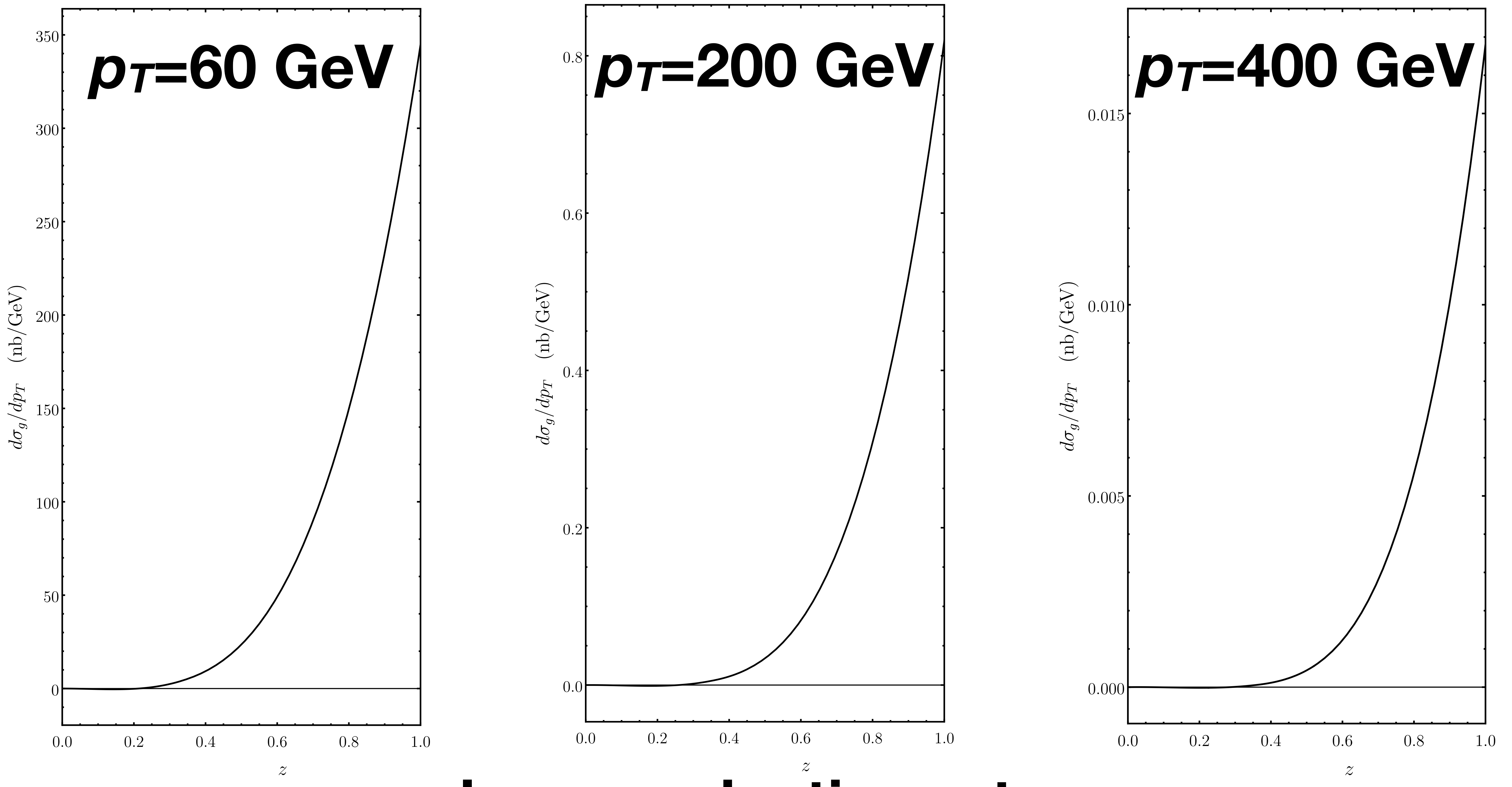
# What happens at large $p_T$

- Similarly, $\chi_{cJ}$ fragmentation function contains singular distributions $\delta(1-z)$, $1/(1-z)_+$, $[\log(1-z)/(1-z)]_+$, … .

- Same problem as $J/\psi$ : contributions from singular distributions become increasingly more important as $p_T$ rises.

- As more singular distributions appear at higher orders in $\alpha_s$, this disturbs convergence of perturbation theory.




gluon production rates

# Threshold logarithms

- Singular distributions in fragmentation functions :

less singular/regular functions

$$D_{g\to {}^3S_1^{[8]}}(z) = \frac{\pi\alpha_s}{3m^3(N_c^2-1)}\left[\delta(1-z) + \frac{\alpha_s C_A}{\pi}\left(\frac{-2\log(1-z)}{1-z}\right)_+ + \cdots\right]$$

Braaten and Lee, NPB 586, 427 (2000)
Ma, Qiu, Zhang, PRD 89, 094029 (2014)

$$D_{g\to {}^3P_J^{[8]}}(z) = \frac{2\alpha_s^2(N_c^2-4)}{9m^5N_c(N_c^2-1)}\left[\frac{1}{(1-z)_+} + \frac{\alpha_s C_A}{\pi}\left(\frac{-4\log^2(1-z)}{1-z}\right)_+ + \cdots\right]$$

Zhang, Meng, Ma, Chao, JHEP 08 (2021) 111

$$D_{g\to {}^3P_J^{[1]}}(z) = \frac{2\alpha_s^2}{9m^5N_c^2}\left[\frac{1}{(1-z)_+} + \frac{\alpha_s C_A}{\pi}\left(\frac{-4\log^2(1-z)}{1-z}\right)_+ + \cdots\right]$$

- The severity of the singularities can be quantified in terms of Mellin moments:

$$\tilde{D}(N) = \int_0^1 dz\, z^{N-1} D(z)$$

Singularities in $D(z)$ at $z=1$ correspond to nonvanishing/divergence of $\tilde{D}(N)$ at $N\to\infty$.

**Mellin transform of a delta function is a constant**

$$\tilde{\delta}(N) = \int_0^1 dz\, z^{N-1}\delta(1-z) = 1$$

**Plus distributions give logarithmically diverging Mellin transforms**

$$\int_0^1 dz\, z^{N-1}\left(\frac{\log^{n-1}(1-z)}{1-z}\right)_+ = \frac{(-1)^n}{n}\log^n N + \cdots$$

# Threshold logarithms

- $N\to\infty$ divergences in Mellin-space fragmentation functions :

$$\tilde{D}_{g\to {}^3S_1^{[8]}}(N) = \frac{\pi\alpha_s}{3m^3(N_c^2-1)}\left[1-\frac{\alpha_s C_A}{\pi}\log^2 N+\cdots\right]$$

$$\tilde{D}_{g\to {}^3P_J^{[8]}}(N) = -\frac{2\alpha_s^2(N_c^2-4)}{9m^5 N_c(N_c^2-1)}\left[\log N-\frac{4}{3}\frac{\alpha_s C_A}{\pi}\log^3 N+\cdots\right]$$

$$\tilde{D}_{g\to {}^3P_J^{[1]}}(N) = -\frac{2\alpha_s^2}{9m^5 N_c^2}\left[\log N-\frac{4}{3}\frac{\alpha_s C_A}{\pi}\log^3 N+\cdots\right]$$

- The most severe singularity at loop level involves $\alpha_s \log^2 N$ compared to LO. Since this divergence is associated with the singularity at the boundary $z=1$, we refer to them as ***threshold double logarithms***.

- Resummation of threshold double logarithms can be accomplished by ***soft factorization→loop calculation of soft function→resummation by exponentiation***

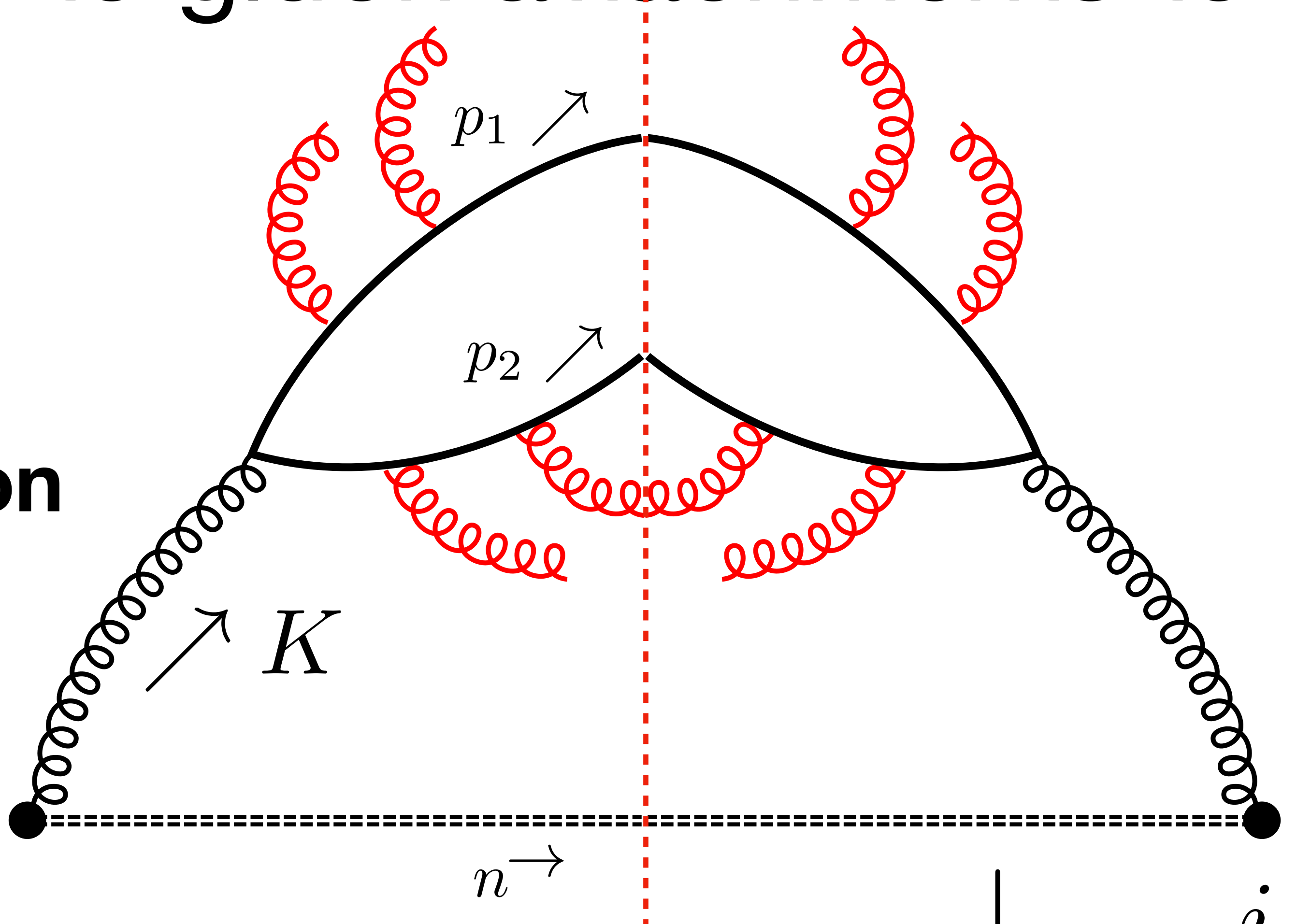
# Soft factorization

- The $z \to 1$ singularities in the gluon FF are identified by taking the ***Grammer-Yennie approximation*** to gluon attachments to Q and Q̄. This turns quark lines to Wilson lines:

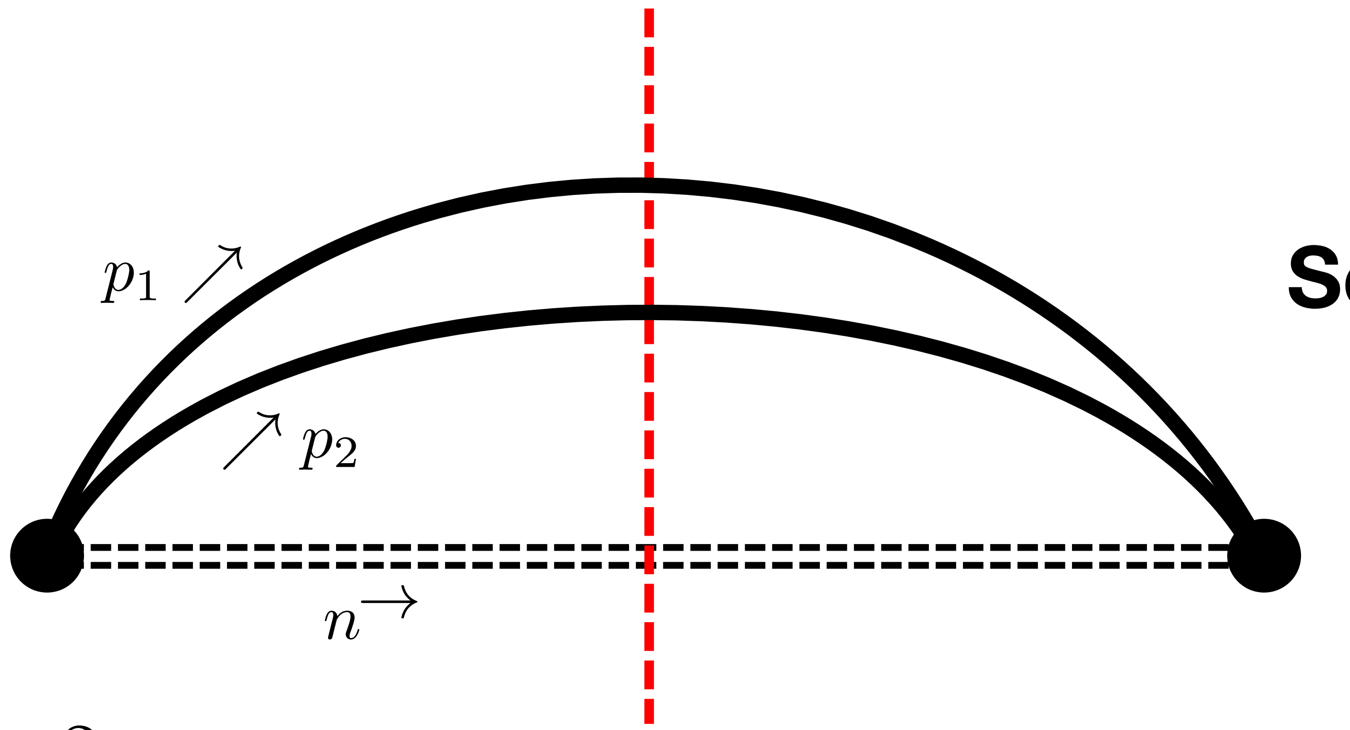
**(soft approximation)**

**Grammer and Yennie, PRD 8, 4332 (1973)**

**Fragmentation function**




***z*→1** →




**Soft function**

$$D_{\rm soft}[g \to Q\bar{Q}] = 2M(-g_{\mu\nu}) C_{\rm frag} \left| \frac{-i}{K^2 + i\varepsilon} + O(\alpha_s) \right|^2 \left( g^{\mu\alpha} - \frac{K^\mu n^\alpha}{K_+} \right) \left( g^{\nu\alpha'} - \frac{K^\nu n^{\alpha'}}{K_+} \right)$$

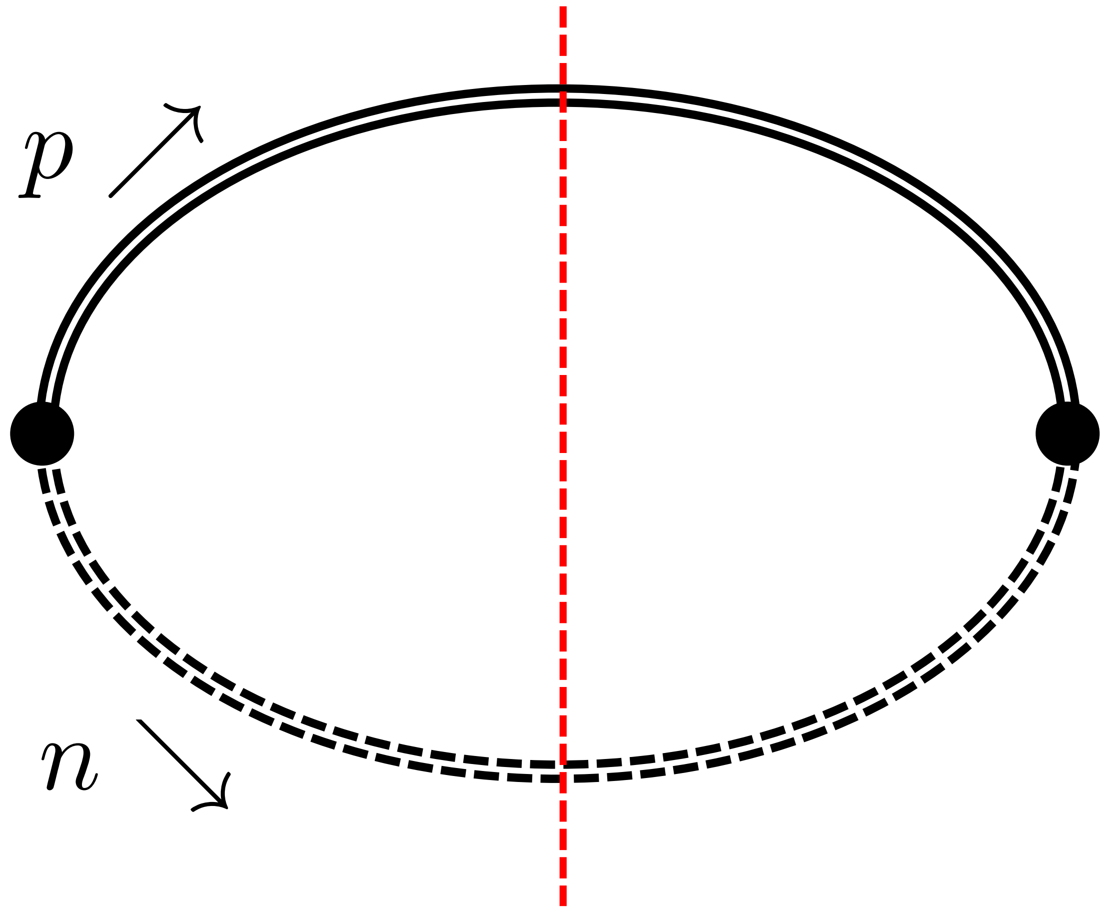
$$\times \langle 0 | \bar{T} \left[ \mathcal{A}^{\alpha',a'}_{\rm soft} \Phi^{ba'}_n(\infty, 0) \right]^\dagger 2\pi\delta(n\cdot\hat{p} - K^+(1-z)) T \left[ \mathcal{A}^{\alpha,a}_{\rm soft} \Phi^{ba}_n(\infty,0) \right] |0\rangle$$

***soft function*** →

$$C_{\rm frag} = z^{d-3} K^+ / [2\pi (N_c^2 - 1)(d-2)] \qquad \mathcal{A}^{\mu,a}_{\rm soft} = T\left[ \bar{u}(p_1) W_{p_1}(\infty,0)(-ig\gamma^\mu T^a) W^\dagger_{p_2}(\infty,0) v(p_2) \right]$$

- This essentially provides the soft factorization with tree-level Wilson coefficient. Loop level coefficient will not be needed until next-to-leading logarithmic accuracy.

# Soft function for ${}^3S_1^{[8]}$




- Soft functions for individual channels are obtained by nonrelativistic expansion and projecting onto spin and color states.

- The ${}^3S_1^{[8]}$ soft function is given by an adjoint Wilson loop with timelike and lightlike segments.

$$S_{{}^3S_1^{[8]}}(z) \equiv \langle 0|[\mathcal{W}({}^3S_1^{[8]})^{cb}]^\dagger 2\pi\delta(n\cdot\hat{p} - P^+(1-z))\mathcal{W}({}^3S_1^{[8]})^{cb}|0\rangle \qquad \mathcal{W}({}^3S_1^{[8]})^{cb} \equiv T\left[\Phi_p^{ca}(\infty,0)\Phi_n^{ba}(\infty,0)\right]$$

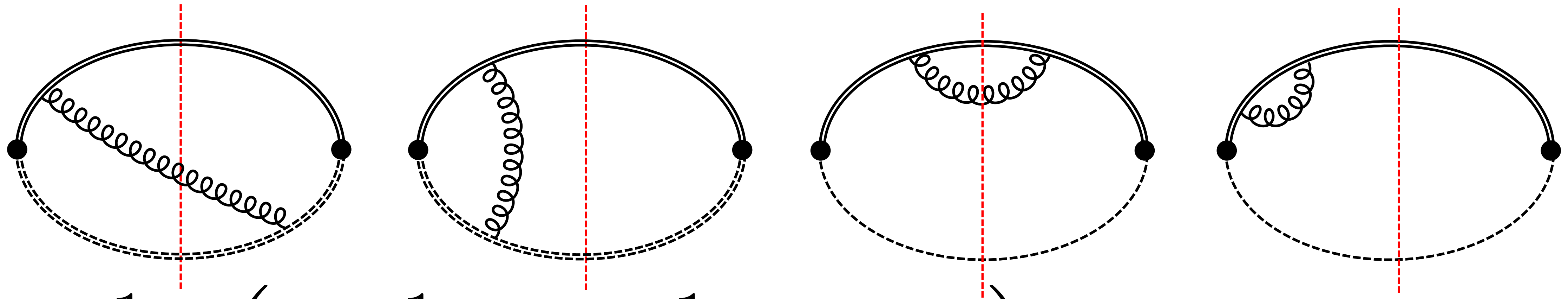
- At tree level,

$$S_{{}^3S_1^{[8]}}(z) = \frac{(N_c^2-1)2\pi}{P^+}\delta(1-z)$$

This reproduces the tree-level result

$$D_{g\to{}^3S_1^{[8]}}(z) = \frac{\pi\alpha_s}{3m^3(N_c^2-1)}\,\delta(1-z)$$

# Soft function for ${}^3S_1^{[8]}$



- ${}^3S_1^{[8]}$ soft function at ***NLO*** :

$$S_{{}^3S_1^{[8]}} = \frac{2\pi(N_c^2-1)}{P^+}\left\{\delta(1-z) + \frac{\alpha_s C_A}{\pi}\left[-\frac{\delta(1-z)}{2\epsilon_{\rm UV}^2} + \frac{1}{\epsilon_{\rm UV}}\left(\frac{1}{(1-z)_+} + \frac{1}{2}\delta(1-z)\right) + \left(\frac{-2\log(1-z)}{1-z} - \frac{1}{1-z}\right)_+ + \cdots\right] + O(\alpha_s^2)\right\}.$$

***anomalous dimension***

- The double pole and the associated double log $[\log(1-z)/(1-z)]_+$ term come only from the ***planar*** *real and virtual diagrams*. The self-energy diagrams only produce single poles.

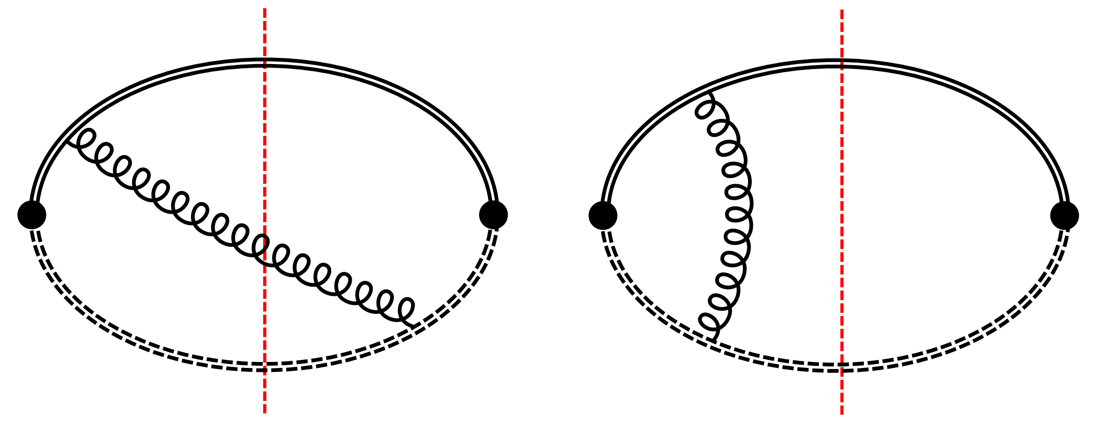
- The ***same soft function*** in the fundamental representation appears in $b \to q+X$, and exactly same NLO result is obtained by setting $C_A \to C_F$. **See e.g. Korchemsky and Sterman, PLB340, 96 (1994)** Especially, the ***cusp anomalous dimension*** $\alpha_s C_{rep}/\pi$ $1/(1-z)_+$ is identified from the coefficient of the UV pole.

- The double log term exactly reproduces the same term in the ${}^3S_1^{[8]}$ fragmentation function.

# Resummation for ${}^3S_1^{[8]}$

- We can write the double logarithmic correction as

$$S_{{}^3S_1^{[8]}}(z)|_{\text{threshold}} = \left\{ 1 + \frac{\alpha_s C_A}{\pi} \left( \frac{-2\log(1-z)}{1-z} \right)_+ + O(\alpha_s^2) \right\} \otimes S_{{}^3S_1^{[8]}}(z)|_{\text{LO}}$$



$$f(z) \otimes g(z) \equiv \int_0^\infty \frac{dz'}{z'} f(z') g(z/z')$$ **(Mellin convolution)**

- This is trivially exponentiated : **See theory of webs in e.g. Laenen, Sterman, Vogelsang, PRD 63, 114018 (2001)**

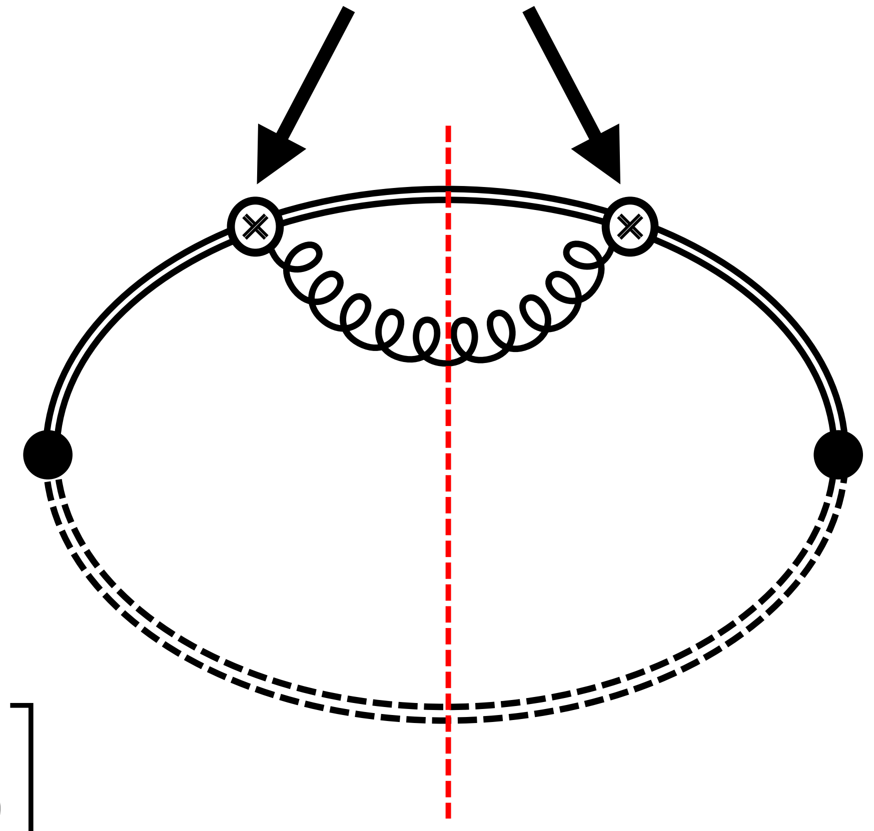
$$\tilde{S}^{\text{resum}}_{{}^3S_1^{[8]}}(N) = \exp\left[ \frac{\alpha_s C_A}{\pi} \int_0^1 dz\, z^{N-1} \left( \frac{-2\log(1-z)}{1-z} \right)_+ \right] \tilde{S}^{\text{LO}}_{{}^3S_1^{[8]}}(N)$$

- While the exponent diverges like $-\alpha_s C_A/\pi \log^2 N$, the exponential ***vanishes faster than any power of N***, so that the resummed soft function has a ***convergent*** *inverse Mellin transform*.

- The resummed expression agrees at double logarithmic level with the resummation in the soft gluon factorization formalism.

**A.-P. Chen, Y.-Q. Ma, and C. Meng, PRD 108, 014003 (2023)**

# Soft function for ${}^3P_J^{[8]}$




- Similarly, we obtain the ${}^3P_J^{[8]}$ soft function by nonrelativistic expansion and projecting onto spin and color states.

$$S_{{}^3P^{[8]}}(z) \equiv \langle 0|[\mathcal{W}^{yx}_{\beta'}({}^3P^{[8]})]^\dagger 2\pi\delta(n\cdot\hat{p} - P^+(1-z))\mathcal{W}^{yx}_{\beta}({}^3P^{[8]})|0\rangle g^{\beta\beta'}$$

$$\mathcal{W}^{yx}_{\beta}({}^3P^{[8]}) \equiv T\left[\int_0^\infty d\lambda\,\lambda\Phi_p^{yc}(\infty,\lambda')p^\mu G^b_{\mu\beta}(p\lambda')d^{bcd}\Phi_p^{da}(\lambda',0)\Phi_n^{xa}(\infty,0)\right]$$

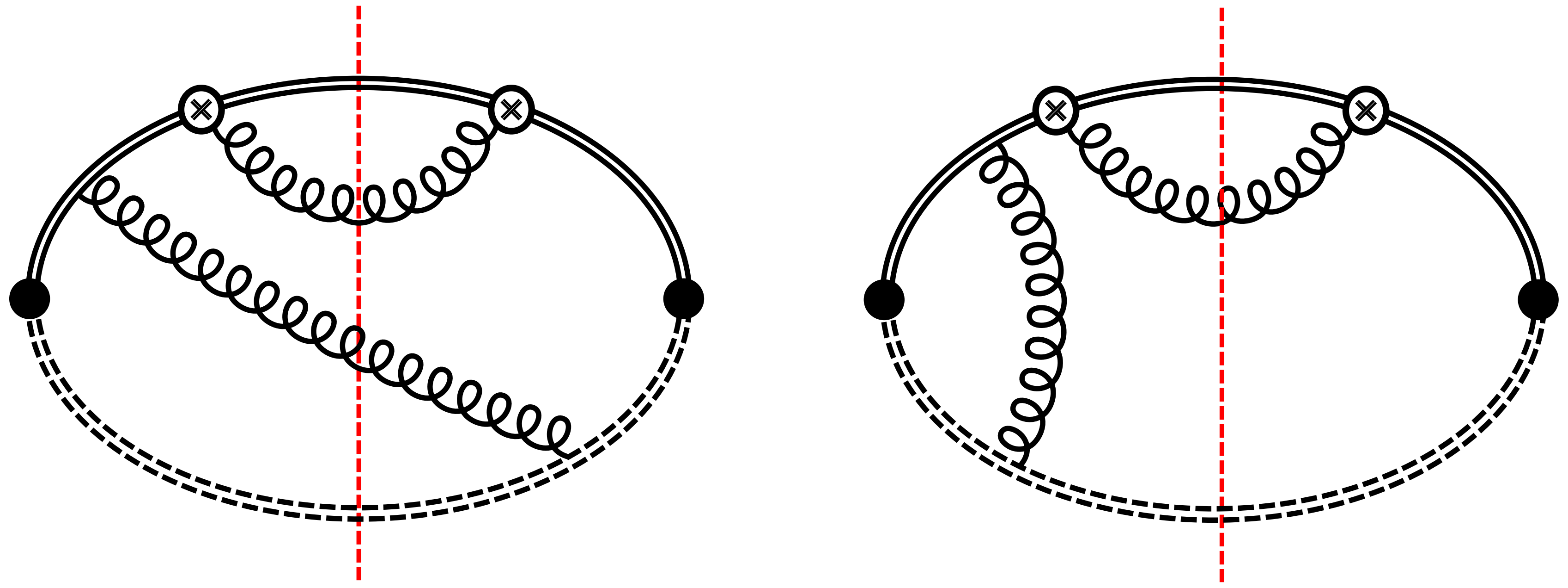
- The ${}^3P_J^{[8]}$ soft function involves field-strength insertions onto the timelike Wilson lines. The leading order result is

$$S^{\rm LO}_{{}^3P^{[8]}}(z) = -\frac{(d-2)4B_F(N_c^2-1)\Gamma(1+\epsilon)}{2\pi^{1-\epsilon}m^2P^+(1-z)^{1+2\epsilon}} \qquad B_F = (N_c^2-4)/(4N_c)$$

- From the identity $\displaystyle \frac{1}{(1-z)^{1+n\epsilon}} = -\frac{1}{n\epsilon_{\rm IR}}\delta(1-z) + \left[\frac{1}{(1-z)^{1+n\epsilon}}\right]_+$

  we reproduce the $1/(1-z)_+$ term in the ${}^3P_J^{[8]}$ fragmentation function.

# Resummation for ${}^3P_J^{[8]}$



- The double logarithmic corrections come from planar real and virtual diagrams.

$$S^{\rm NLO}_{{}^3P^{[8]}}(z) = \frac{\alpha_s C_A}{\pi} \frac{4B_F(N_c^2-1)\epsilon_{\rm UV}^{-2}}{2\pi m^2[P^+(1-z)]^{1+4\epsilon}} + \cdots$$

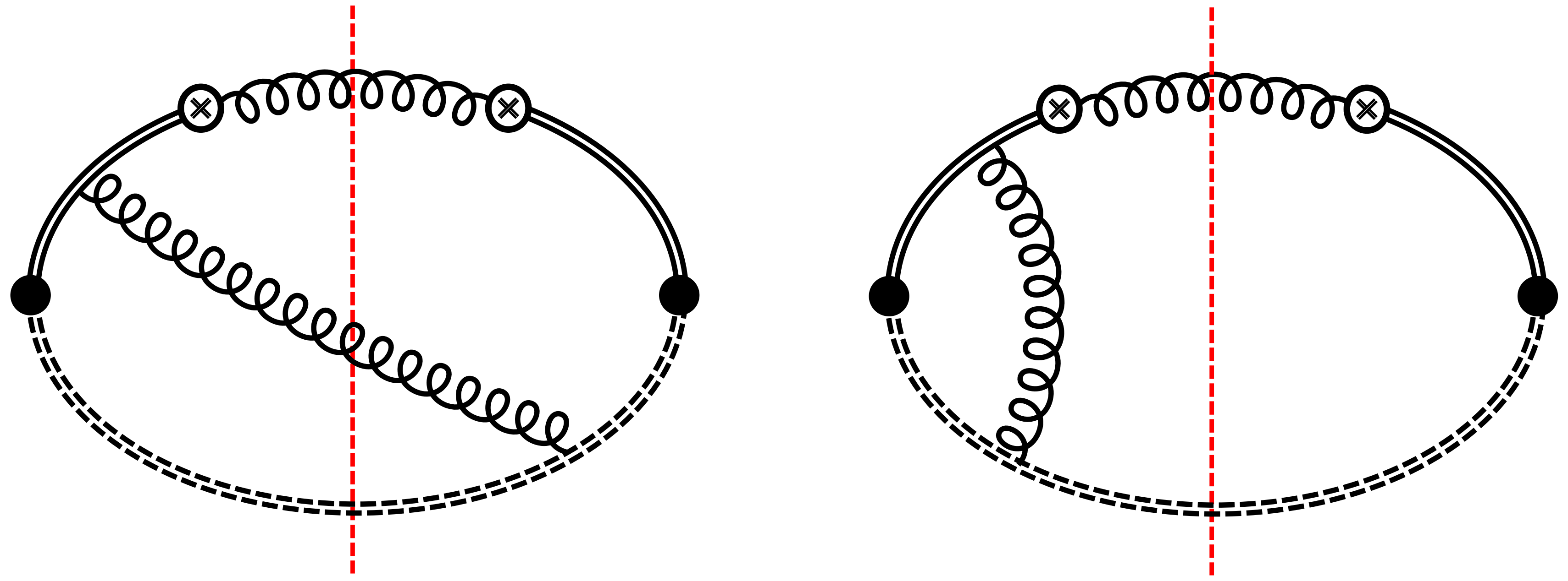
- From the identity $\frac{1}{(1-z)^{1+n\epsilon}} = -\frac{1}{n\epsilon_{\rm IR}}\delta(1-z) + \left[\frac{1}{(1-z)^{1+n\epsilon}}\right]_+$

  we obtain the same $[\log^2(1-z)/(1-z)]_+$ term in the ${}^3P_J^{[8]}$ fragmentation function.

- Again the double logarithm is trivially exponentiated:

$$\tilde{S}^{\rm resum}_{{}^3P^{[8]}}(N) = \exp\left[\frac{4}{3}\frac{\alpha_s C_A}{\pi}\int_0^1 dz\, z^{N-1}\left(\frac{-2\log(1-z)}{1-z}\right)_+\right]\tilde{S}^{\rm LO}_{{}^3P^{[8]}}(N)$$

# Resummation for ${}^3P_J^{[1]}$



- The soft function for ${}^3P_J^{[1]}$ is almost identical to the ${}^3P_J^{[8]}$ case, except for final-state color.
- The resummation is carried out in the same way :

$$\tilde{S}^{\rm resum}_{{}^3P^{[1]}}(N) = \exp\left[\frac{4}{3}\frac{\alpha_s C_A}{\pi}\int_0^1 dz\, z^{N-1}\left(\frac{-2\log(1-z)}{1-z}\right)_+\right]\tilde{S}^{\rm LO}_{{}^3P^{[1]}}(N)$$

- There is an additional soft function arising from the anisotropic contribution that produces *J*-dependent single logarithms; this can be discarded at the double logarithmic level.
- The ${}^3P_J^{[1]}$ soft function is essentially equivalent to the $\chi_{cJ}$ shape function.

$$\mathcal{S}^{\chi_{Q0}}_{{}^3S_1^{[8]}}(l_+) = \langle \chi^\dagger \sigma^i T^a \psi \Phi_\ell^{\dagger ab}(0) \mathcal{P}_{\chi_{Q0}} \delta(l_+ - iD_+) \Phi_\ell^{bc}(0) \psi^\dagger \sigma^i T^c \chi \rangle$$ **HSC, JHEP 07 (2023) 007**

# Resummation

- The resummed formula for the fragmentation function is

Fixed-order NLO

$$\tilde{D}^{\rm resum}_{g\to Q\bar{Q}(\mathcal{N})}(N) = \exp\left[J^N_{\mathcal{N}}\right] \times \left(\tilde{D}^{\rm FO}_{g\to Q\bar{Q}(\mathcal{N})}(N) - J^N_{\mathcal{N}}\tilde{D}^{\rm LO}_{g\to Q\bar{Q}(\mathcal{N})}(N)\right)$$

$$J^N_{^3S_1^{[8]}} = \frac{\alpha_s C_A}{\pi}\int_0^1 dz\, z^{N-1}\left[\frac{-2\log(1-z)}{1-z}\right]_+$$

$$J^N_{^3P^{[8]}} = J^N_{^3P^{[1]}} = \frac{4}{3}J^N_{^3S_1^{[8]}}.$$

All double logarithms are accounted for by the resummed exponential. The subtraction term removes the double log in fixed-order NLO expression to avoid double counting.

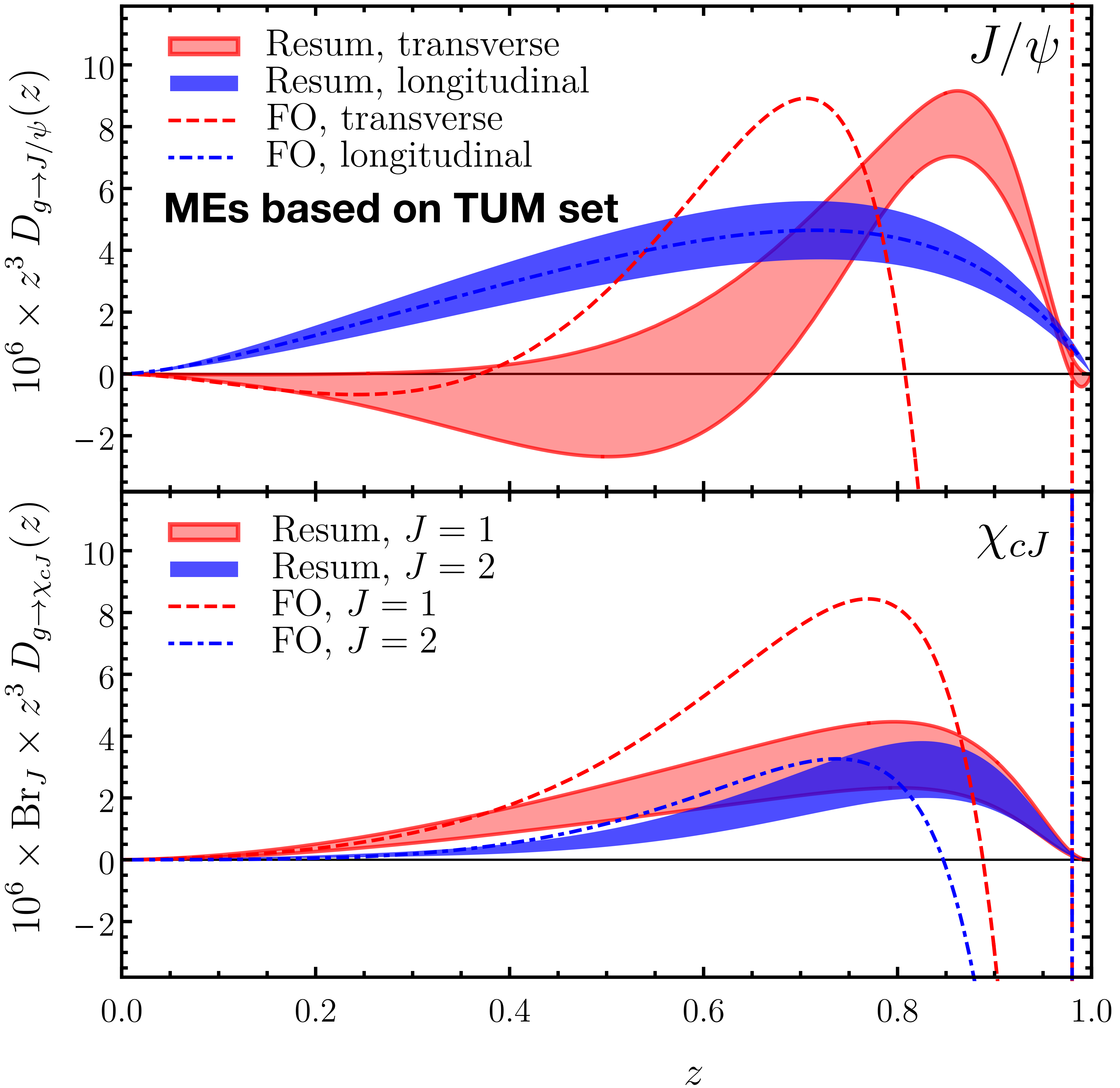
- While the exponent diverges double logarithmically, the ***resummed exponential vanishes faster than any power of N***, so that the resummed soft function has a ***convergent*** *inverse Mellin transform.*

# Fragmentation Functions

- The resummed fragmentation functions are now smooth functions that vanish at $z = 1$.

- The resummed fragmentation functions are semi-positive definite, which ensures positivity of cross sections.
  ***This essentially resolves the negative cross section problem.***

- The polarized fragmentation functions lead to the estimate $-0.25 \lesssim \lambda_\theta \lesssim +0.15$ for direct $J/\psi$ and $\psi(2S)$ polarization at $p_T$=100 GeV at midrapidity.
  Shapes of transverse and longitudinal FFs suggest very slow rise of $\lambda_\theta$ with $p_T$.



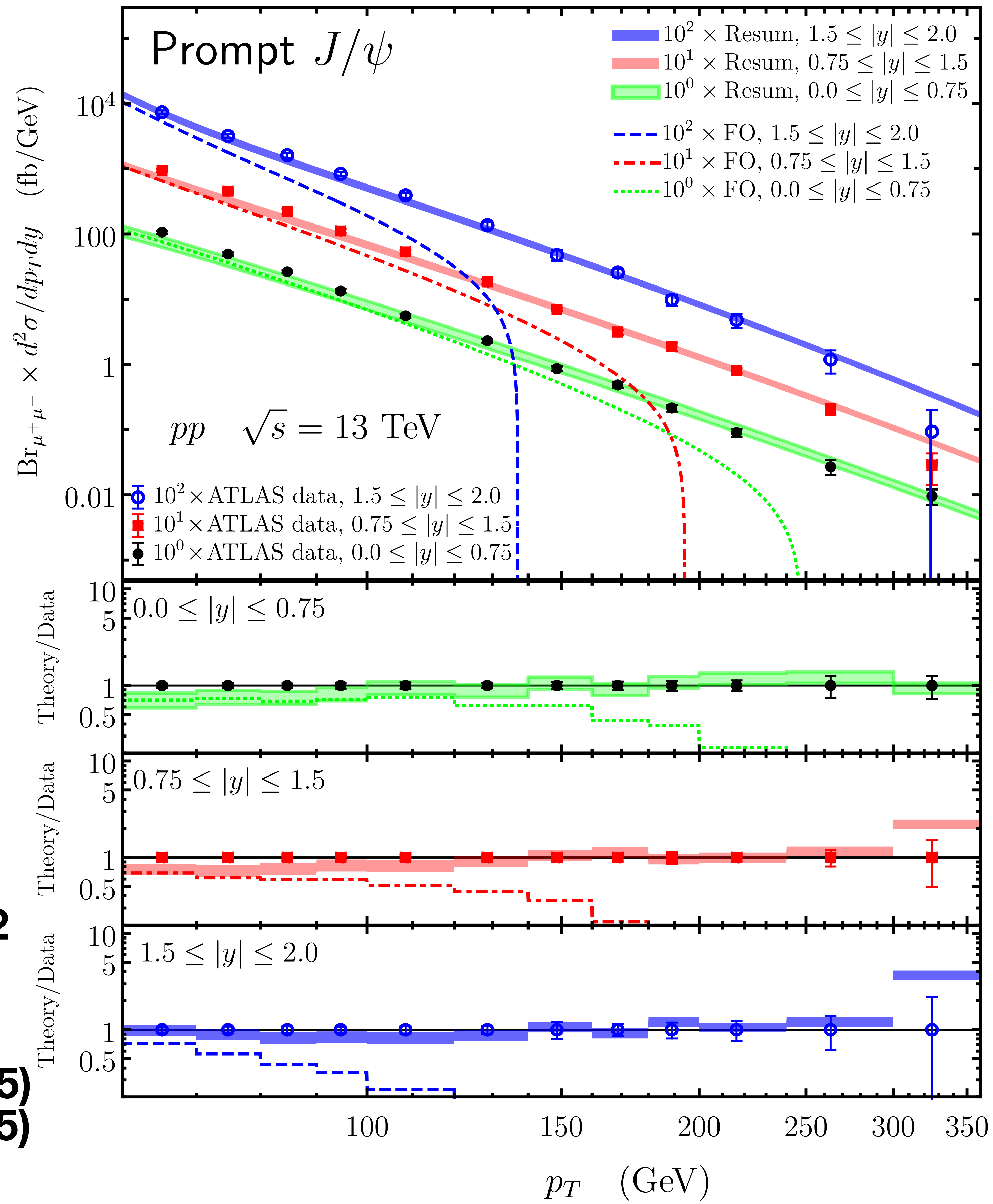

With DGLAP evolution from 3 GeV to 50 GeV

# Large-$p_T$ Cross Sections

- The resummed cross sections agree well with ATLAS data at large $p_T$.
- Predictions for prompt $J/\psi$ include feeddowns from $\psi(2S)$ and $\chi_{cJ}$.
- Fragmentation (leading power) contributions are included to NLO accuracy with threshold resummation and DGLAP resummation. Next-to-leading power contributions are also included to NLO.
- Results shown from MEs based on TUM set.

**$J/\psi$ and $\psi(2S)$ : Brambilla, *HSC*, Vairo, Wang, JHEP03 (2023) 242**
**$\chi_{cJ}$ : Brambilla, *HSC*, Vairo, JHEP 09 (2021) 032**

Similar results for direct $J/\psi$ are obtained from MEs based on $J/\psi$ *and* $\eta_c$ data. **K.T.Chao et al, PRL114, 092005 (2015)** **W.L.Sang et al, PRL114, 092006 (2015)**



# Summary

- We resummed ***threshold logarithms*** that appear in $J/\psi$, $\psi(2S)$ and $\chi_{cJ}$ inclusive production cross sections at the leading double logarithmic level.
  This ***resolves the negative cross section problem*** in fixed-order calculations in NRQCD.

- Resummation leads to ***solid*** predictions of prompt $J/\psi$ production rates that *agree well with ATLAS data* at very large $p_T$.

- Resummation *removes the singularities* in the fragmentation functions near the boundary.
  Resummation may also be important for observables involving *kinematical cuts* near the boundary, such as photoproduction rates and fragmenting jet functions.

- Theshold resummation may be extended to *single logarithmic level*, as well as to *double-parton (next-to-leading power)* fragmentation contributions.
  Resummation for non-singular fragmentation functions will require *next-to-leading logarithmic* accuracy.

# Backup

# What happens at large $p_T$

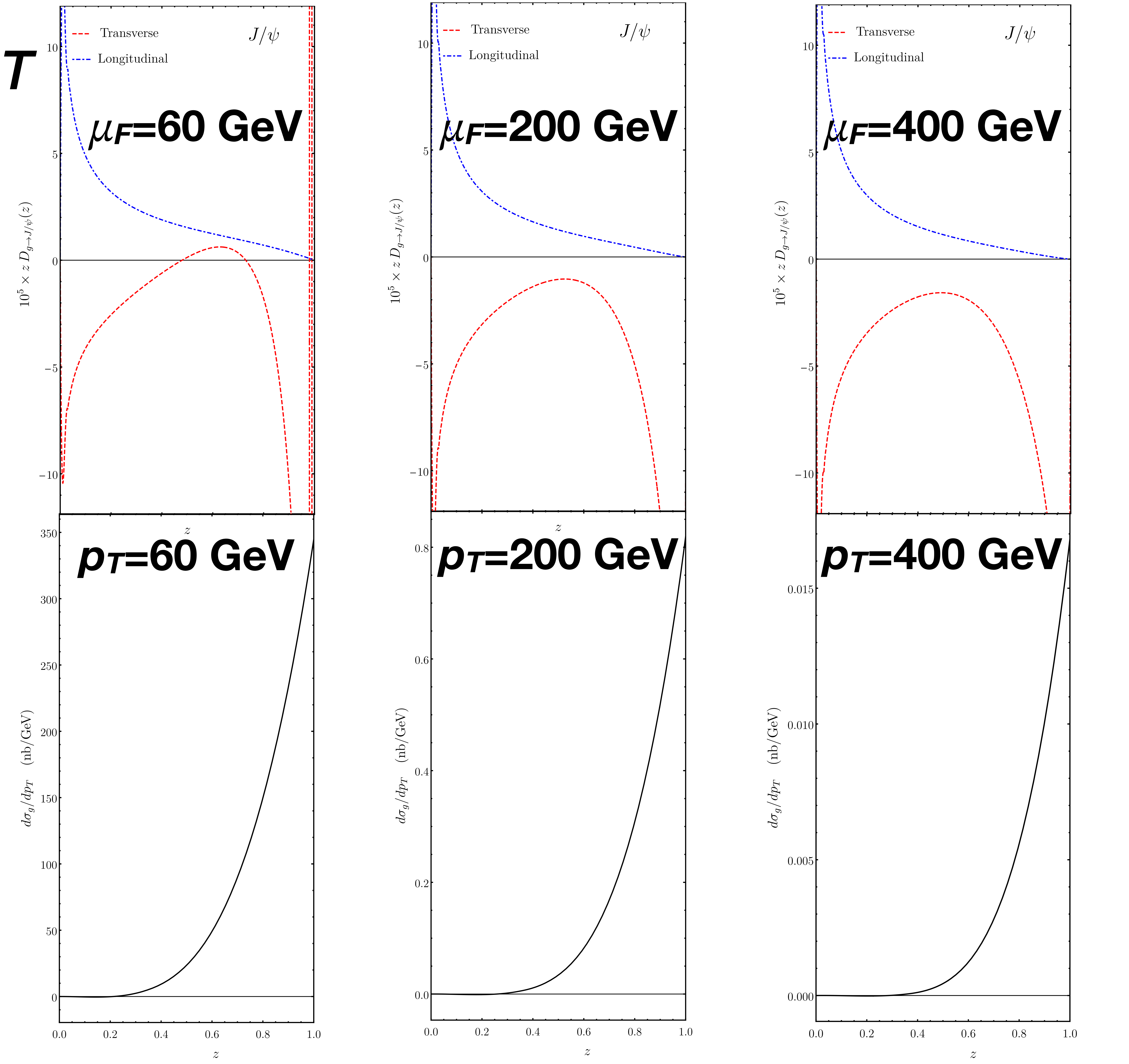
- DGLAP evolution makes the problem worse.

**$J/\psi$ fragmentation function**
**DGLAP leading logs resummed**
**MEs from TUM set**

- Note that FFs still contain distributions singular at $z=1$ after DGLAP evolution.

**See e.g. *PRD* 93, 034041 (2016)**

**gluon production rates**



# What happens at large $p_T$

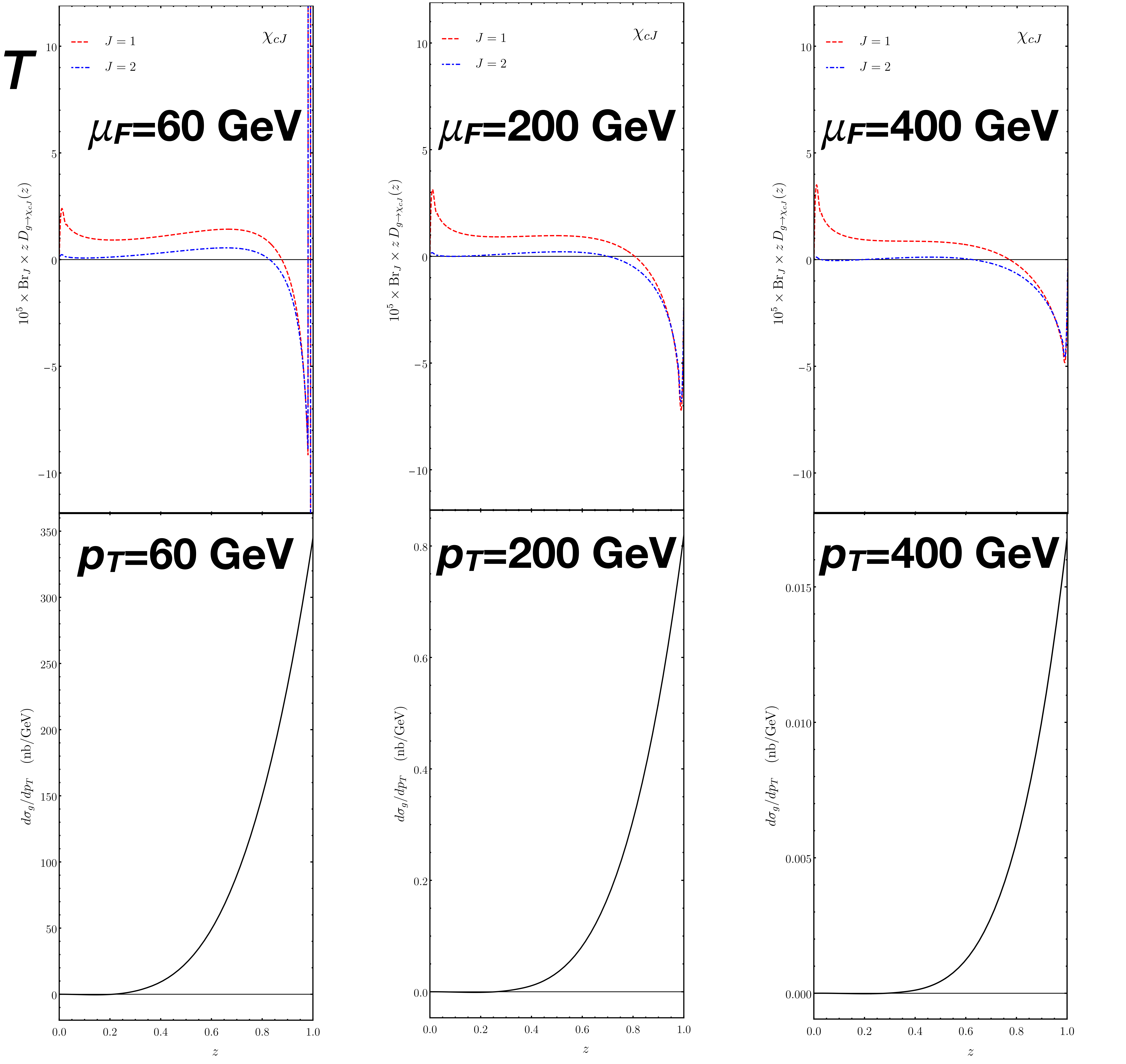
- DGLAP evolution makes the problem worse.

**$\chi_{cJ}$ fragmentation functions**
**DGLAP leading logs resummed**
**MEs from TUM set**

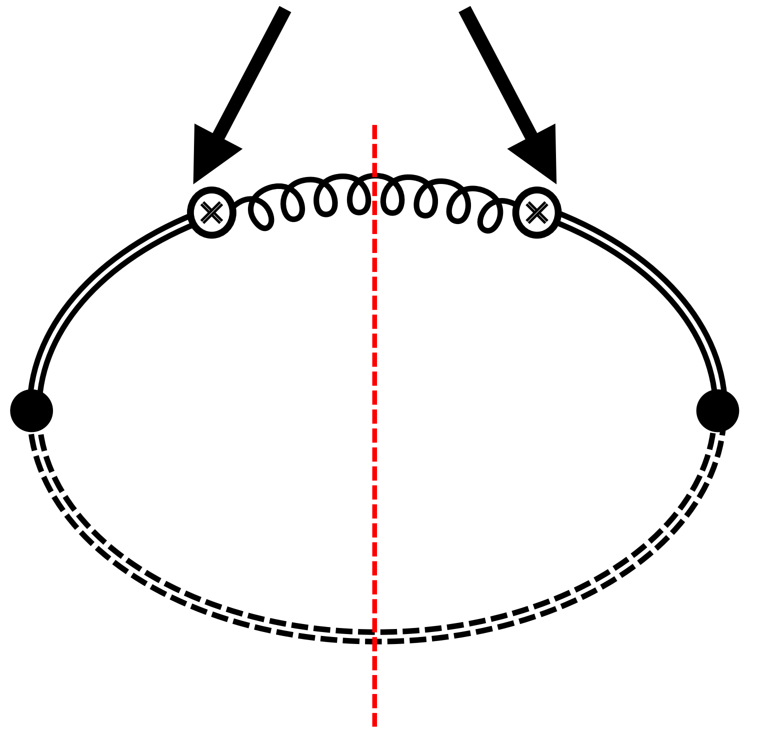
- Note that FFs still contain distributions singular at $z = 1$ after DGLAP evolution.

**See e.g. *PRD* 93, 034041 (2016)**

**gluon production rates**



# Soft function for ${}^3P_J^{[1]}$




- The ${}^3P_J^{[1]}$ soft function is

$$S_{{}^3P^{[1]}} = \langle 0 | [\mathcal{W}^b_{\beta'}({}^3P^{[1]})]^\dagger 2\pi\delta(n\cdot\hat{p} - P^+(1-z))\mathcal{W}^b_\beta({}^3P^{[1]})|0\rangle g^{\beta\beta'}$$

$$\mathcal{W}^b_\beta({}^3P^{[1]}) \equiv \int_0^\infty d\lambda\,\lambda\, p^\mu G^d_{\mu\beta}(p\lambda)\Phi_p^{da}(\lambda,0)\Phi_n^{ba}(\infty,0)$$

which is essentially the $\chi_{cJ}$ shape function in NRQCD

$$\mathcal{S}^{\chi_{Q0}}_{{}^3S_1^{[8]}}(l_+) = \langle \chi^\dagger\sigma^i T^a\psi\Phi_\ell^{\dagger ab}(0)\mathcal{P}_{\chi_{Q0}}\delta(l_+ - iD_+)\Phi_\ell^{bc}(0)\psi^\dagger\sigma^i T^c\chi\rangle$$ **HSC, JHEP 07 (2023) 007**

with the $\chi_{cJ}$ wavefunction factored out.

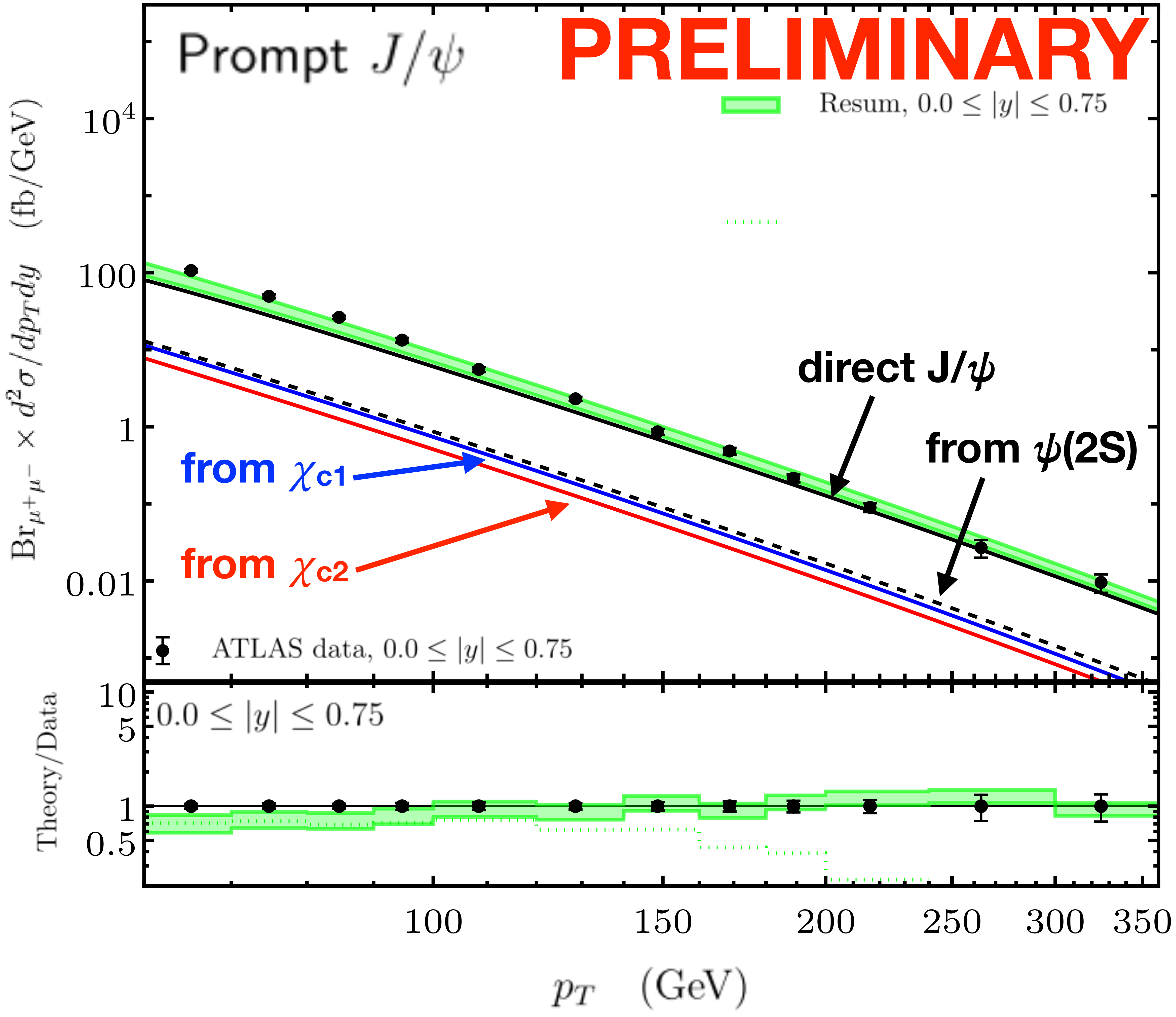
- There is an additional soft function arising from the anisotropic contribution

$$S^{TT}_{{}^3P^{[1]}} = \langle 0 | [\mathcal{W}^b_{\beta'}({}^3P^{[1]})]^\dagger 2\pi\delta(n\cdot\hat{p} - P^+(1-z))\mathcal{W}^b_\beta({}^3P^{[1]})|0\rangle \left(\frac{p^2 n_\beta n_{\beta'}}{p_+^2} + \frac{g_{\beta\beta'}}{d-1}\right)$$

The anisotropic term is suppressed by $\epsilon=(4-d)/2$ compared to the isotropic term, so that it only has a single UV pole at NLO, and is IR finite consistently with NRQCD factorization. This does not produce double logs, but is necessary to reach single log accuracy.

# Large-$p_T$ Cross Sections

- Feeddown contributions




- Feeddown fractions