Heavy meson Lightcone distribution amplitudes from Lattice QCD

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Based on 2403.17492 and work in progress

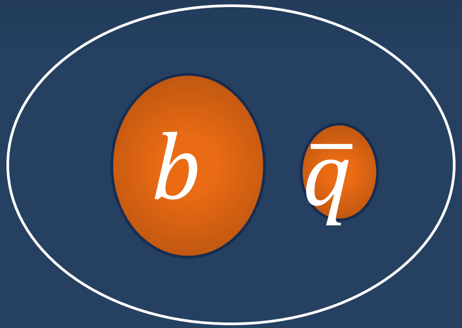
XVIth Quark Confinement and the Hadron Spectrum Conference

Outline

- Heavy meson LCDAs
- Two-step matching method
- Lattice QCD calculations of LCDAs
- Summary and outlook

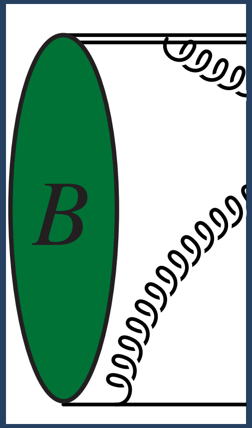
What is heavy meson LCDA?

- Heavy meson: B meson

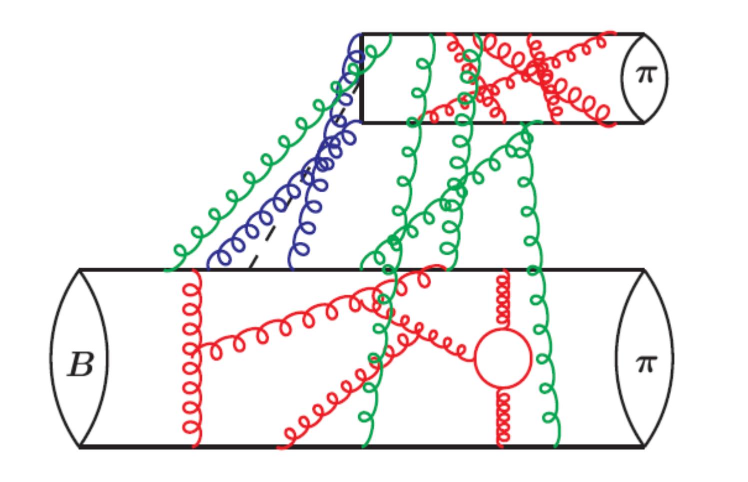


- LCDA: light-ray HQET matrix element [Grozin, Neubert, PRD55, 272 (1997)]

$$\langle H(p_H)|\bar{h}_v(0)\not{n}_+\gamma_5[0,tn_+]q_s(tn_+)|0\rangle = -i\tilde{f}_H m_H n_+\cdot v\int_0^\infty d\omega e^{i\omega t n_+\cdot v}\varphi_+(\omega;\mu)$$



Why is LCDA important?



$$\langle \pi(p')\pi(q)|Q_i|\bar{B}(p)\rangle = f^{B\to\pi}(q^2)\int_0^1 dx T_i^{\mathrm{I}}(x)\phi_\pi(x) + \int_0^1 d\xi dx dy T_i^{\mathrm{II}}(\xi,x,y)\phi_B(\xi)\phi_\pi(x)\phi_\pi(y)$$

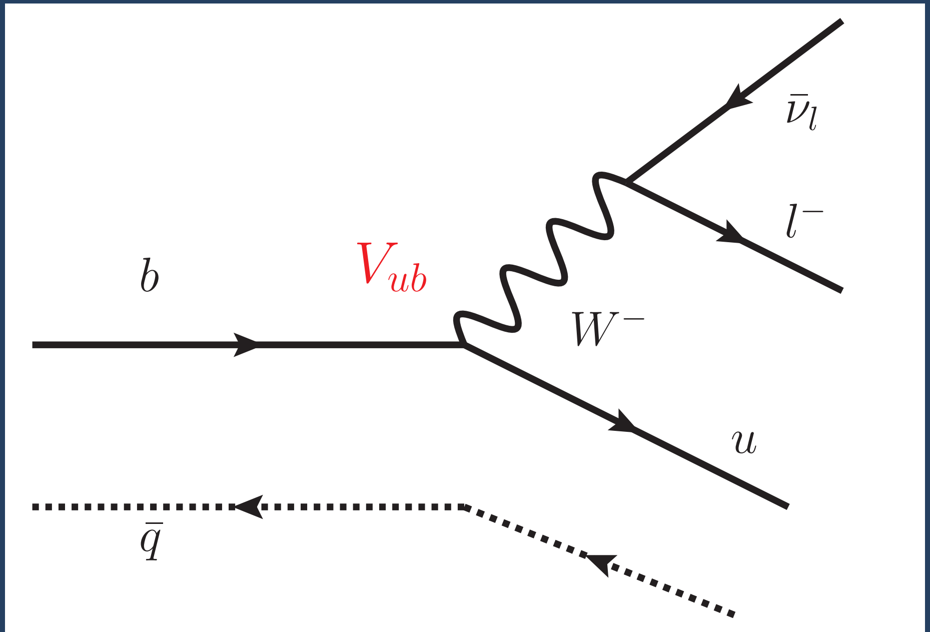
$B\to\pi$ form factor

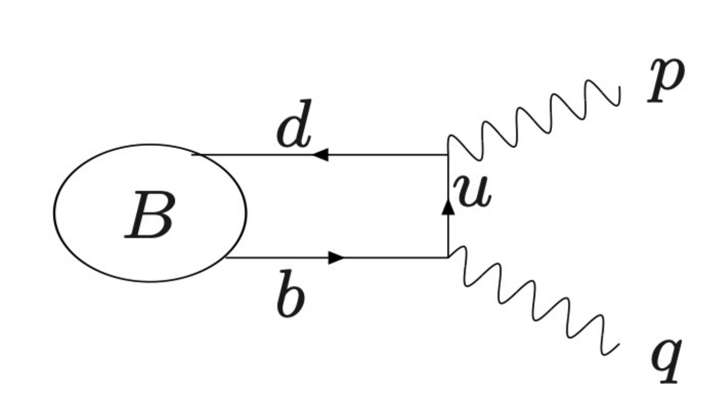
Hard kernel

B-meson LCDA

QCD Factorization: BBNS, PRL 83, 1914 (1999)
For PQCD, See: Keum, Li, Sanda PRD 63,054008 (2001)

Why is LCDA important?





B meson LCSR:
De Fazio, Feldmann, Hurth, NPB 733, 1 (2006)
Khodjamirian, Mannel, Offen, PLB620,52 (2005)

What do we know about HM LCDA?

- Equation of Motion: *[Kawamura, Kodaira, Qiao, Tanaka, PLB523, 111 (2001)]*

- Evolution equations: *[Lange, Neubert, 2003; Bell, Feldmann, 2008]*

$$\frac{d}{d\ln\mu}\phi_B^+(\omega,\mu) = -\frac{\alpha_s C_F}{4\pi}\int_0^\infty d\omega' \gamma_+^{(1)}(\omega,\omega',\mu)\,\phi_B^+(\omega',\mu) + \mathcal{O}(\alpha_s^2)\,.$$

$$\gamma_+^{(1)}(\omega,\omega',\mu) = \left(\Gamma_{\rm cusp}^{(1)}\,\ln\frac{\mu}{\omega} - 2\right)\delta(\omega-\omega') - \Gamma_{\rm cusp}^{(1)}\,\omega\left[\frac{\theta(\omega'-\omega)}{\omega'(\omega'-\omega)} + \frac{\theta(\omega-\omega')}{\omega(\omega-\omega')}\right]_+ .$$

  - Solution of evolution equations. *[Bell, Feldmann, Wang, Yip, 2013; Braun, Manashov, 2014]*
  - RG equations of $\phi_B^+(\omega,\mu)$ at two-loops. *[Braun, Ji, Manashov, 2019; Liu, Neubert, 2020]*
  - RG equations of the higher-twist B-meson distribution amplitudes. *[Braun, Ji, Manashov, 2017]*

- Perturbative constraint for large ω *[Lee, Neubert, PRD72 (2005) 094028]*

$$\phi_+(\omega,\mu) = \frac{C_F\alpha_s}{\pi\omega}\left[\left(\frac{1}{2} - \ln\frac{\omega}{\mu}\right) + \frac{4\bar{\Lambda}}{3\omega}\left(2 - \ln\frac{\omega}{\mu}\right)\right]$$

But…

Models for heavy meson LCDAs

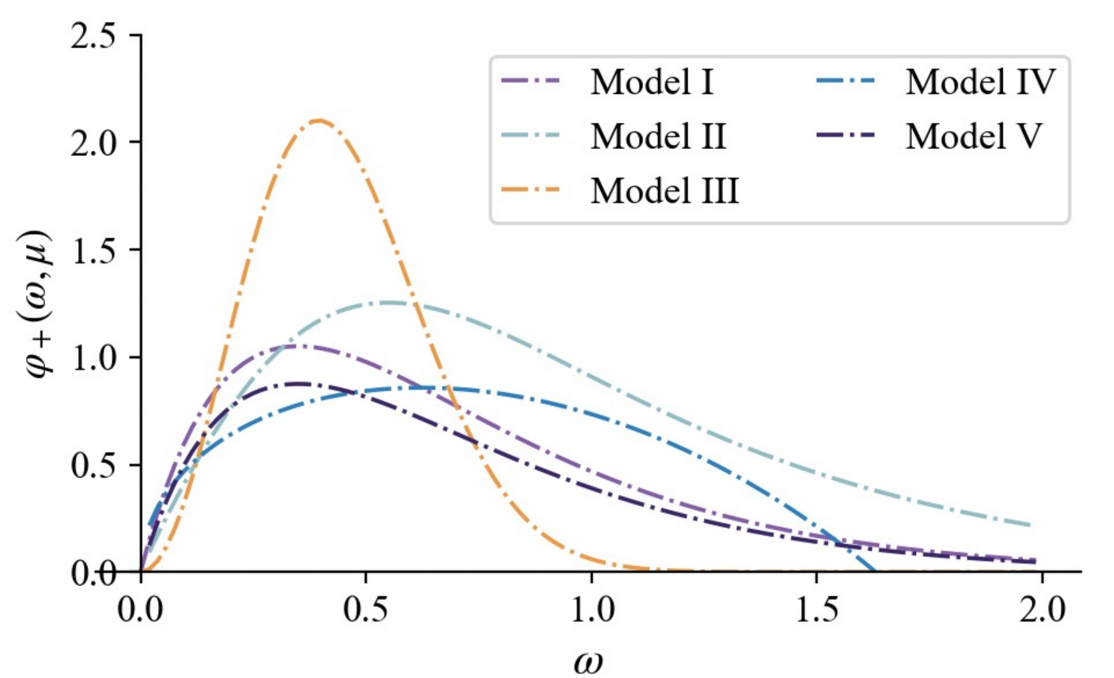
$$\varphi_{\mathrm{I}}^{+}(\omega,\mu_0)=\frac{\omega}{\omega_0^2}e^{-\omega/\omega_0}\,,$$

$$\varphi_{\mathrm{II}}^{+}(\omega,\mu_0)=\frac{4}{\pi\omega_0}\frac{k}{k^2+1}\left[\frac{1}{k^2+1}-\frac{2\left(\sigma_B^{(1)}-1\right)}{\pi^2}\ln k\right]$$

$$\varphi_{\mathrm{III}}^{+}(\omega,\mu_0)=\frac{2\omega^2}{\omega_0\omega_1^2}e^{-(\omega/\omega_1)^2}\,,$$

$$\varphi_{\mathrm{IV}}^{+}(\omega,\mu_0)=\frac{\omega}{\omega_0\omega_2}\frac{\omega_2-\omega}{\sqrt{\omega(2\omega_2-\omega)}}\theta(\omega_2-\omega)\,,$$

$$\varphi_{\mathrm{V}}^{+}(\omega,\mu_0)=\frac{\Gamma(\beta)}{\Gamma(\alpha)}\frac{\omega}{\omega_0^2}e^{-\omega/\omega_0}U(\beta-\alpha,3-\alpha,\omega/\omega_0)\,,$$

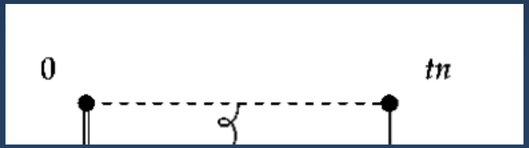


$$f^{+}_{B\to\pi}(0)=0.122\times\left[1\pm0.07\big|_{S_0^\pi}\pm0.11\big|_{\bar\Lambda_q}\pm0.02\big|_{\lambda_E^2/\lambda_H^2}{}^{+0.05}_{-0.06}\big|_{M^2}\pm0.05\big|_{2\lambda_E^2+\lambda_H^2}\right.$$
$$\left.{}^{+0.06}_{-0.10}\big|_{\mu_h}\pm0.04\big|_{\mu}{}^{+1.36}_{-0.56}\big|_{\lambda_{B_q}}{}^{+0.25}_{-0.43}\big|_{\hat\sigma_1,\hat\sigma_2}\right]$$

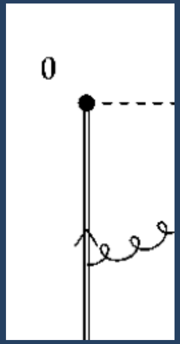
Cui, et.al, JHEP 03 (2023) 140

Difficulties in first-principle determinations

Lightcone:

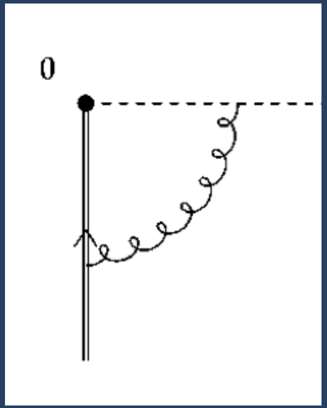


Heavy quark Field:



$$\frac{1}{iv \cdot k}$$

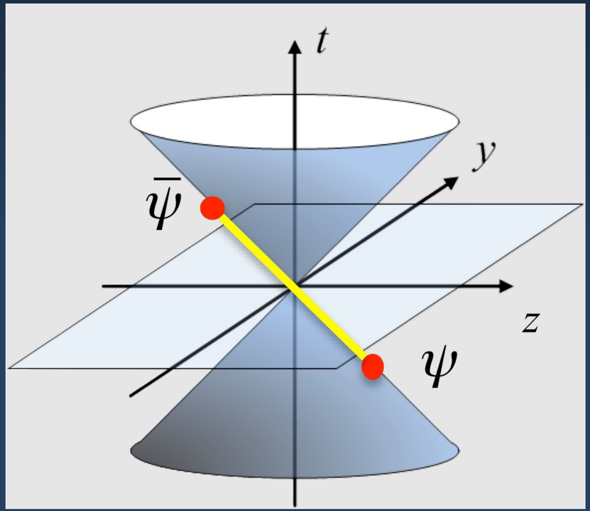
Cusp divergence:

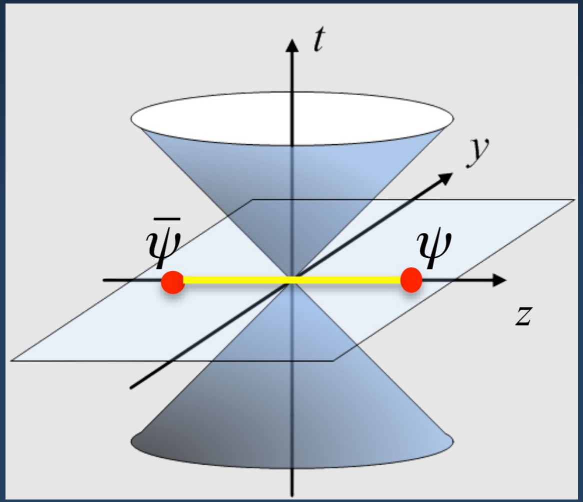


$$O_+^{\rm ren}(t,\mu) = O_+^{\rm bare}(t) + \frac{\alpha_s C_F}{4\pi}\left\{\left(\frac{4}{\hat\epsilon^2} + \frac{4}{\hat\epsilon}\ln(it\mu)\right) O_+^{\rm bare}(t) - \frac{4}{\hat\epsilon}\int_0^1 du \frac{u}{1-u}\left[O_+^{\rm bare}(ut) - O_+^{\rm bare}(t)\right]\right.$$

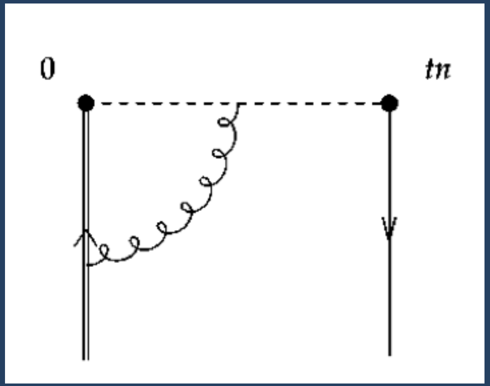
Braun, Ivanov, Korchemsky, PRD 69, 034014 (2004)

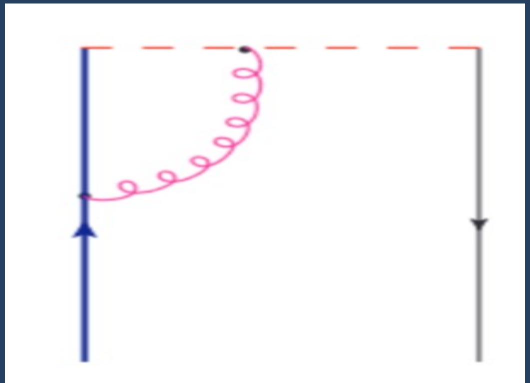
How to solve this problem?





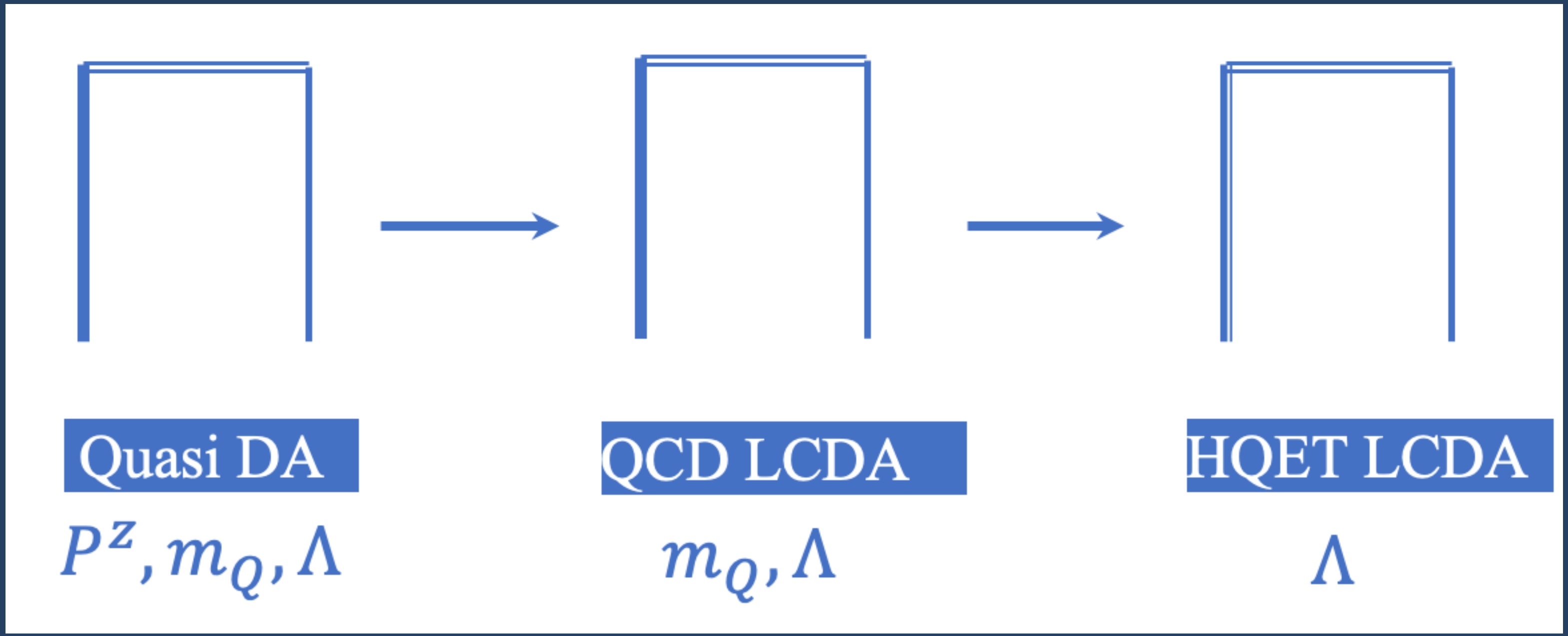
LaMET[*Ji, PRL 110, 262002 2013*]:
lightcone can be accessed by simulating correlation functions with a large but finite P^z





HQET fields can be accessed by simulating correlation functions with a large but finite mQ

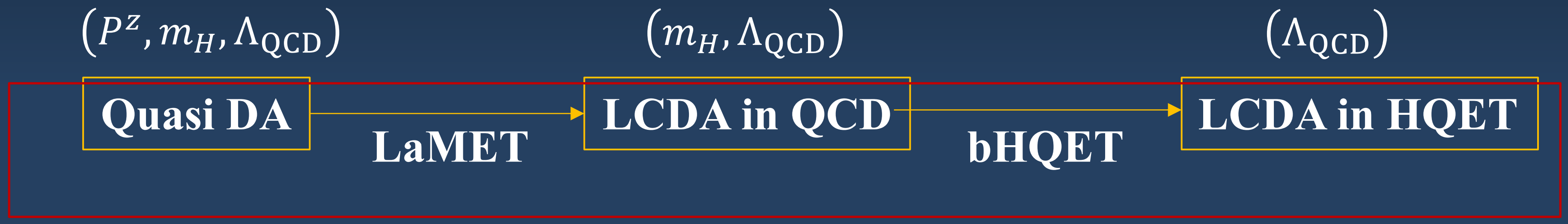
How to solve this problem?



Two-step method: ***Han, et.al, 2403.17492***

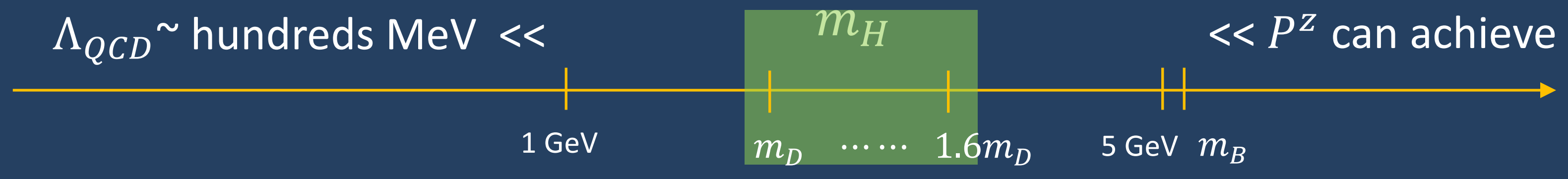
A two-step matching method

➢ Start from Quasi DA, calculable from LQCD



- A multi-scale processes:
  1. LaMET requires $\Lambda_{\rm QCD}, m_H \ll P^z$ and finally integrate out P^z;
  2. bHQET requires $\Lambda_{\rm QCD} \ll m_H$ and integrate out m_H;

$\Rightarrow$ **Hierarchy $\boldsymbol{\Lambda_{\rm QCD} \ll m_H \ll P^z}$:** A big challenge for lattice simulation but still calculable on the lattice

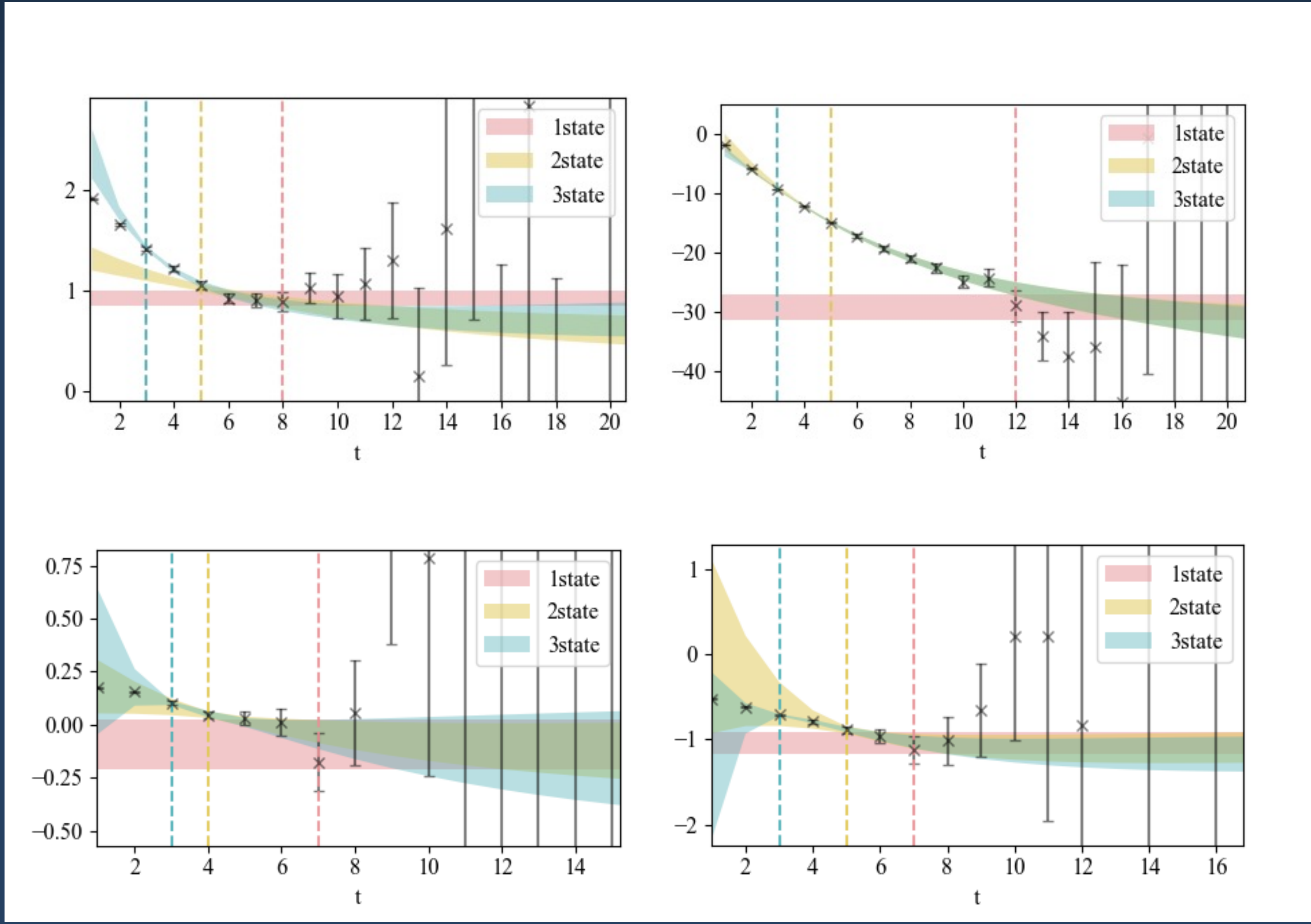


Step 0: Lattice setup

- H48P32, $n_s^3 \times n_t = 48^3 \times 144$, $a = 0.05187$fm;
- $m_\pi \simeq 317$MeV, $m_{\eta_s} = 700$MeV;
- Determine the charm quark mass by tuning $m_{J/\psi}$ to its physical value, then $m_D \simeq 1.90$GeV;
- Coulomb gauge fixed grid source with grid $= 1 \times 1 \times n_s$; 549 configurations $\times$ 8 measurements.

Han, et.al,2403.17492, updated

Step 1: Fit strategy of nonlocal correlation functions

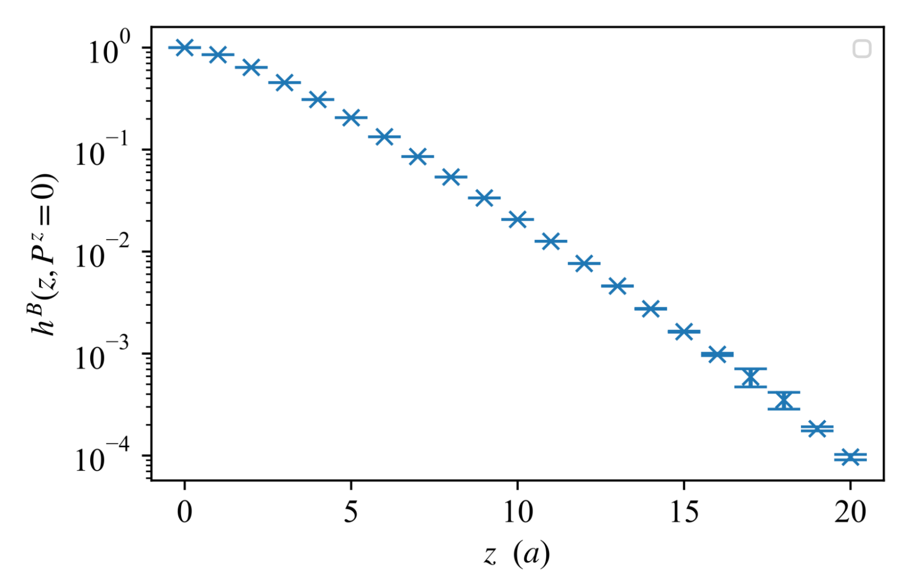


- We compare the 1, 2, 3-state fits. All the fit results are consistent with each other.
- Different fit strategy valid at different t-range.
- To balance the signal of data and reliability of the multi-state fits, we prefer the 1-state fit, and select the 2-state fit when the former one is inadequate to describe the data. Result from 3-state fit is only used as a reliability check for the first two strategy.

Step 1: Renormalization in the hybrid-ratio scheme

$$\tilde{h}^R(z,P^z) = \begin{cases} \frac{\tilde{h}^B(z,P^z)}{\tilde{h}^B(z,P^z=0)} & |z| < z_s \\ e^{(\delta m + m_0)(z-z_s)} \frac{\tilde{h}^B(z,P^z)}{\tilde{h}^B(z_s,P^z=0)} & |z| \geq z_s \end{cases},$$

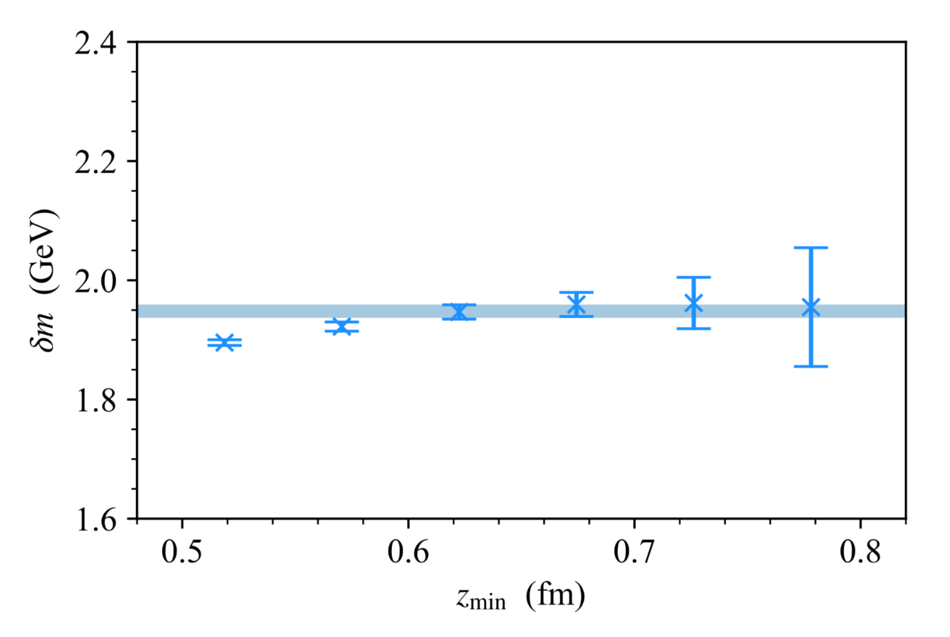
- We use the zero momentum matrix element $\tilde{h}^B(z, P^z = 0)$ to renormalize the bare ones.
- The Dirac structure of zero momentum matrix element is $\gamma^t\gamma_5$, it contains same UV divergence as the one with $\gamma^z\gamma_5$.



Ji, Liu, Schäfer, Wang, Yang, Zhang, Zhao, NPB 964, 115311 (2021)

Step 1: Renormalization in the hybrid-ratio scheme

- The linear divergence factor δm can be extracted from fitting the long-range correlations of the zero momentum matrix element.
- We fit the exponential decay behavior $Ae^{-\delta m z}$ of the matrix element at large-z range to extract δm. We choose the fit range $z \in [z_{\mathrm{min}}, z_{\mathrm{min}} + 4a]$ with varying z_{min} from $10a$ to $15a$ to check the stability of the fits.
- We find the fit of δm becomes stable at $z_{\mathrm{min}} > 0.6\mathrm{fm}$, that suggests the extraction of δm is universal. By performing a constant fit, we obtain the value of $\delta m = 1.948(11)\mathrm{GeV}$.



Step 1: Renormalization in the hybrid-ratio scheme

- The renormalon ambiguity m_0 can be extracted from the matching between perturbative and lattice results:

$$H^B(z,0;a)e^{(\delta m(a,\tau)+m_0(\tau))z} = \underline{C_0(z,\mu_0;\tau)}\,\underline{e^{-\mathcal{I}(\mu_0)}}\,\underline{e^{\mathcal{I}^{\rm lat}(a^{-1})}},$$

C_0: perturbative Wilson coefficient up to NLO. We also try the LRR improvement.

$$C_0^{\rm NLO}(z,\mu) = 1 + \frac{\alpha_s C_F}{2\pi}\left[\frac{3}{2}\log\left(\frac{1}{4}e^{2\gamma_E}z^2\mu^2\right) + A\right],$$

$A = \frac{5}{2}$ for $\gamma^t/\gamma^t\gamma_5$, $\frac{7}{2}$ for $\gamma^z/\gamma^z\gamma_5$

$$C_0^{\rm NLO+LRR}(z,\mu) = C_0^{\rm NLO}(z,\mu) + z\mu\left(C_{\rm PV}(z,\mu) - \sum_{i=0}^{k-1}\alpha_s^{i+1}(\mu)r_i\right),$$

Yao, Ji, Zhang, JHEP 11 (2023) 021

$\mathfrak{I}(\mu_0)$: RGE from $\mu_0 = 2e^{-\gamma_E}z^{-1}$ to $\overline{\rm MS}$ scale μ.

$$\mathcal{I}(\mu) = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d\alpha\,\frac{\gamma(\alpha)}{\beta(\alpha)}$$

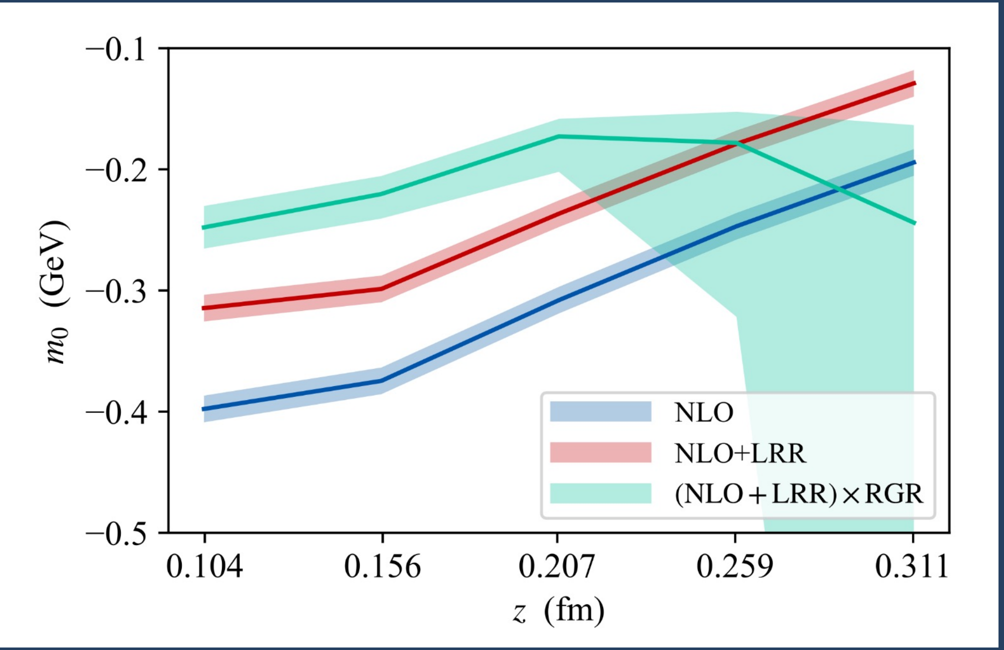
$\mathfrak{I}^{\rm lat}(a^{-1})$: For a single lattice spacing, this term is a constant.

Step 1: Renormalization in the hybrid-ratio scheme

The extraction of m_0:

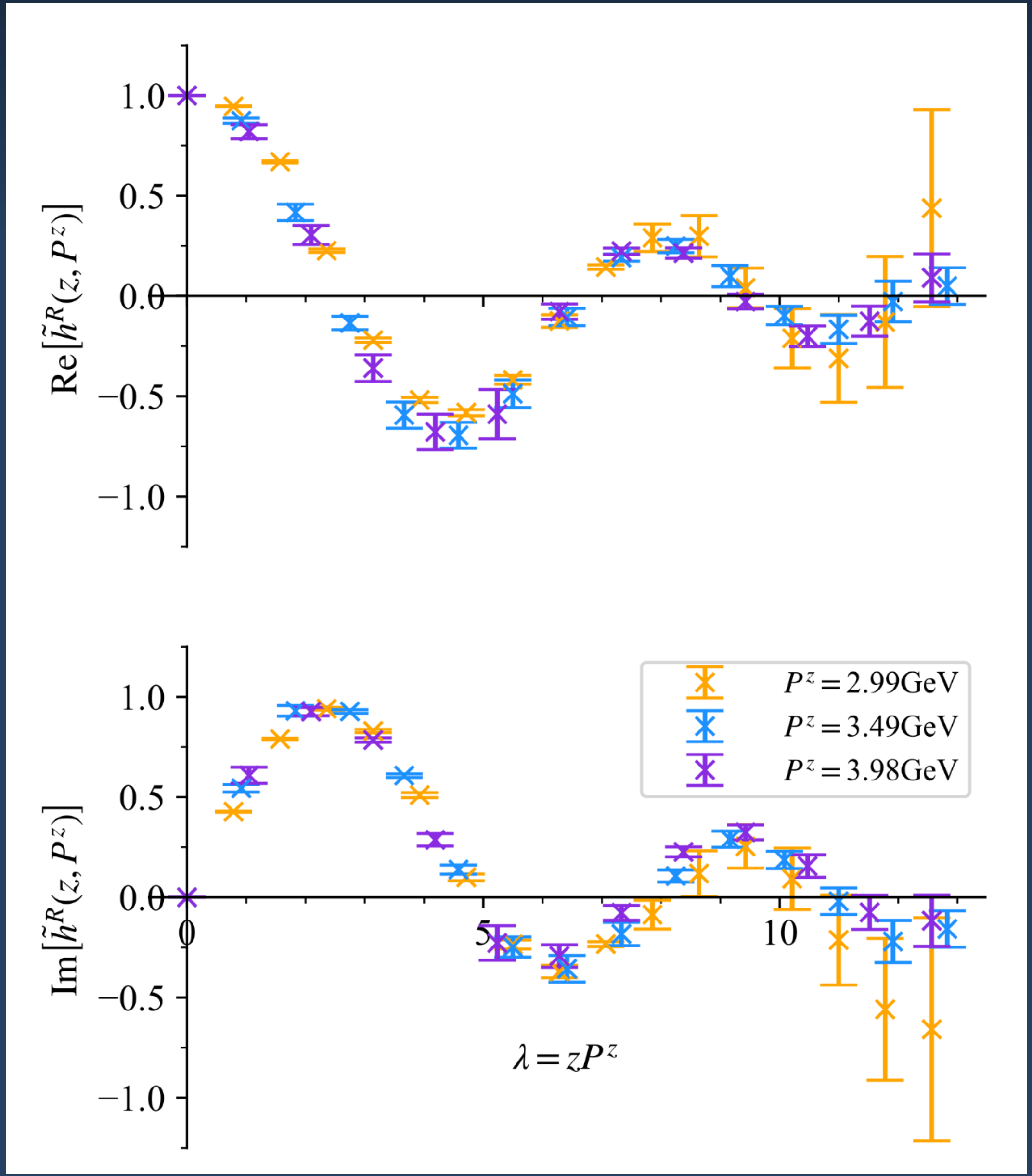
- The fit is performed at region $0 \ll z < z_s$ and in range $[z-a, z+a]$ for each z.
- We compare the schemes 'NLO', 'NLO+LRR' and '(NLO+LRR)×RGR'.
- We choose $\mu = 2$GeV, and vary μ_0 with a factor from 0.8-1.2 to introduce the scaling uncertainty.
- The fixed-order results contain visible dependence on z, and these z-dependence will be removed by the RGR improvement at $z \simeq 0.2$fm.
- We take the result $m_0 = -0.173^{+0.014}_{-0.029}$ GeV at $z = 0.207$fm. Accordingly, the choice of z_s should be larger than 0.3fm.

$$(m_0 + \delta m)z - I_0 = \ln\left[\frac{C_0^{\mathrm{NLO(+LRR)(\times RGR)}}(z,\mu)}{H^B(z,0)}\right],$$



Step 1: Renormalized matrix elements

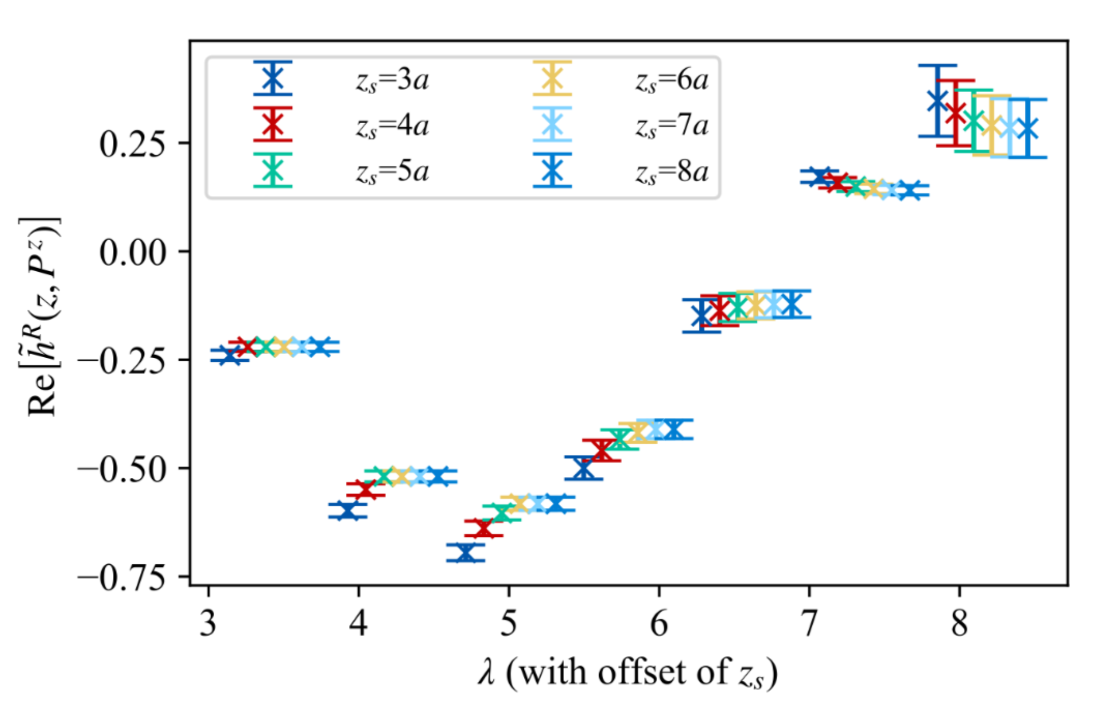
- The renormalized matrix elements at different momenta are basically consistent with each other.



Step 1: Renormalized matrix elements

z_s dependence of renormalized matrix elements:

$$\tilde{h}^R(z,P^z) = \begin{cases} \frac{\tilde{h}^B(z,P^z)}{\tilde{h}^B(z,P^z=0)} & |z| < z_s \\ e^{(\delta m + m_0)(z-z_s)} \frac{\tilde{h}^B(z,P^z)}{\tilde{h}^B(z_s,P^z=0)} & |z| \geq z_s \end{cases},$$



- z_s denotes the boundary between perturbative and nonperturbative regions.
- $\mathcal{O}(z_s^{-1})$ typically chosen to be around hundreds MeV to few GeV.
- We find that at $z_s \geq 6a \simeq 0.31$fm, the renormalized matrix elements become consistent with each other.
- Differences between $z_s = (6,7,8)a$ are small. We choose $z_s = 6a$.

Step 1: λ-extrapolation

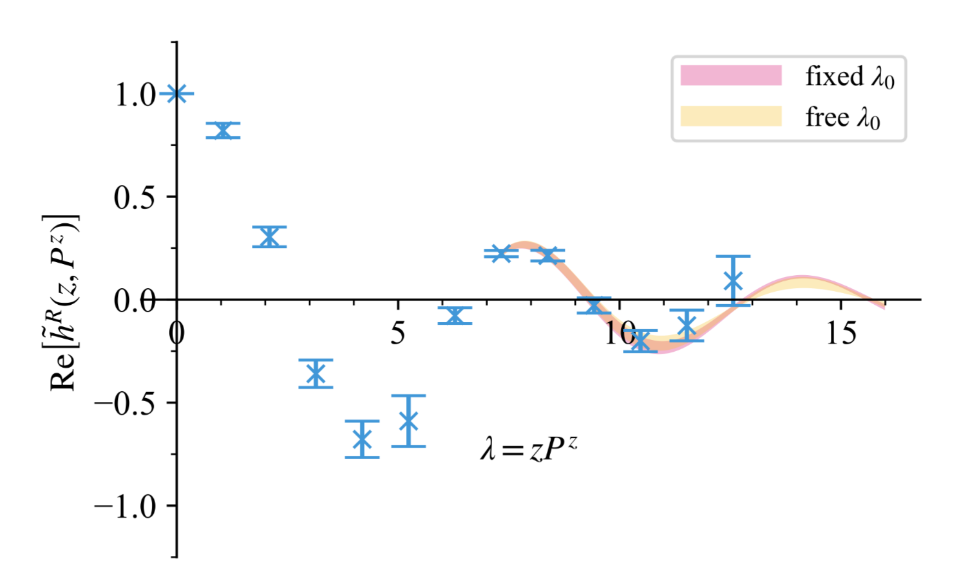
We extrapolate the renormalized matrix elements to infinity based on the data at large λ:

- The parameterization inside the square brackets account for the algebraic behavior and motivated by the Regge behavior of the light-cone distributions at endpoint regions.

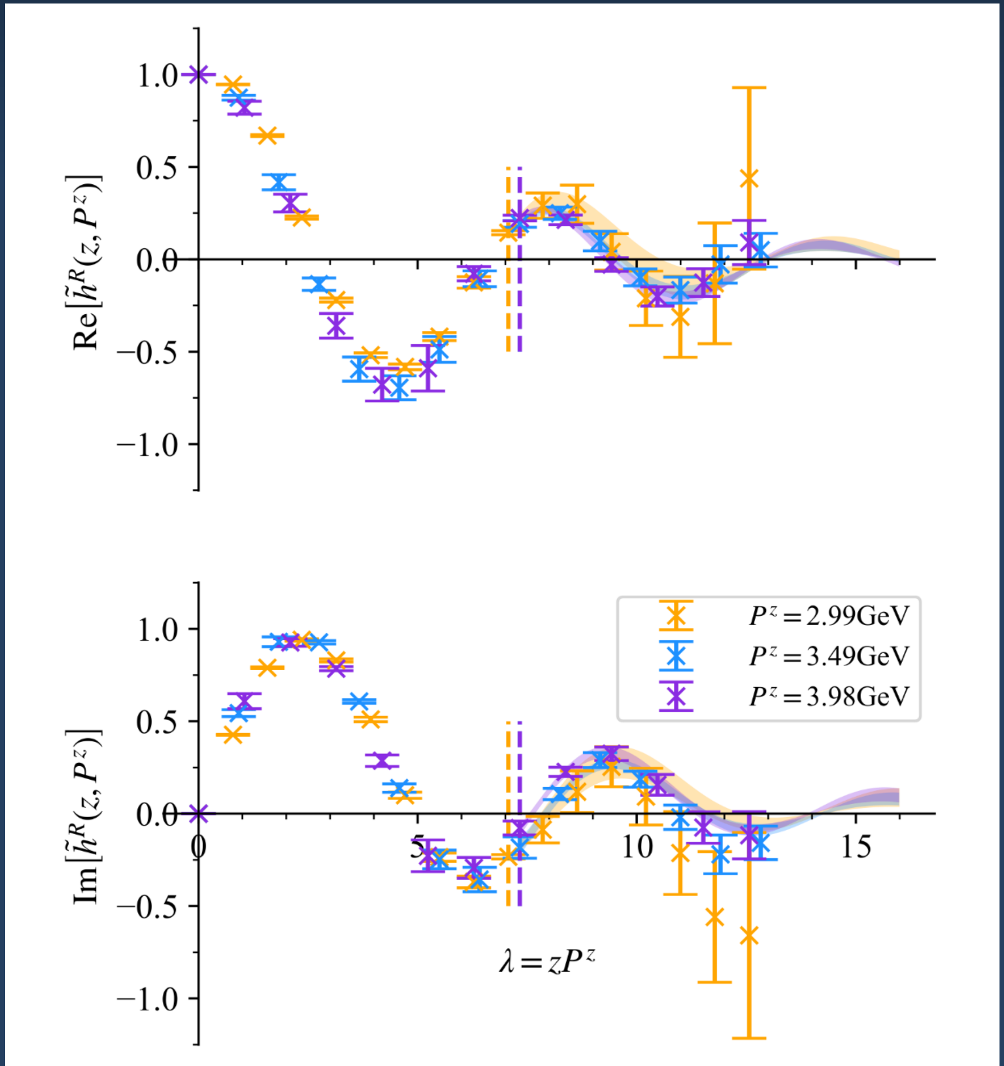
$$\tilde{h}^R(\lambda) = \left[\frac{c_1}{(-i\lambda_1)^{d_1}} + e^{i\lambda}\frac{c_2}{(i\lambda_2)^{d_2}}\right] e^{-\lambda/\lambda_0},$$

- The exponential decay behavior is governed by the decaying $\propto e^{-\delta m z}$ at long-tail region. Based on the definition of hybrid ratio scheme, the renormalized matrix elements decaying with $e^{m_0(z-z_s)}$, which related to the finite correlation length $\lambda_0 \sim -P^z/m_0$.

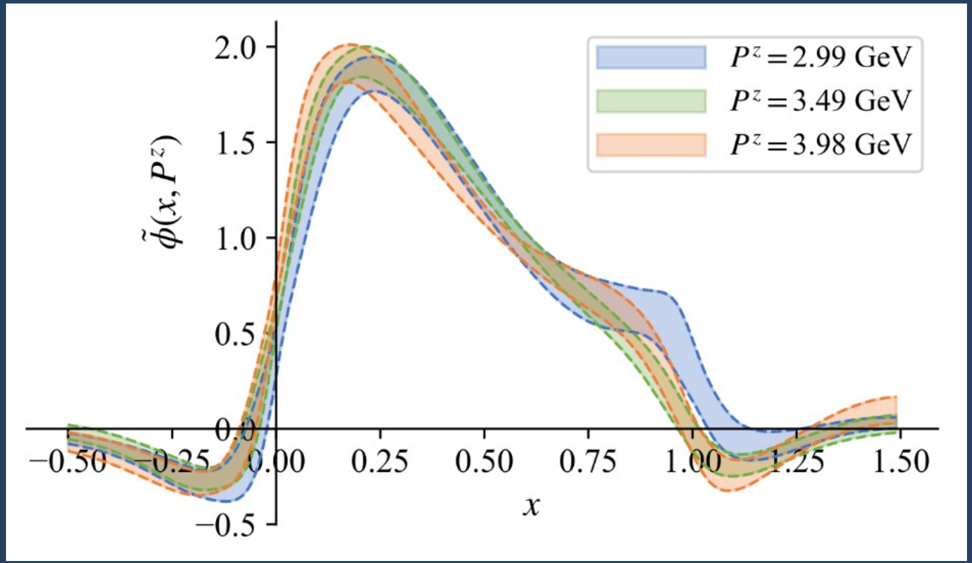
- We compare the extrapolation from "fixed λ_0" and "free λ_0". The results from two strategies are consistent with each other.



Step 1: λ-extrapolation and quasi DAs



- We extrapolate the renormalized matrix elements to infinity, and then Fourier transform them to momentum space to obtain the quasi DA.
- We use the "free λ_0" strategy for conservative and adopt $\lambda_L = \{7.07, 7.34, 7.32\}$ for $P^z = \{2.99, 3.49, 3.98\}$GeV.



Step 1: LaMET matching and QCD LCDA

- The matching formula in LaMET:

$$\tilde{\phi}(x, P^z) = \int dy C(x, y, P^z)\phi(y),$$

the perturbative matching kernel up to NLO at leading power:

$$\begin{aligned} C^{(0)}(x,y) &= \delta(x-y), \\ C^{(1)}(x,y) &= C_B^{(1)} - C_{\text{CT}}^{(1)}. \end{aligned}$$

$$C_B^{(1)}\left(x,y,\frac{P^z}{\mu}\right) = \frac{\alpha_s C_F}{2\pi}\begin{cases} [H_1(x,y)]_{+(y)} & x<0<y<1 \\ \left[H_2\left(x,y,\frac{P^z}{\mu}\right)\right]_{+(y)} & 0<x<y<1 \\ \left[H_2\left(1-x,1-y,\frac{P^z}{\mu}\right)\right]_{+(y)} & 0<y<x<1 \\ [H_1(1-x,1-y)]_{+(y)} & 0<y<1<x \end{cases}$$

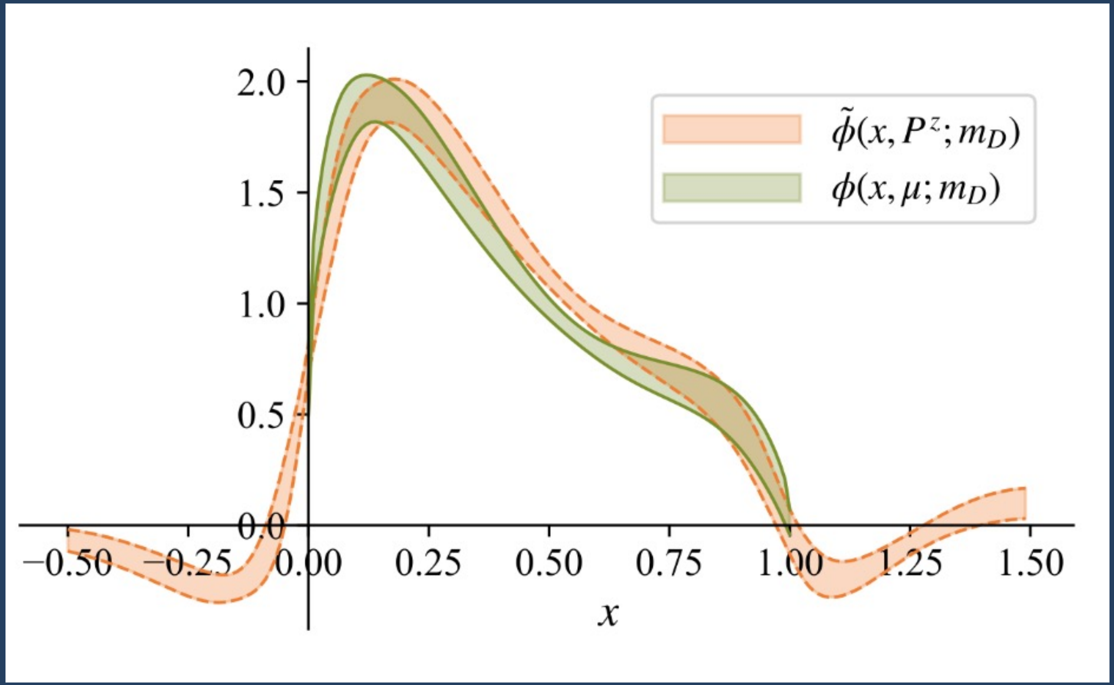
with

$$\begin{aligned} H_1(x,y) &= \frac{1+x-y}{y-x}\frac{1-x}{1-y}\ln\frac{y-x}{1-x} + \frac{1+y-x}{y-x}\frac{x}{y}\ln\frac{y-x}{-x} \\ H_2\left(x,y,\frac{P^z}{\mu}\right) &= \frac{1+y-x}{y-x}\frac{x}{y}\ln\frac{4x(y-x)(P^z)^2}{\mu^2} + \frac{1+x-y}{y-x}\left(\frac{1-x}{1-y}\ln\frac{y-x}{1-x} - \frac{x}{y}\right), \end{aligned}$$

and the counter-term is

$$\begin{aligned} C_{\text{CT}}^{(1)} &= -\frac{3\alpha_s C_F}{4\pi}\left|\frac{1}{x-y}\right|_+ \\ &= \frac{\alpha_s C_F}{2\pi}\begin{cases} -\frac{3}{2}\left(\frac{1}{x-y}\right)_+ & x>y \\ -\frac{3}{2}\left(\frac{1}{y-x}\right)_+ & x<y \end{cases}. \end{aligned}$$

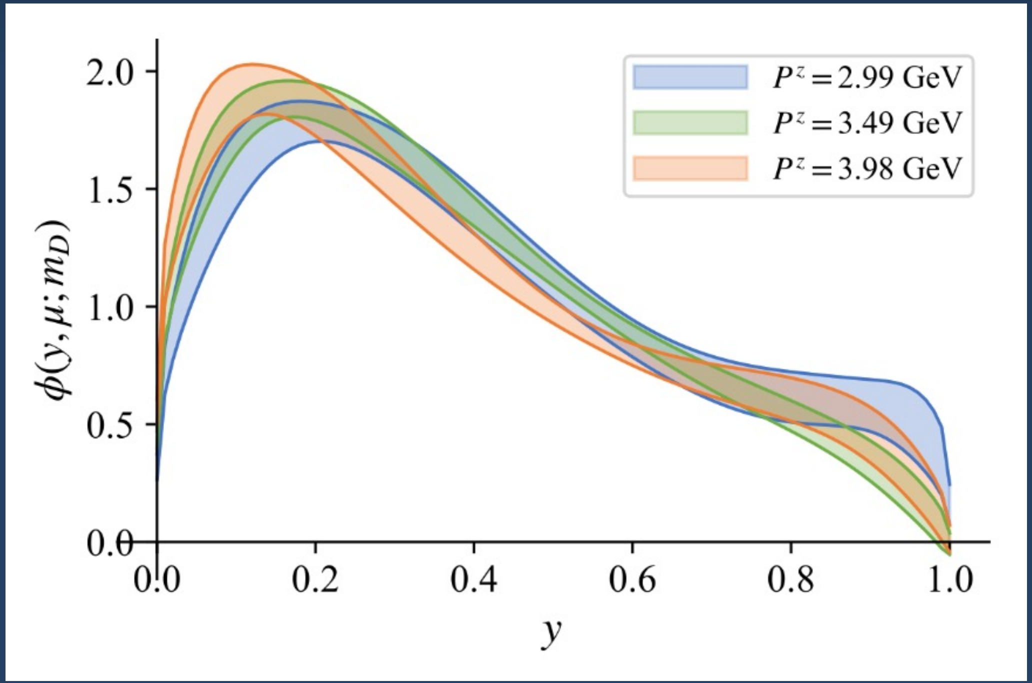
$P^z = 3.98\text{GeV}, \mu = 2\text{GeV}$



Step 1: LaMET matching and QCD LCDA

The results with different momenta are consistent within 1-σ.

The matching kernel without renormalon resummation still contains some large $\log P^z$ terms, these terms will give the more major contribution than the polynomial P^z terms at the limit of $P^z \to \infty$.

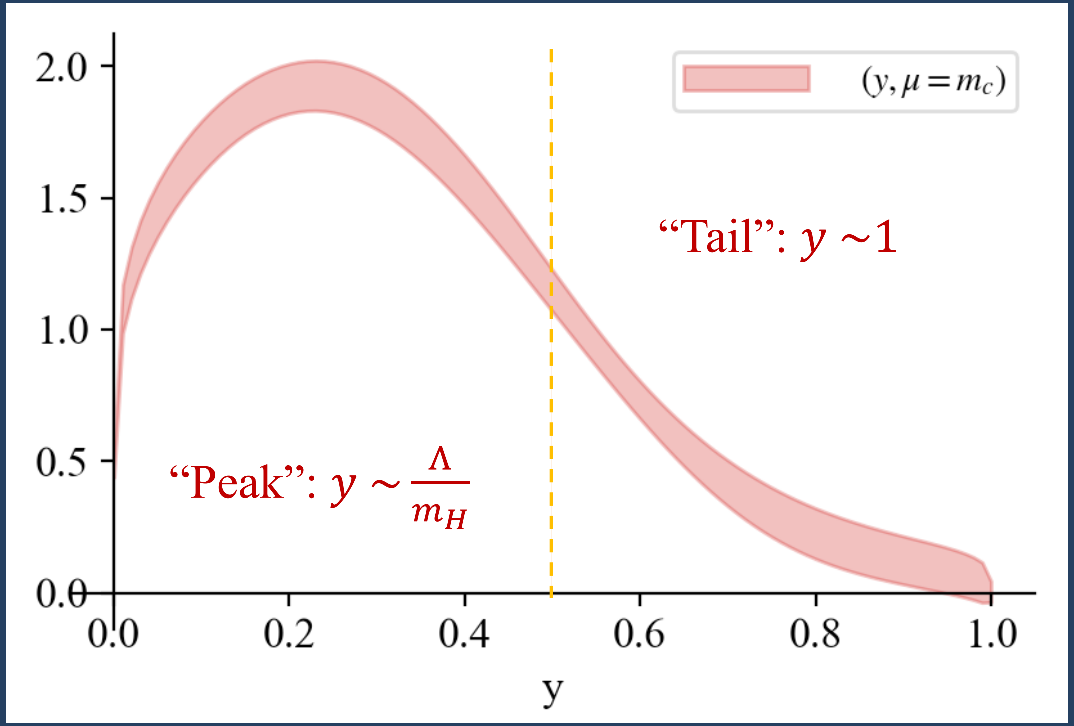


Step 2: LCDAs in QCD and in HQET

➢ The LCDAs in QCD defined as:

$$\langle 0|\bar{q}(tn_+)\not{n}_+\gamma_5 W_c(tn_+,0)Q(0)|H(P_H)\rangle$$
$$= i f_H n_+ \cdot P_H \int_0^1 dy e^{iyP_H\cdot tn_+}\phi(y,\mu),$$

can be divided into 2 parts based on the hierarchy of y:



- For very large scale $\mu \gg m_Q$, $\phi(y,\mu)$ will tend to asymptotic form;

- For the scale $\mu \lesssim m_Q$,

  ➯ Light quark carries small momentum fraction $y \sim \Lambda/m_H$
  ⇒ peak region, related to the HQET LCDA;

  ➯ $y \sim O(1)$ region be suppressed in LCDA:

  P_q is soft-collinear, $\ll P_Q$, only contribute through power corrections;

  SCET renormalized matrix element in this region contain only hard-collinear physics, and starts at the one-loop level.

Step 2: LCDAs in QCD and in HQET

- The leading twist heavy meson LCDA in HQET:

$$\langle 0|\bar{q}(tn_+)\not{n}_+\gamma_5 W_c(tn_+,0)h_v(0)|H(v)\rangle = iF_H(\mu)n_+\cdot v\int_0^\infty d\omega e^{i\omega t n_+\cdot v}\varphi_+(\omega,\mu),$$

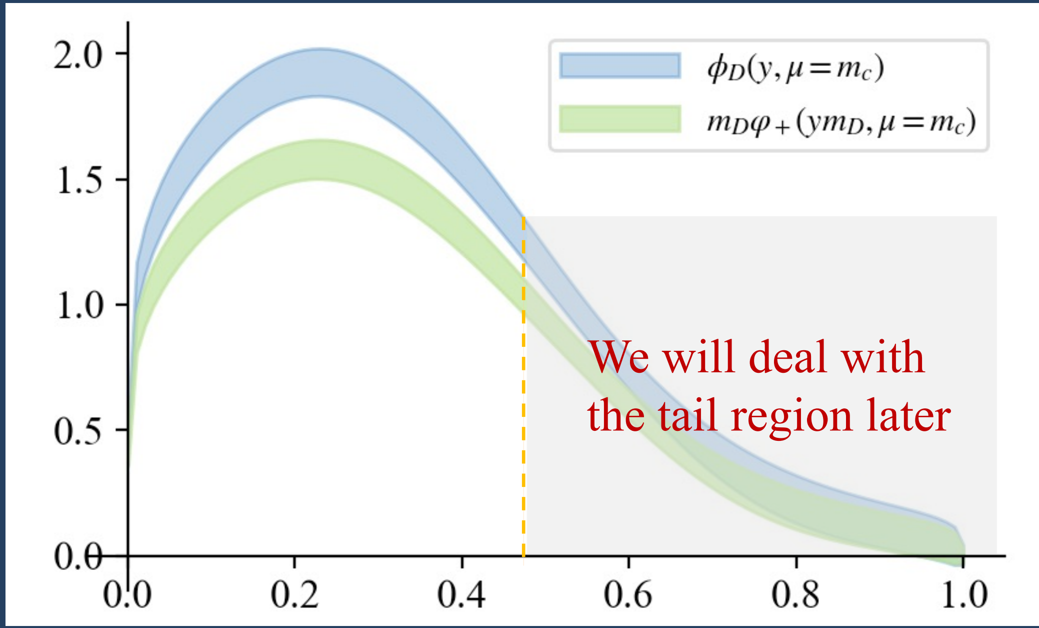
is connected with the QCD LCDA through a multiplicative factorization in the peak region:

$$\phi(u,m_H) = \frac{\tilde{f}_H}{f_H} J_{\text{peak}}\, m_H\, \varphi_+(\omega=u m_H),$$

$$J_{\text{peak}} = 1+\frac{\alpha_s C_F}{4\pi}\left(\frac{1}{2}\ln^2\frac{\mu^2}{m_H^2}+\frac{1}{2}\ln\frac{\mu^2}{m_H^2}+\frac{\pi^2}{12}+2\right)+O(\alpha_s^2),$$

$$F_H = \tilde{F}_H(\mu)\left[1-\frac{\alpha_s C_F}{4\pi}\left(\frac{3}{2}\ln\frac{\mu^2}{m_H^2}+2\right)+O(\alpha_s^2)\right],$$

[Beneke, Finauri, Keri Vos, Wei, JHEP 09, 066 (2023)]



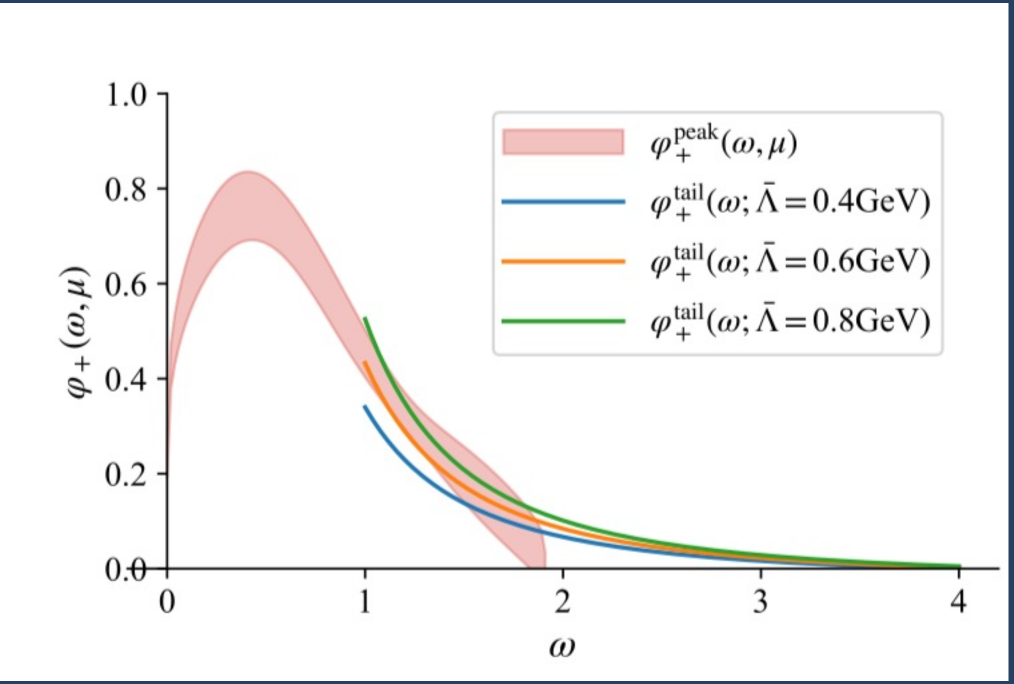
Step 2: Tail of HQET LCDA

➢ The tail region of HQET LCDA is perturbative and its 1-loop result: *[Lee, Neubert, PRD72 (2005) 094028]*

$$\phi_+(\omega,\mu) = \frac{C_F\alpha_s}{\pi\omega}\left[\left(\frac{1}{2} - \ln\frac{\omega}{\mu}\right) + \frac{4\bar{\Lambda}}{3\omega}\left(2 - \ln\frac{\omega}{\mu}\right)\right]$$

where $\bar{\Lambda} \equiv m_H - m_Q^{\mathrm{pole}}$ reflect the power correction, and usually be chosen as $400{\sim}800\mathrm{MeV}$.

- $\bar{\Lambda} = 0$: neglect the power correction;
- We use the difference between the lines to estimate the power correction.



The final results of HQET LCDA will merge the peak (from LQCD) and tail region (from 1-loop calculation).

Step 2: Final Results for HQET LCDA

- Models for HQET LCDAs

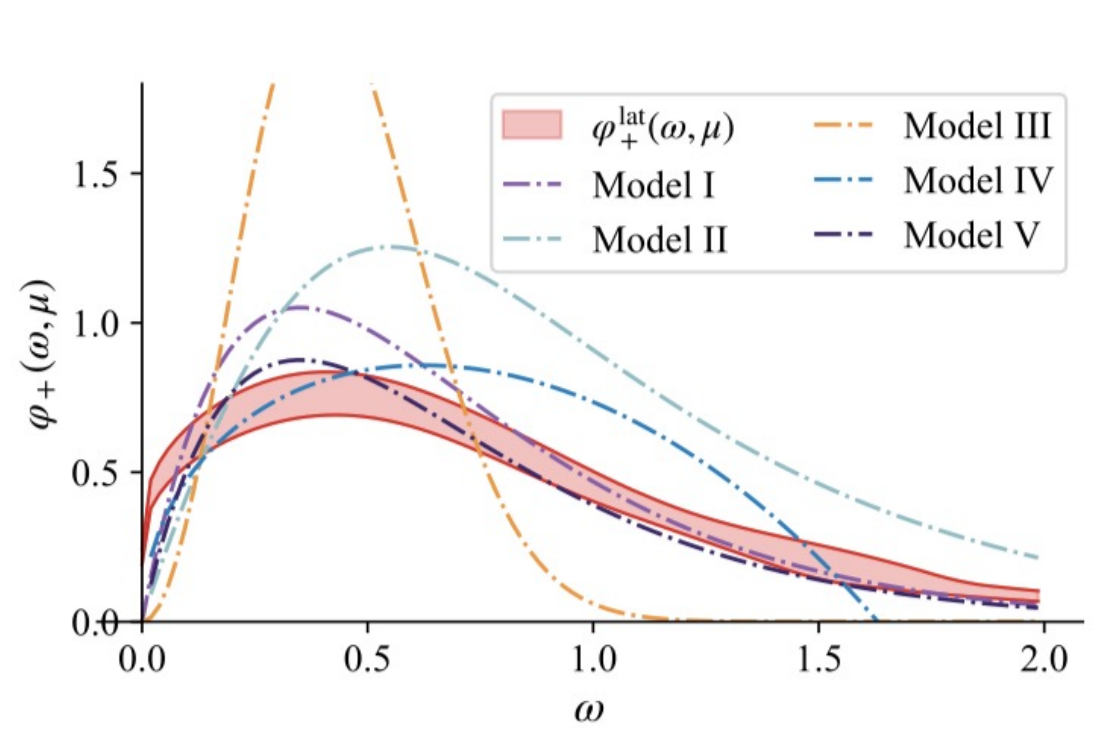
$$\varphi_{\mathrm{I}}^{+}(\omega,\mu_0) = \frac{\omega}{\omega_0^2} e^{-\omega/\omega_0}\,,$$

$$\varphi_{\mathrm{II}}^{+}(\omega,\mu_0) = \frac{4}{\pi\omega_0}\frac{k}{k^2+1}\left[\frac{1}{k^2+1} - \frac{2\left(\sigma_B^{(1)}-1\right)}{\pi^2}\ln k\right]$$

$$\varphi_{\mathrm{III}}^{+}(\omega,\mu_0) = \frac{2\omega^2}{\omega_0\omega_1^2} e^{-(\omega/\omega_1)^2}\,,$$

$$\varphi_{\mathrm{IV}}^{+}(\omega,\mu_0) = \frac{\omega}{\omega_0\omega_2}\frac{\omega_2-\omega}{\sqrt{\omega(2\omega_2-\omega)}}\theta(\omega_2-\omega)\,,$$

$$\varphi_{\mathrm{V}}^{+}(\omega,\mu_0) = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\frac{\omega}{\omega_0^2} e^{-\omega/\omega_0} U(\beta-\alpha, 3-\alpha, \omega/\omega_0)\,,$$



Han, et.al, 2403.17492 and updated

Summary and Outlook

- ✓ We proposed a two-step method to determine heavy meson LCDA from Lattice QCD.
- ✓ We use the finest CLQCD ensemble (H48P32) to simulate the heavy (D) meson quasi DAs with largest momentum up to 4GeV.
- ✓ We consider a hybrid renormalization on lattice and λ-extrapolation scheme.
- ✓ The obtained (preliminary) results for LCDAs are consistent with model parametrizations

Summary and Outlook

Theory

- Heavy quark spin symmetry
- $1/P^z$ corrections
- $1/m_Q$ corrections
- m_Q dependence
- …

Lattice

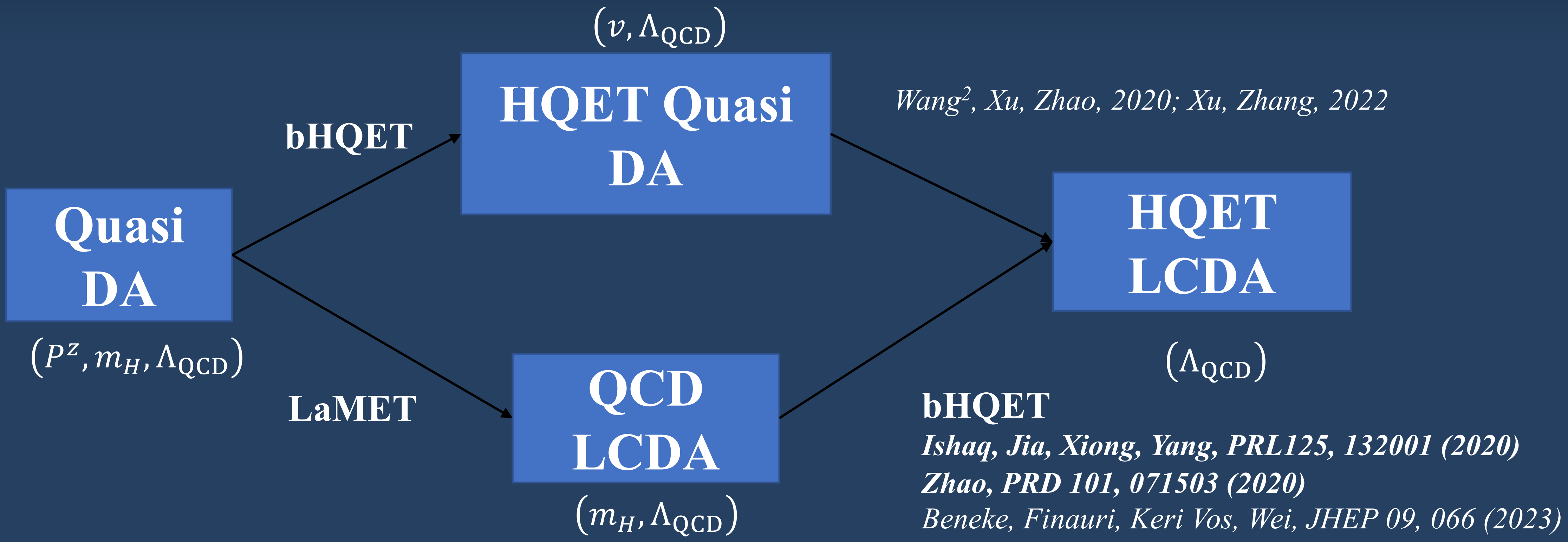
- Finer Lattices
- Renormalization
- Different sources
- …

Precise results on heavy meson LCDAs

Thank you for your attention!

Backup

How to solve this problem?



Two-step method: Han, et.al,2403.17492

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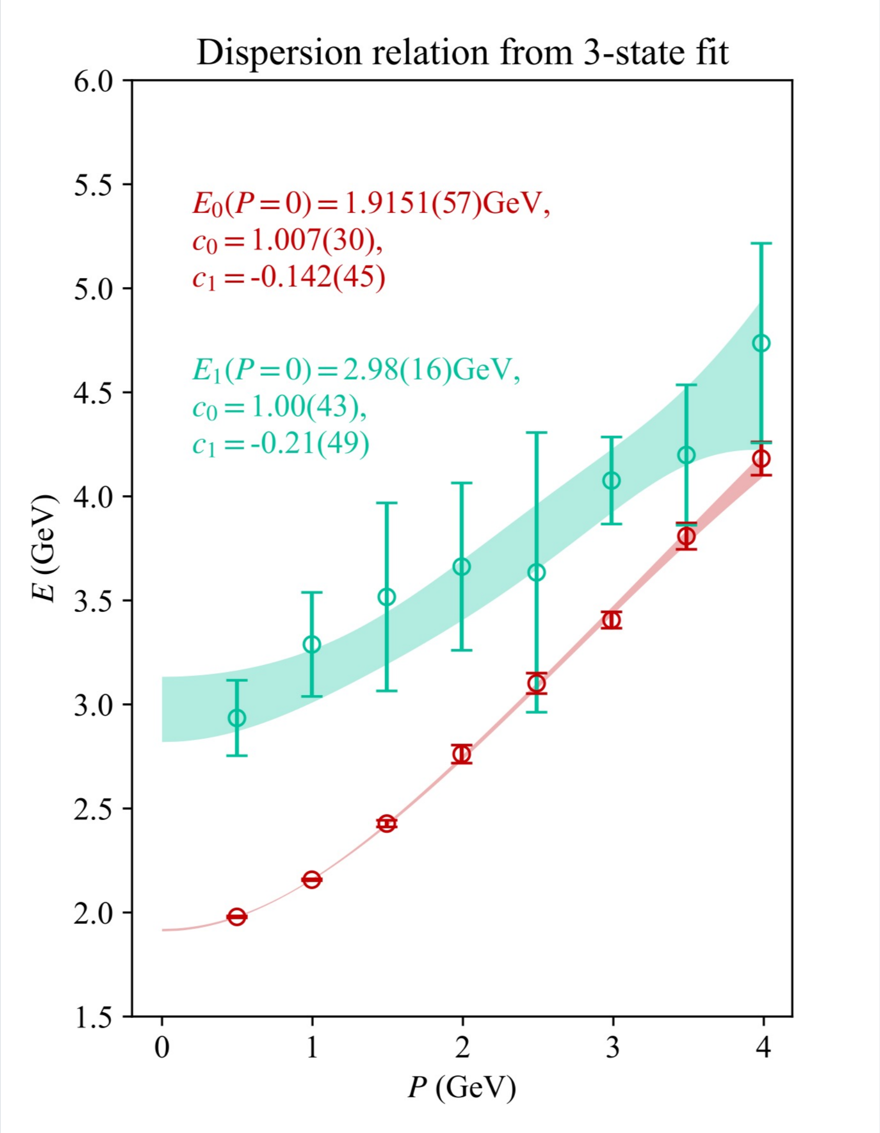
Step 0: Dispersion relation

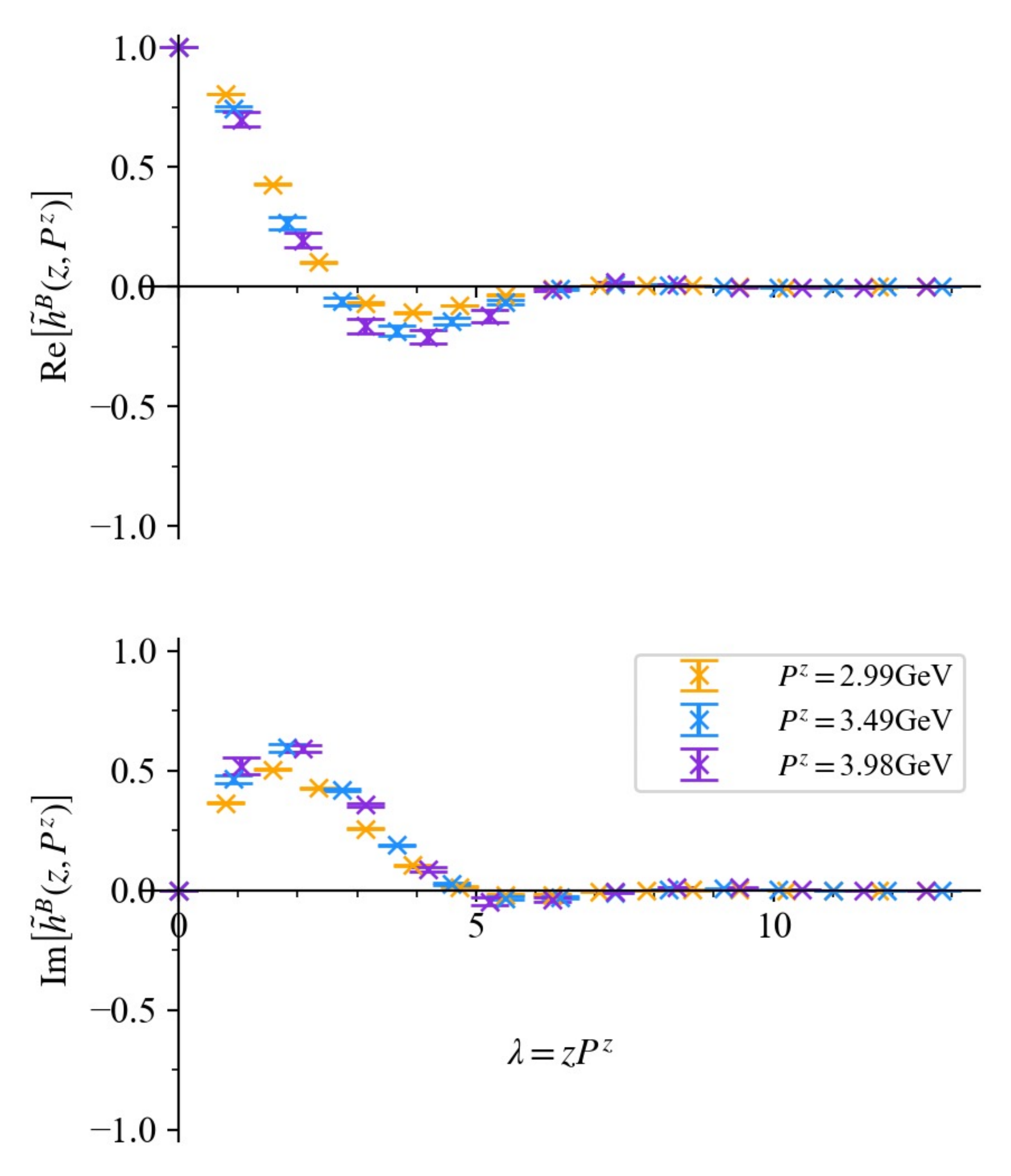
➢ **Dispersion relation**

$$E^2 = \sqrt{m^2 + c_0P^2 + c_1P^4a^2}$$

- The signals of local matrix elements are good, it allows us to perform a 3-state fit and verify the dispersion relation of the ground-state and first excited state.
- We use the model averaging method to improve the fit quality.
- We adopt the Dirac structure of the current as $\Gamma = \gamma^z\gamma_5$.



Step 1: Results of the bare matrix elements



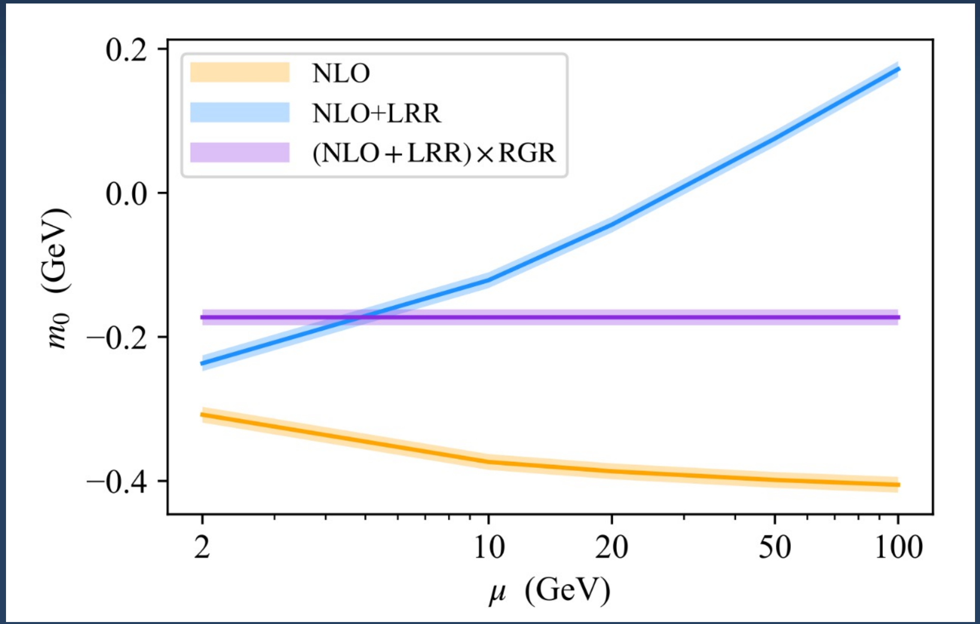
Fit strategies and ranges we used.

z			
0	1-state, [11, 16]	2 -state, [6, 15]	2-state, [6, 13]
1	1-state, [11, 16]	2-state, [6, 15]	2-state, [6, 13]
2	1-state, [11, 16]	2-state, [6, 15]	2-state, [6, 13]
3	1-state, [11, 16]	2-state, [5, 15]	2-state, [5, 13]
4	1-state, [11, 16]	2-state, [5, 14]	2-state, [5, 13]
5	1-state, [11, 14]	2-state, [5, 13]	2-state, [5, 13]
6	1-state, [11, 14]	2-state, [5, 13]	2-state, [5, 13]
7	1-state, [11, 14]	2 -state, [5, 12]	2-state, [5, 12]
8	1-state, [11, 14]	2-state, [5, 10]	1-state, [8, 12]
9	1-state, [11, 14]	2 -state, [5, 9]	1-state, [8, 11]
10	1-state, [11, 14]	2-state, [5, 9]	1-state, [8, 11]
11	1-state, [11, 14]	1-state, [8, 13]	1-state, [8, 11]
12	1-state, [10, 14]	1-state, [8, 11]	1-state, [8,11]
13	1-state, [10, 14]	1-state, [8, 11]	
14	1-state, [10, 14]	1-state, [7, 11]	
15	1-state, [10, 14]		
16	1 -state, [10, 14]		

Step 1: Renormalization in the hybrid-ratio scheme

Scale dependence of m_0:

- The scale dependence reflect the contamination effects from the uncounted higher-order terms in C_0.
- We compare the extracted results of m_0 from fixed-order C_0^{NLO}, $C_0^{\mathrm{NLO+LRR}}$, and with RGR improvement $C_0^{(\mathrm{NLO+LRR})\times\mathrm{RGR}}$.
- The RGR method significantly improves the stability after scale variation.

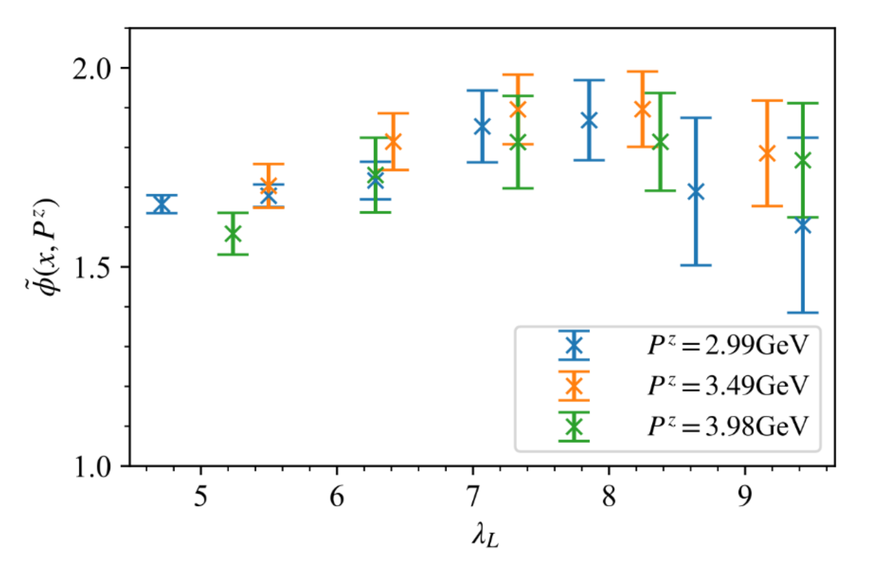


Step 1: λ-extrapolation

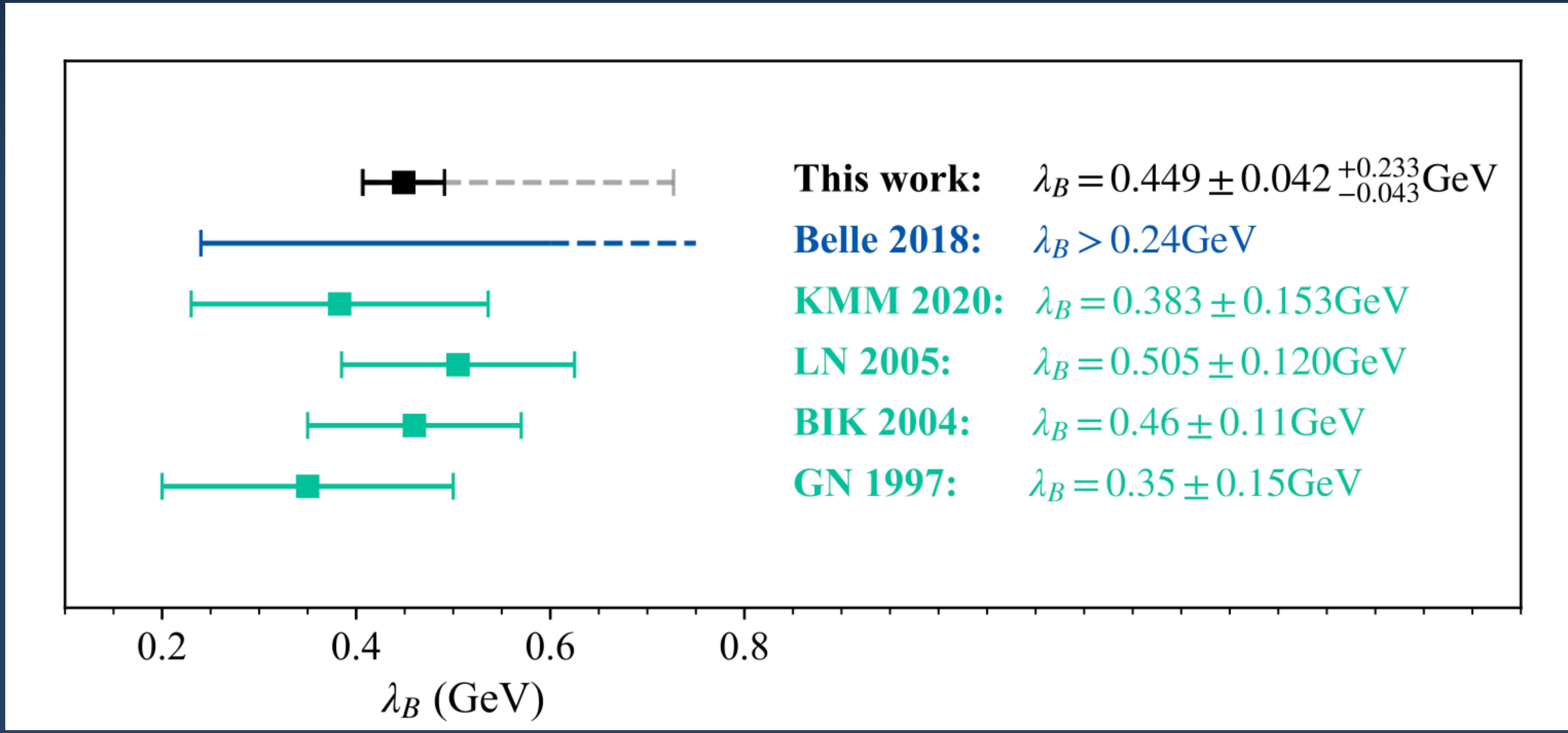
- We use the data at $\lambda \geq \lambda_L$ to perform the extrapolation. The choice of λ_L should neither too small nor too large.

- To estimate the λ_L dependence, we perform the fits with data start from different λ_L, extrapolate them to infinity (we adopt 200a as the infinity) and then Fourier transform to the momentum space.

- We compare the results of quasi DAs $\tilde{\phi}(x = 0.25, P^z)$ from extrapolated data with different λ_L. One can see that the results trend to stabilize after $\lambda_L \simeq 7$ at each momenta, with only the errors increasing.



Step 2: Final Results for HQET LCDA



KMM 2020: Khodjamirian, Mandal , Mannel, JHEP 10, 043 (2020)
LN 2005: Lee, Neubert, PRD 72, 094028 (2005)
BIK2004: Braun, Ivanov, Korchemsky, PRD69, 034014 (2004)
GN 1997: Grozin,Neubert, Phys. Rev. D 55, 272- 290 (1997)