Di-J/psi structures from the quark Pauli-blocking effect

Sachiko Takeuchi
(Japan College of Social Work)
Makoto Takizawa
(Showa Pharmaceutical Univ.)
Yasuhiro Yamaguchi
(Nagoya Univ.)
Atsushi Hosaka
(RCNP)

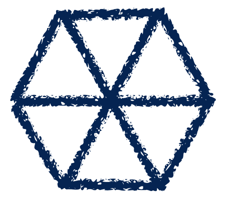
I'd like to begin by acknowledging the Traditional Owners of the land on which we meet today.

Today's point

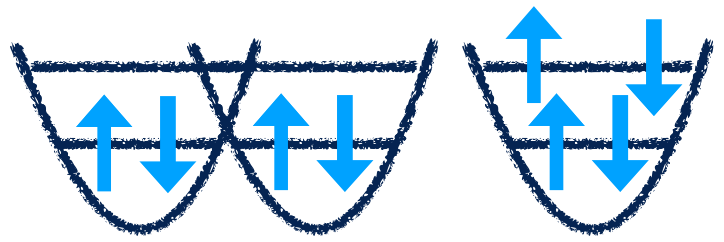
Q: Where and how can we see the quark degrees of freedom in the low-energy region?

A: By looking into the symmetry.

Where can we see the quark degrees of freedom more clearly?



- There are 10 spin-3/2 baryons but the number of spin-1/2 baryons is 8.
- Some of the two-baryon systems have a large short-range repulsion caused by the Pauli-blocking.
  (Experiments, Quark models, the Lattice QCD)



- Two-hadron systems get a large short-range attraction because hadrons are composed of multiple quarks (many-body effect).

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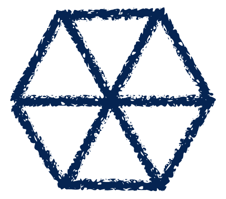

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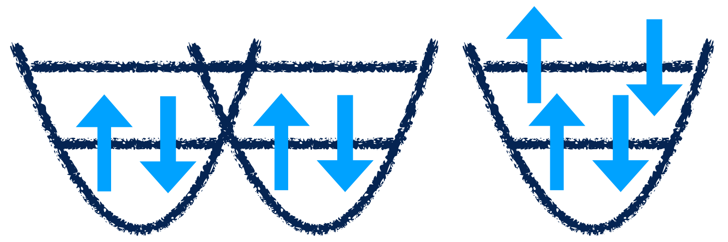


- Two-hadron systems get a large short-range attraction because hadrons are composed of multiple quarks (many-body effect).

may be seen in the di-J/ψ mode !?

Contents

diJpsi

Motivation

- (a) Pauli-Blocking effects between quarks, and (b) Quark many-body effects

Hadron potential from the Quark antisymmetrization

- How to derive potentials to describe the effects (a) and (b)

Rough size of the effects and their channel dependence

Examples

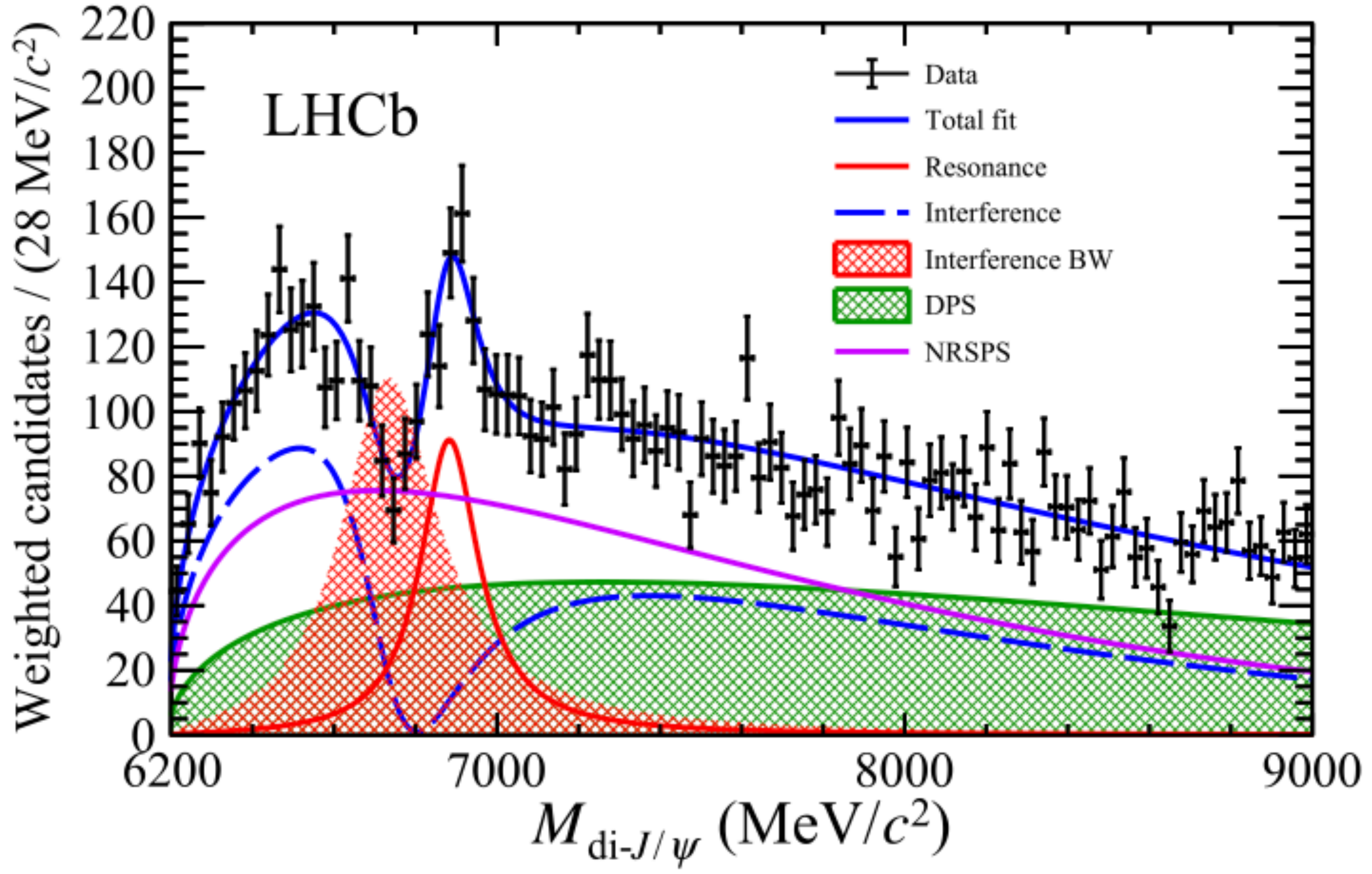
- $c\bar{c}c\bar{c}$ $(J^P) = (0,1,2^+)$

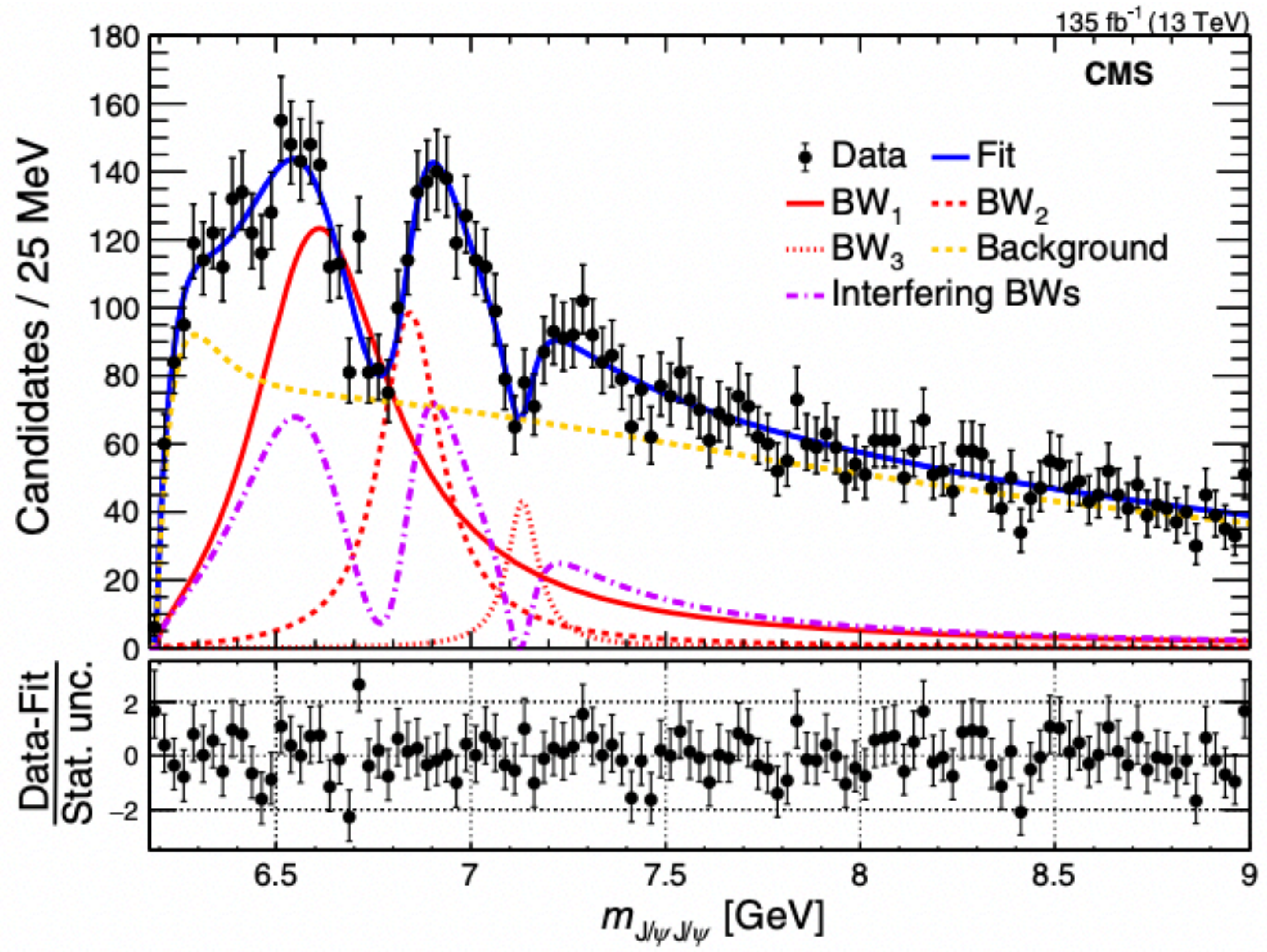
  Dynamic calculation for the $c\bar{c}$-$c\bar{c}$ scattering phase shifts, S matrix poles

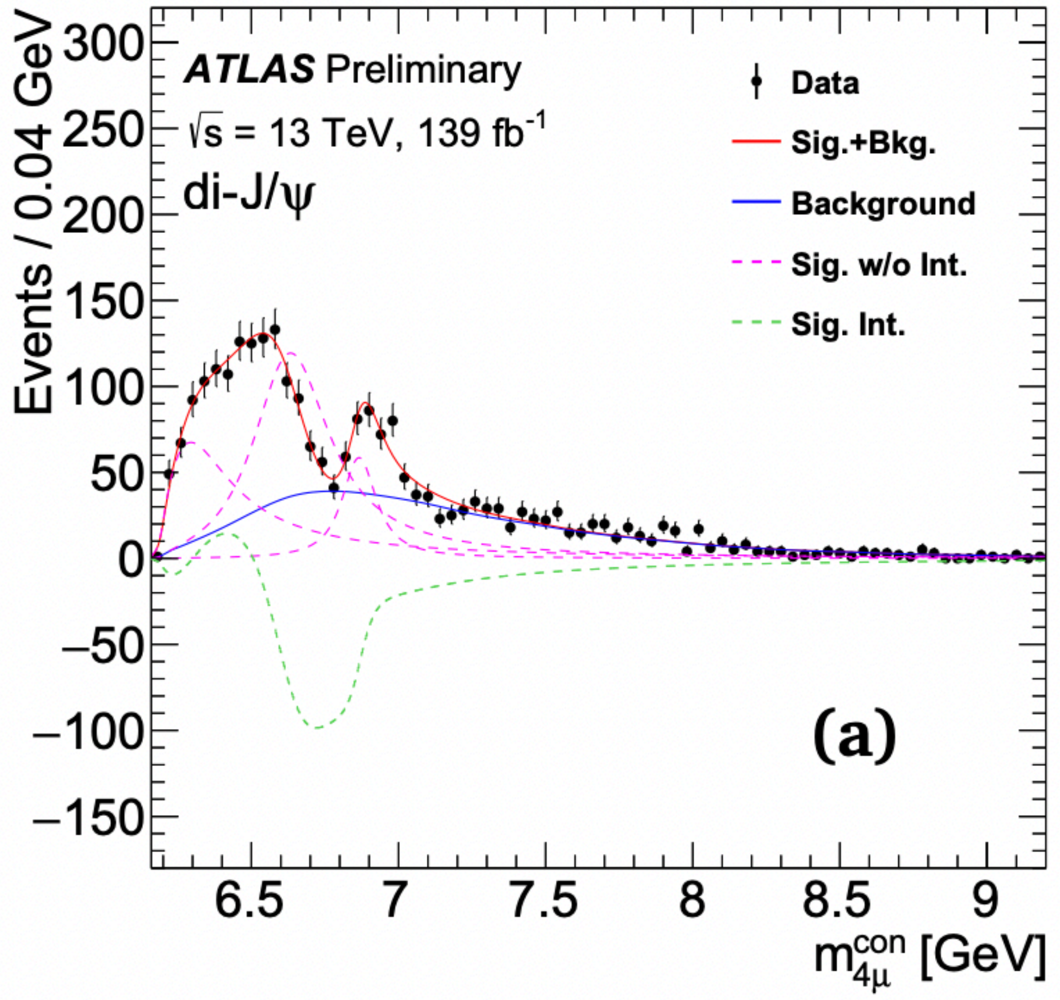
Summary and Outlook

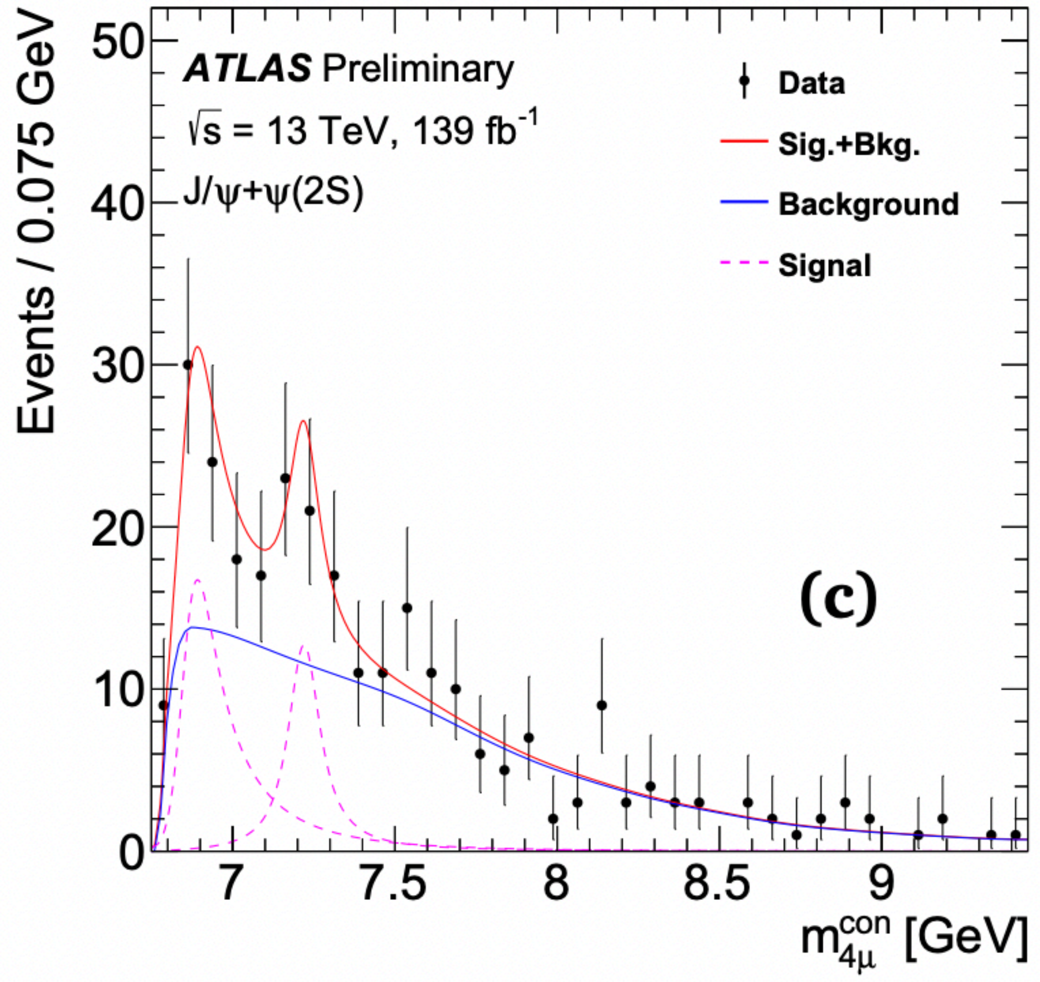
diJ/ψ experiments

$J/\psi J/\psi$ or $J/\psi\psi(2S)$ mass spectra









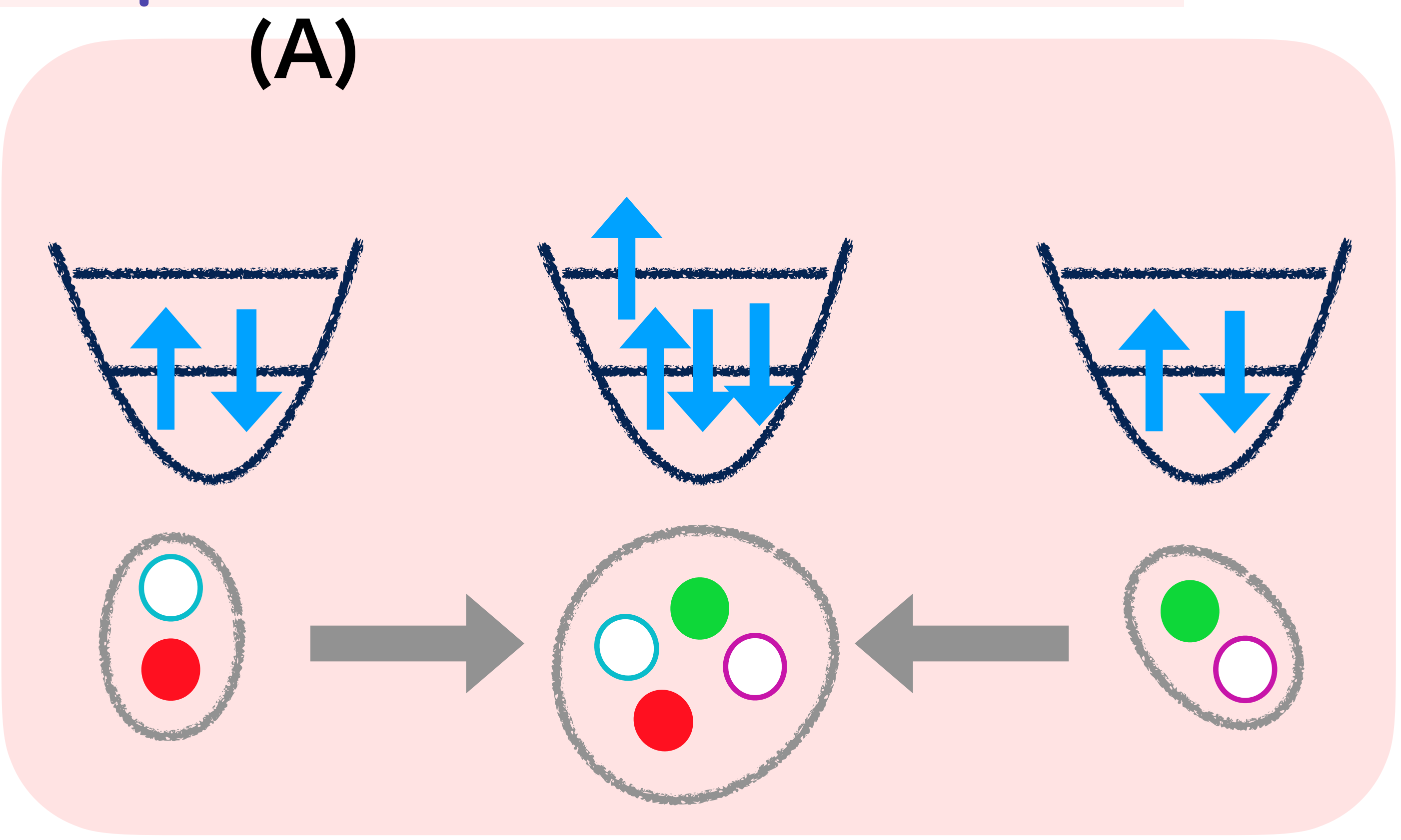
R. Aaij *et al.* (LHCb), Sci. Bull. **65**, 23, 1983 (2020), [arXiv:2006.16957].
A. Hayrapetyan *et al.* (CMS) (2023), Phys. Rev. Lett. **132**, 111901
Y. Xu (ATLAS), Acta Phys. Polon. Supp. **16**, 3, 21 (2023), [arXiv:2209.12173].

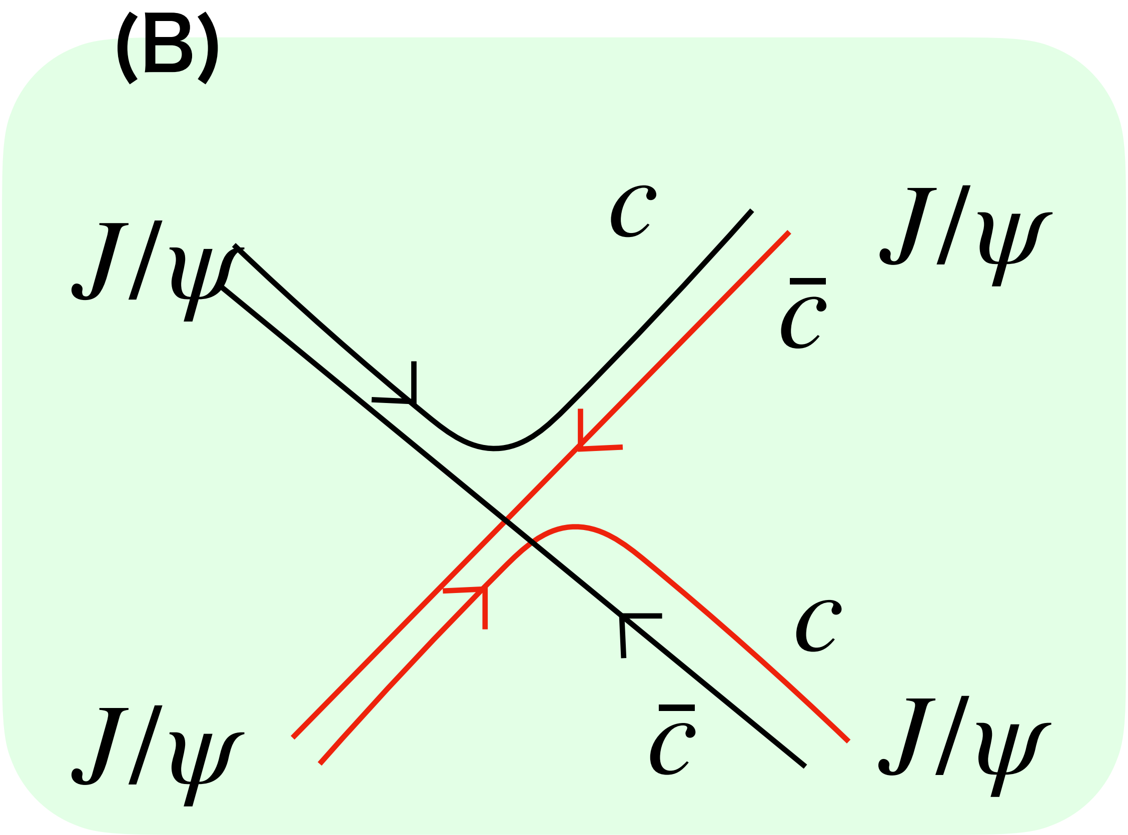
Motivation

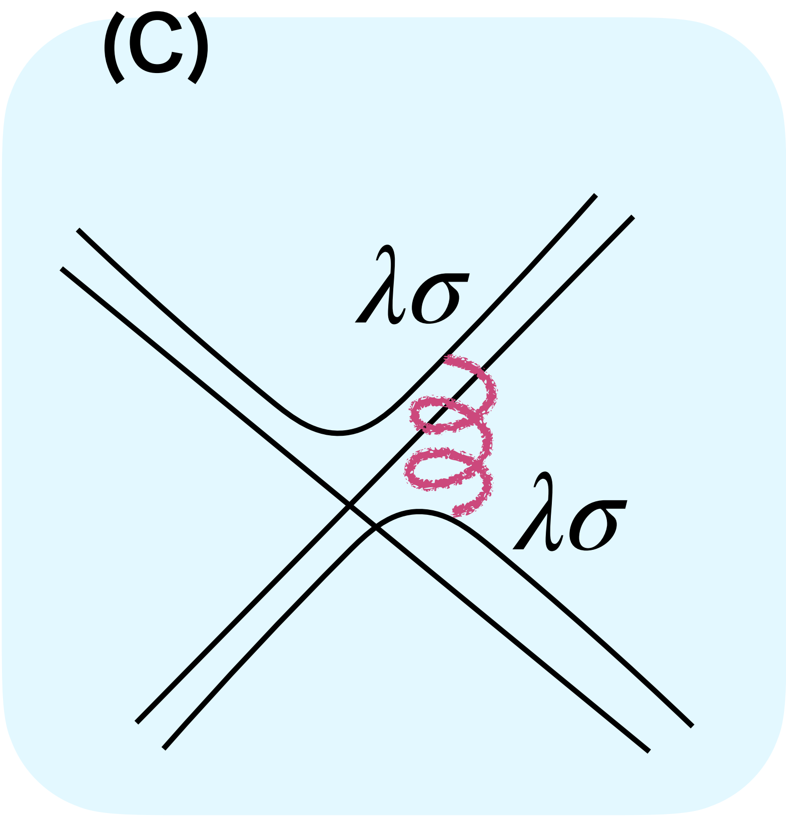
The interaction between the hadrons originated from the quark degrees of freedom consists of

di-J/ψ systems

(A) Pauli-Blocking effects between quarks



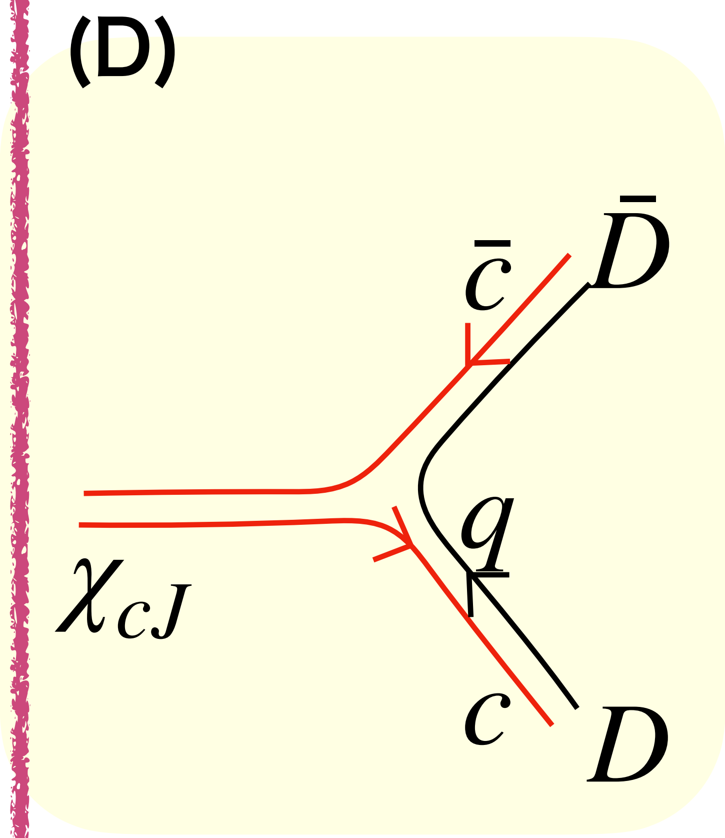




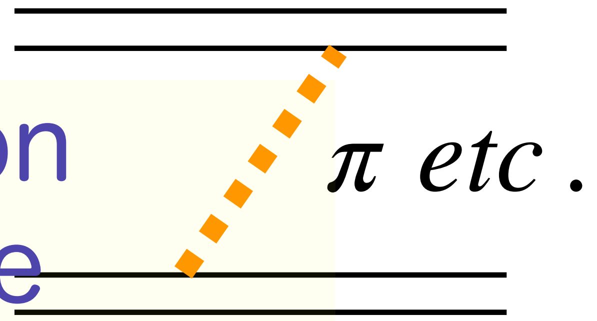
(B) Interference of the hadrons by the quark rearrangement

(C) Gluon color-spin interaction (as a residual int of the gluon effects)

(D) $q\bar{q}$-MM coupling



(E) meson exchange



Quark potential to Hadron potential (How to derive (a) and (b))

Makoto Oka, Kiyotaka Shimizu, Koichi Yazaki
PTP Supplement, Volume 137, 2000, Pages 1–20,

Hamiltonian for quarks:

- Kinetic + color-Coulomb + confinement + color-spin

$$H_q = H_0 + V_q$$

$$\boxed{H_0 = \sum_i (m_i + \frac{p_i^2}{2m_i}) - \frac{p_G^2}{2m_G}}$$

$$V_q = \sum_{ij} \left(\frac{\lambda_i \cdot \lambda_j}{4} \frac{\alpha_s}{r_{ij}} - \lambda_i \cdot \lambda_j a_{\text{conf}} r_{ij} - \frac{\lambda_i \cdot \lambda_j \sigma_i \cdot \sigma_j}{4} \alpha_s \frac{2\pi}{3m_i m_j} \delta^3(r_{ij}) \right)$$

This comes from the OgE. However, since we assume the quark exchange occurs only in the (0s) state, one does not actually have to assume the shape of the potentials.

Wave functions:

- J/ψ and η_c mesons consist of $c\bar{c}$, confined in the meson:

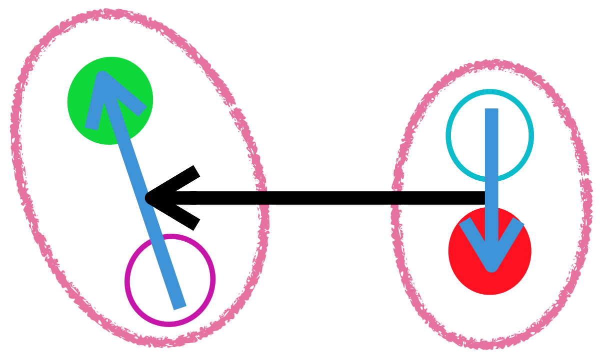
$$\phi_{0s}(b_{12}, \vec{r}_{12}) \,|\, q_{\alpha_1} \bar{q}_{\alpha_2}; \alpha_a \rangle$$

- two $c\bar{c}$ mesons with the relative motion wave function, antisymmetrized over the quarks:

$$\Psi(\alpha) = \mathscr{A}_q \sum_{\alpha_a \alpha_b}^{(spin,color,flavor)} \psi(\vec{r}_{ab}) \phi_{0s}(b_{12}, \vec{r}_{12}) \,|\, q_{\alpha_1} \bar{q}_{\alpha_2}; \alpha_a \rangle \phi_{0s}(b_{34}, \vec{r}_{34}) \,|\, q_{\alpha_3} \bar{q}_{\alpha_4}; \alpha_b \rangle \Big|_{\alpha}$$

Quark potential to Hadron potential (How to derive (a) and (b))

- Nonlocal hamiltonian for the mesons is obtained by integrating the internal motions of the hadrons:



$$\mathcal{N}(\vec{p}, \vec{q}) = \int d^3\vec{p}_{12} d^3\vec{p}_{34} d^3\vec{p}_a \phi^\dagger(b_{12}, \vec{p}_{12})\phi^\dagger(b_{34}, \vec{p}_{34})\delta(\vec{p}_a - \vec{p})$$

non-orthogonal

$$(1 - P_{24})\phi(b_{12}, \vec{p}_{12})\phi(b_{34}, \vec{p}_{34})\delta^3(\vec{p}_a - \vec{q})$$

Assumption: the hadrons are quark clusters

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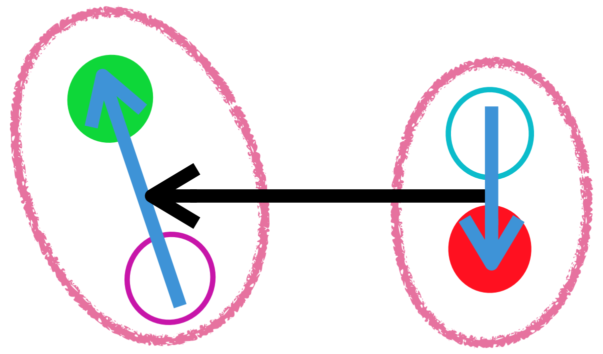


$$\mathscr{H}_0(\vec{p},\vec{q}) = \int d^3\vec{p}_{12} d^3\vec{p}_{34} d^3\vec{p}_a \phi^\dagger(b_{12},\vec{p}_{12})\phi^\dagger(b_{34},\vec{p}_{34})\delta(\vec{p}_a - \vec{p})$$

$$\left(\sum_i m_i + \frac{p_{12}^2}{2\mu_{12}} + \frac{p_{34}^2}{2\mu_{34}} + \frac{p_a^2}{2\mu_a}\right)(1 - P_{24})\phi(b_{12},\vec{p}_{12})\phi(b_{34},\vec{p}_{34})\delta^3(\vec{p}_a - \vec{q})$$

$$= \left(\sum_i m_i + K_{12} + K_{34}\right)\mathcal{N}(\vec{p},\vec{q}) + \mathscr{K}(\vec{p},\vec{q})$$

non-orthogonal

Assumption: the hadrons are quark clusters

- Schrödinger equation written by the hadron coordinates becomes

$$\left(M\mathcal{N} + \mathscr{K} - E\mathcal{N}\right)\psi = 0$$

or

$$\left(M + \mathcal{N}^{-1/2}\,\mathscr{K}\,\mathcal{N}^{-1/2} - E\right)\mathcal{N}^{1/2}\psi = 0$$

$$\left(M + K + V_K - E\right)\bar{\psi} = 0$$

$$\bar{\psi} = \mathcal{N}^{1/2}\psi$$

$$V_K \equiv \mathcal{N}^{-1/2}\,\mathscr{K}\,\mathcal{N}^{-1/2} - K$$

potential written by the hadron coordinates

Quark potential to Hadron potential (How to derive (a) and (b))

- The kinetic term can be expanded by ns-harmonic oscillator wave function of the hadron size as

$$K = \frac{\hbar\omega_0}{2}\left[\cdots + \sqrt{\frac{3}{2}}\left(\phi_{0s}(b,\vec{p})\phi^\dagger_{1s}(b,\vec{q}) + \phi_{1s}(b,\vec{p})\phi^\dagger_{0s}(b,\vec{q})\right) + \cdots\right]$$

Kinetic term mix the 0s and 1s states. (function of the relative hadron coodinates)

- Or

$$K = \begin{array}{cccc} 0s & 1s & 2s & \\ \end{array}$$

$$K = \begin{pmatrix} \frac{3}{2} & \boxed{\sqrt{\frac{3}{2}}} & 0 & \cdots \\ \boxed{\sqrt{\frac{3}{2}}} & \frac{5}{2} & \cdots & \\ 0 & \cdots & \frac{2n+\ell+3}{2} & \cdots \end{pmatrix} \frac{\hbar\omega_0}{2}$$

ω_0 are the free parameters of the model

Quark potential to Hadron potential (How to derive (a) and (b))

- $V_K \equiv \mathcal{N}^{-1/2}\,\mathcal{K}\,\mathcal{N}^{-1/2} - K$ can be expanded by ns-harmonic oscillator wave function of the hadron size as

$$V_K(\vec{p},\vec{q}) = \frac{\hbar\omega_0}{2}(\sqrt{\nu}-1)\sqrt{\frac{3}{2}}\left(\phi_{0s}(b,\vec{p})\phi^\dagger_{1s}(b,\vec{q}) + \phi_{1s}(b,\vec{p})\phi^\dagger_{0s}(b,\vec{q})\right)$$

$$\nu = \langle\alpha|1 - P^{sfc}_{24}|\beta\rangle$$

Pauli effect pushes the particle from 0s to 1s, which means the effect can be expressed by modifying the 0s-1s mixing in the kinetic term.

- Or

$$V_K = \mathcal{N}^{-1/2}\,\mathcal{H}_0\,\mathcal{N}^{-1/2} - \mathcal{H}_0 = \begin{array}{c} \begin{array}{ccc} 0s & 1s & 2s \end{array} \\ \begin{pmatrix} 0 & \boxed{\sqrt{\frac{3}{2}}(\sqrt{\nu}-1)} & 0 & \cdots \\ \boxed{\sqrt{\frac{3}{2}}(\sqrt{\nu}-1)} & 0 & \cdots & \\ 0 & \cdots & 0 & \cdots \end{pmatrix} \end{array} \frac{\hbar\omega_0}{2}$$

$$\begin{cases} \text{attractive} & \nu > 0 \\ \text{no effect} & \nu = 1 \\ \text{repulsive} & \nu < 0 \end{cases}$$

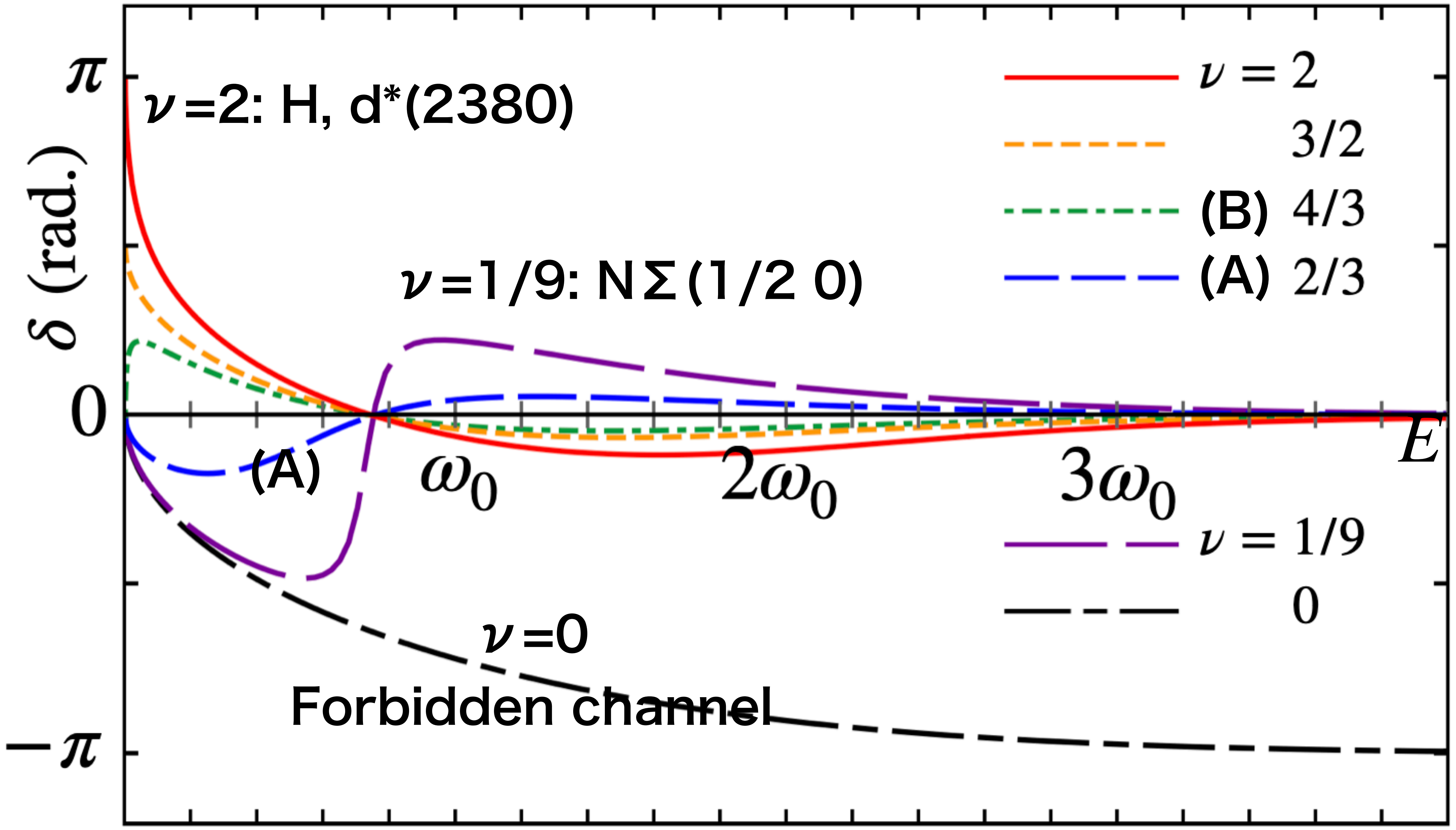
ν depends on the systems, and ω_0 are the free parameters of the model.
When all the 4 quark masses are equal, the above expansion is exact.

The effects (a)+(b) give a node in the phase shift

- When we assume $1/mb^2 = \omega_0$, the effects are written by ω_0.

(will be shown later)

$\nu = \langle \alpha | 1 - P_{24}^{sfc} | \beta \rangle$



we have to check ths effects on di-J/ψ …

(A): $c\bar{c}c\bar{c}$ total spin 2, 1, and total spin 0-cc-spin-1 component

(B): $c\bar{c}c\bar{c}$ total spin 0-cc-spin-0 component

LHCb, Science Bulletin 65 (2020) 1983–1993

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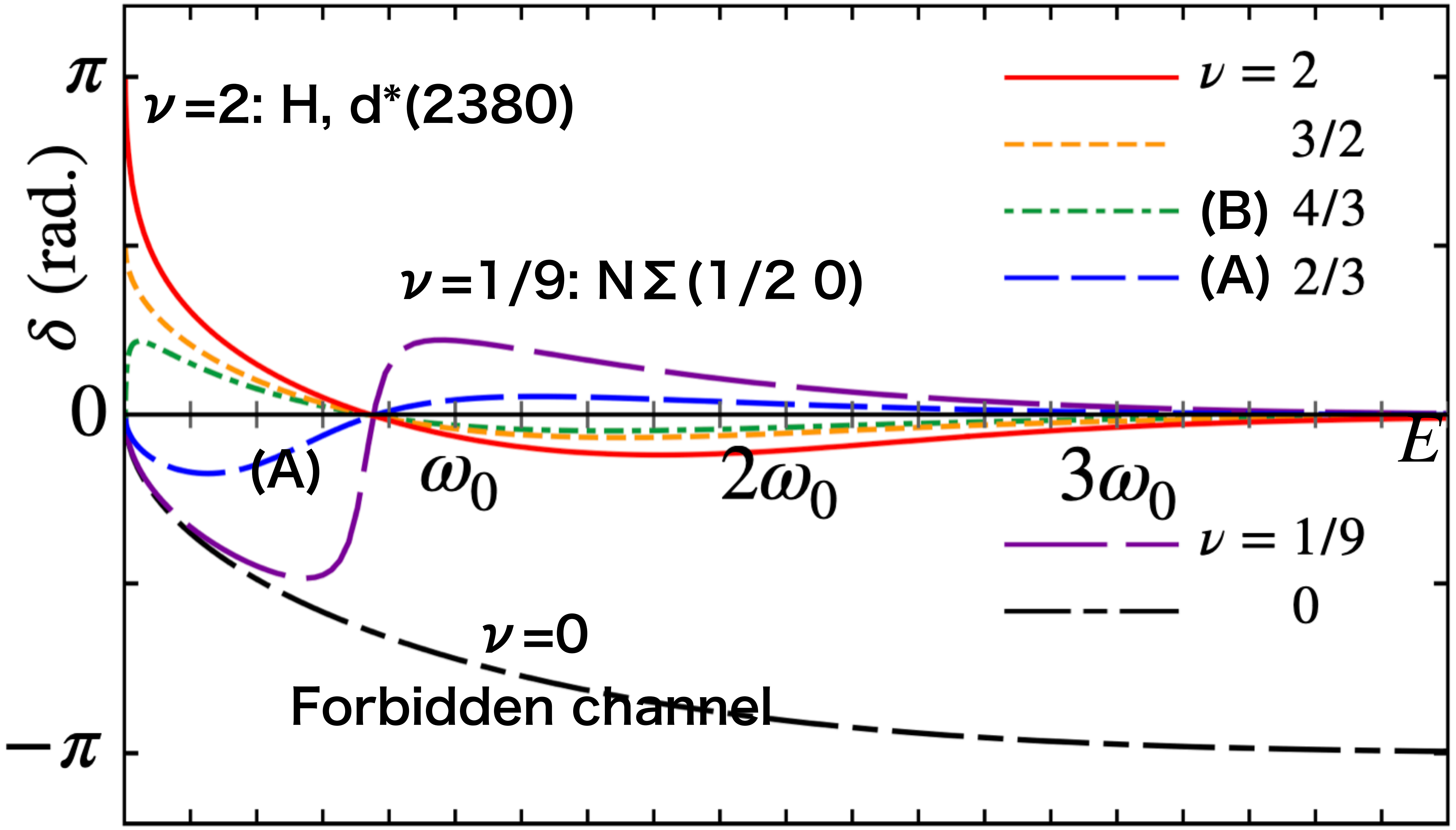

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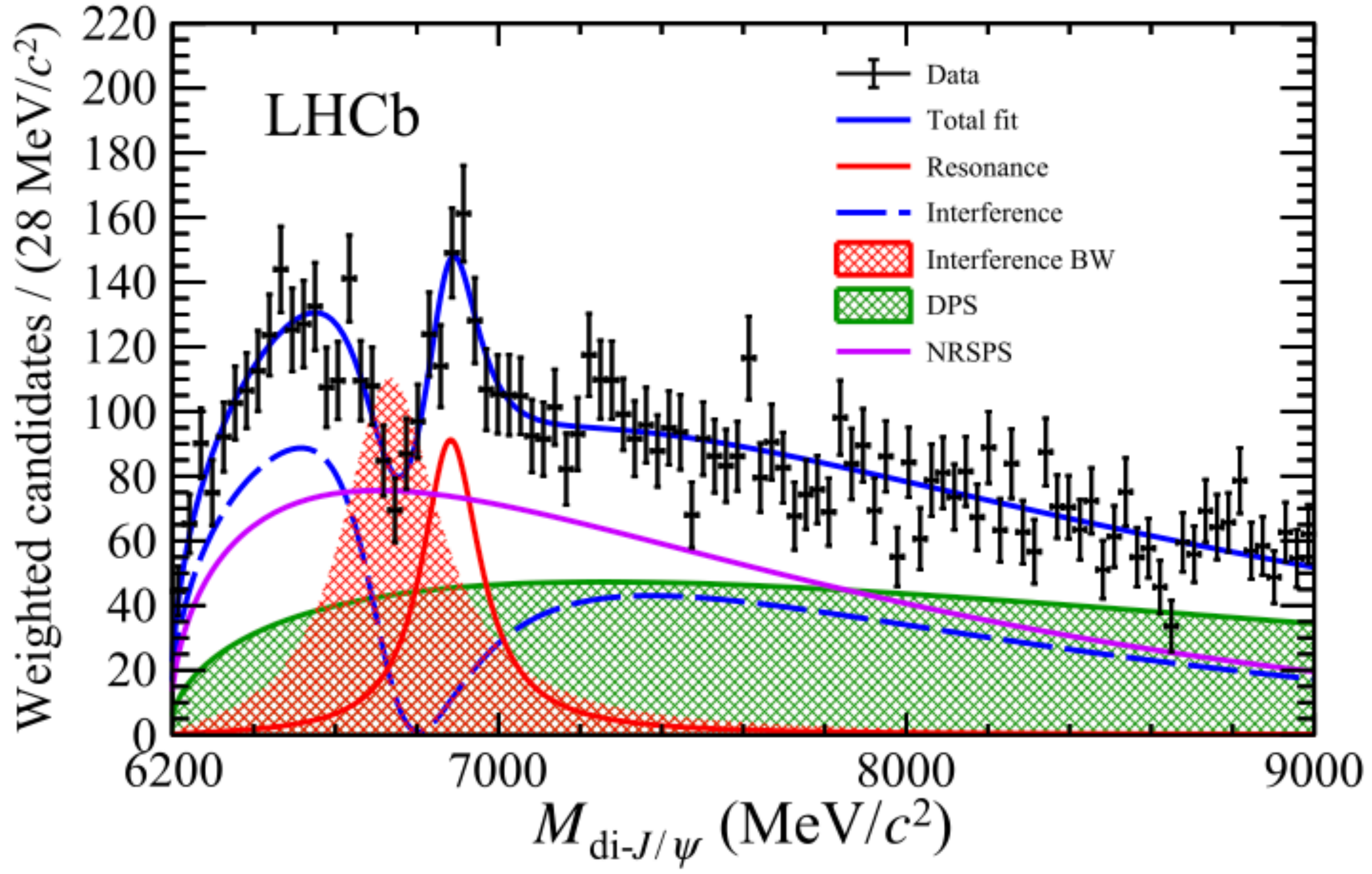


(A): $c\bar{c}c\bar{c}$ total spin 2, 1, and total spin 0-cc-spin-1 component

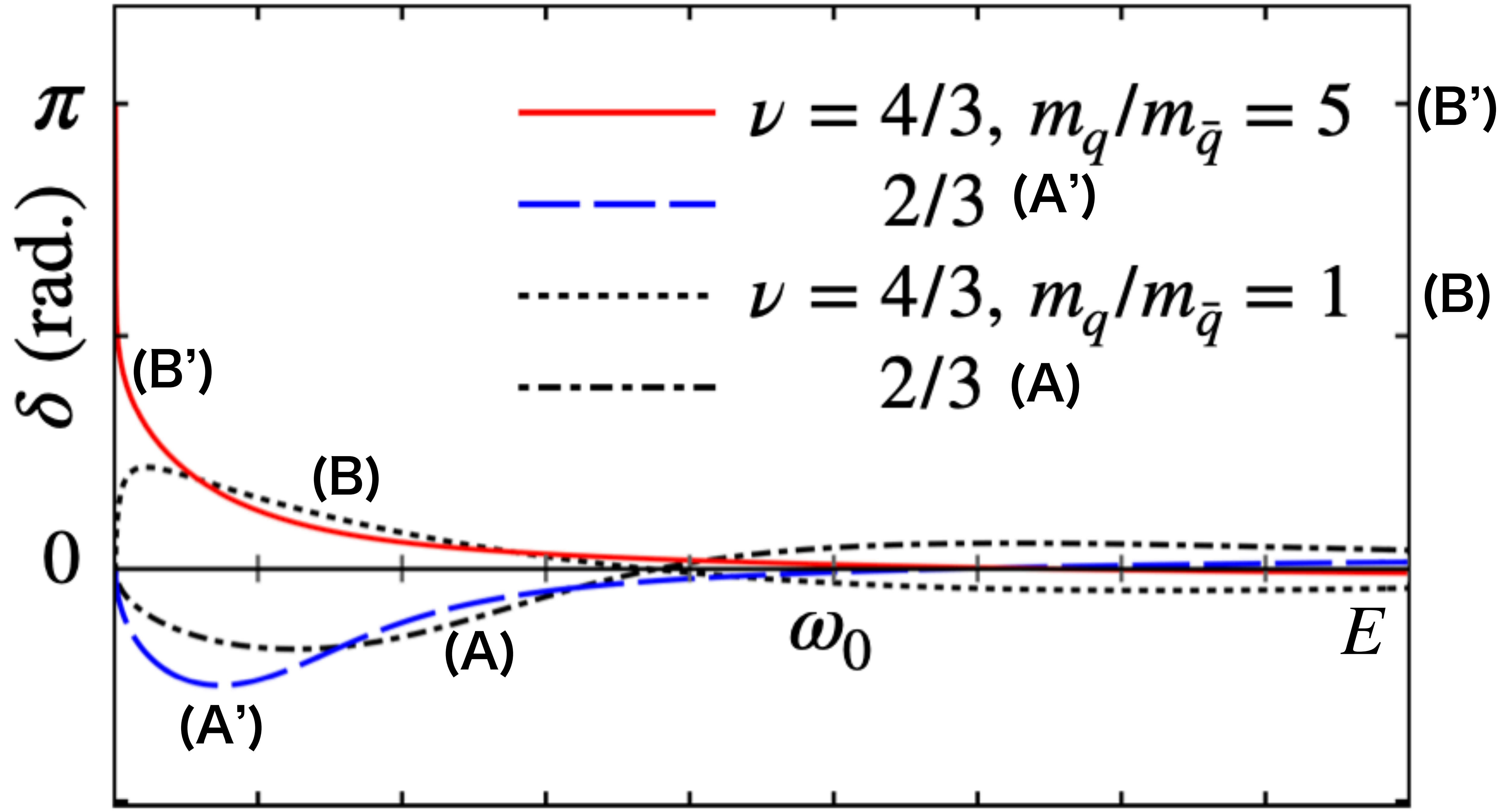
(B): $c\bar{c}c\bar{c}$ total spin 0-cc-spin-0 component

LHCb, Science Bulletin 65 (2020) 1983–1993

The effects (a)+(b) may give a bound state in some case.

- $m_q/m_{\bar{q}} = 5$

diJ/ψ v.s. Tcc



Tcc

diJ/ψ

(A'):mc/mu=5 Tcc, cc spin-1 component

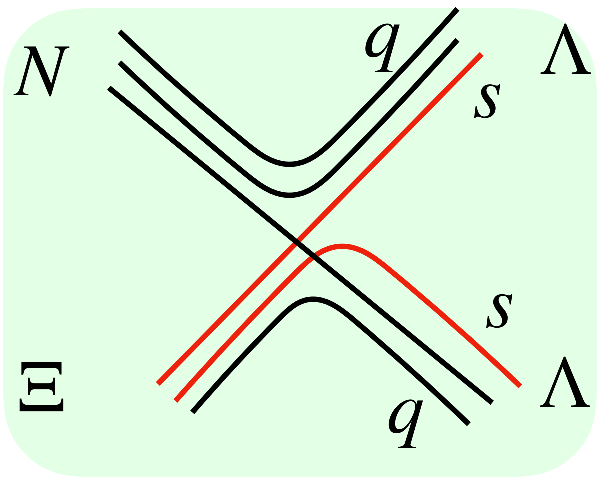
(B'):mc/mu=5 Tcc, cc spin-0 component

Size of the effects (a)+(b) is size of ν

Wave function normalization is taken to be 1 when $R \to \infty$.

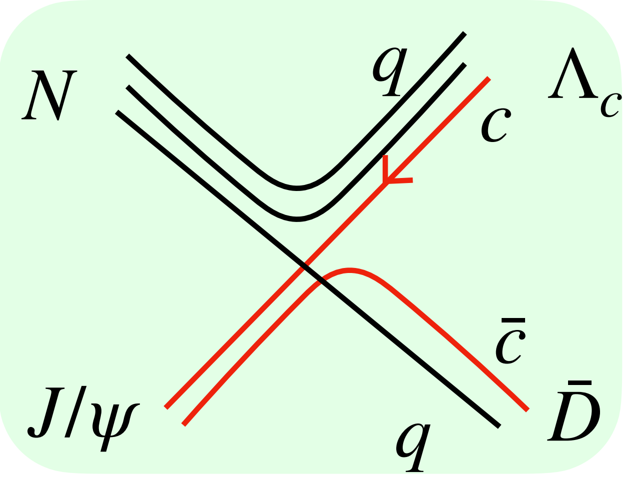
- q^6 systems: $6! = 3!3!2! \times 10,\ \langle P^c_{36}\rangle = 1/3, \langle P^{sf}_{36}\rangle \geq \langle [33] | P^{sf}_{36} | [33]\rangle = -\frac{1}{3}$

$$\nu = \langle q^6 | \mathscr{A}_6 | q^6 \rangle = \langle BB | (1 - 9P_{36}) | BB \rangle \leq 2$$



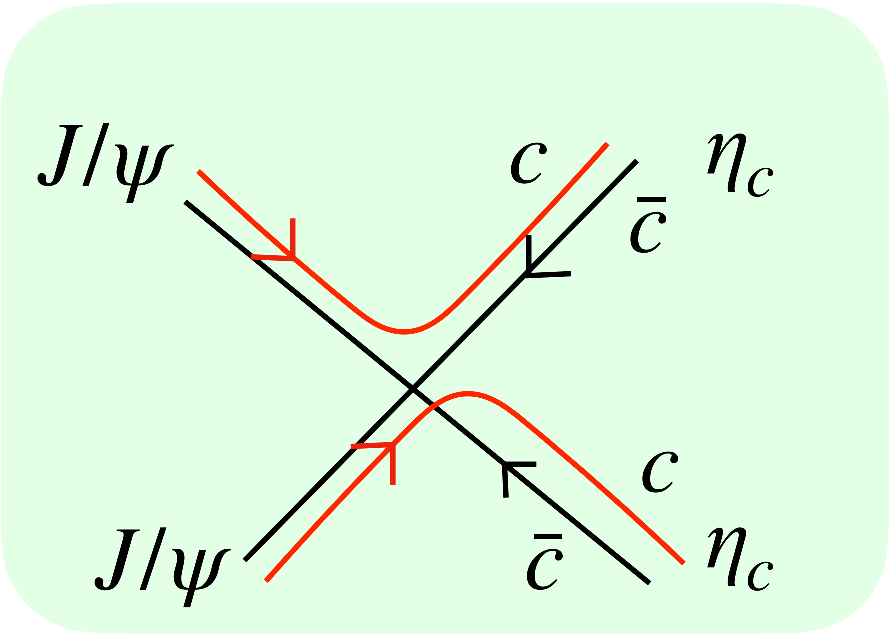
- $q^4\bar{q}$ systems: $4! = 3! \times 4,\ \langle P^{sf}_{34}\rangle \geq \langle [31] | P^{sf}_{34} | [31]\rangle = -\frac{1}{3}$

$$\nu = \langle q^4\bar{q} | \mathscr{A}_4 | q^4\bar{q} \rangle = \langle BB | (1 - 3P_{34}) | BB \rangle \leq \frac{4}{3}$$



- $q\bar{q}q\bar{q}$ systems: $\mathscr{A}_2\overline{\mathscr{A}}_2 = (1 - P_{13})(1 - P_{24}) = (1 - P_{24})(1 + P_{24}P_{13})$

$$\nu = \langle q^2\bar{q}^2 | \mathscr{A}_2\overline{\mathscr{A}}_2 | q^2\bar{q}^2 \rangle = \langle MM | (1 - P_{13}) | MM \rangle$$



$$= \frac{1}{2}\langle (1 - P^c_{13})(1 + P^{fs}_{13}) + (1 + P^c_{13})(1 - P^{fs}_{13})\rangle \Big|_{\text{color singlet}}$$

$$= \frac{1}{3}\langle (1 + P^{fs}_{13})\rangle + \frac{2}{3}\langle (1 - P^{fs}_{13})\rangle \leq \frac{4}{3}$$

$2/3 \leqq \nu \leqq 4/3$ for ccbar-ccbar

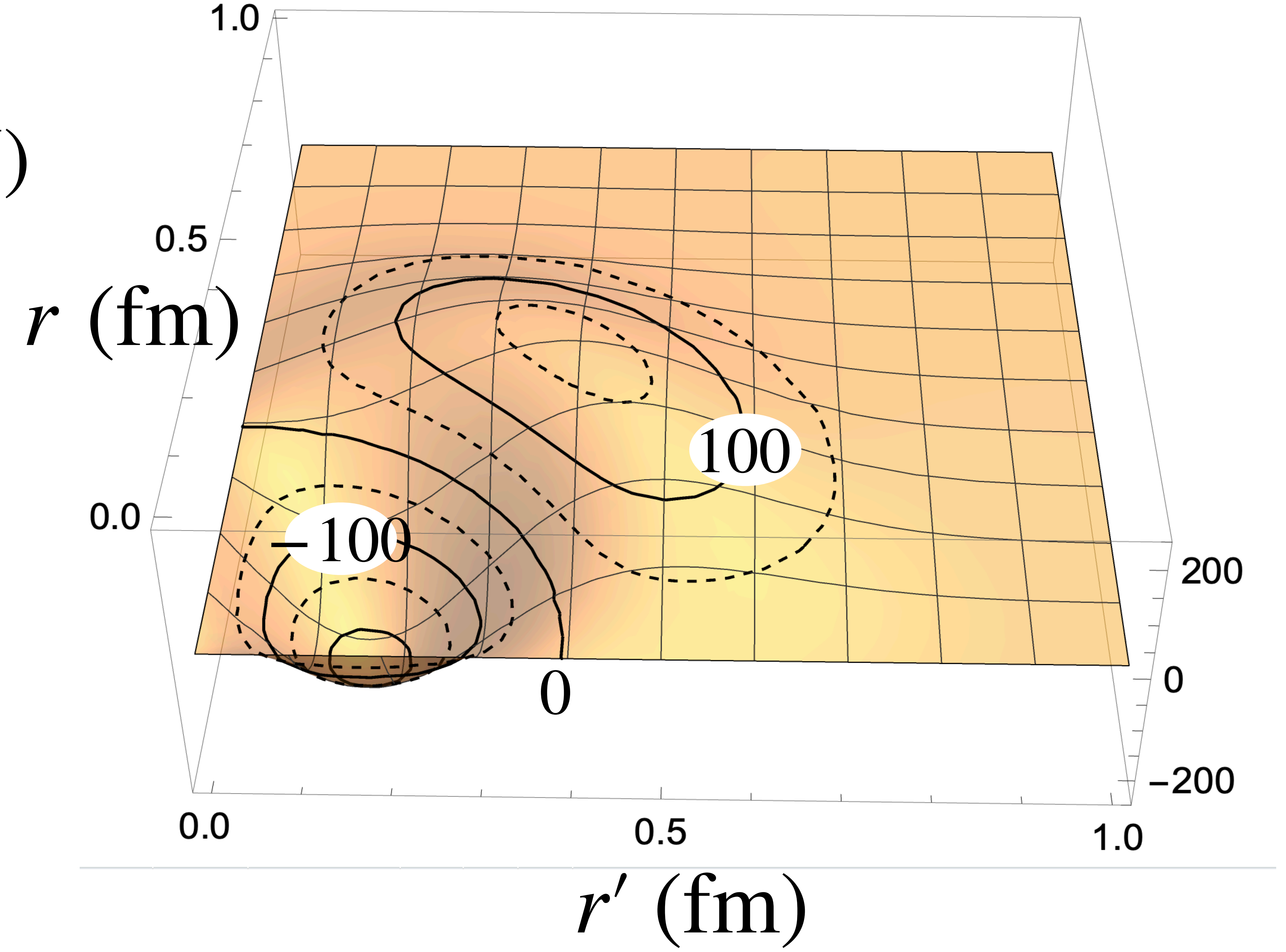
V_K Potential shape (nonlocal)

$c\bar{c}c\bar{c}(J = 1,\ 2)$

$$V_K(r, r') = \frac{\hbar\omega_0}{2}\left(\sqrt{\frac{2}{3}} - 1\right)\sqrt{\frac{3}{2}}\left(\phi_{0s}(b, r)\phi^{\dagger}_{1s}(b, r') + \phi_{1s}(b, r)\phi^{\dagger}_{0s}(b, r')\right)$$

with $\omega_0 = 500MeV, b = 1/\sqrt{m_c\omega_0}$

$V_K(r, r')\, rr'b$ (MeV)



For $J = 0$ $(\eta_c\eta_c, J/\psi J/\psi)$, $\nu - 1 = \begin{pmatrix} -1/6 & 1/\sqrt{12} \\ \sqrt{12} & 1/6 \end{pmatrix}$

the orbital dependence is the same.

Born-phase-shift equivalent local potential

Phase shift with Born approximation is essentially Fourier transformation of the potential. By the inverse transformation, we can obtain a local potential (something familiar) from the nonlocal potential (something not so familiar).

$$\tan\delta^B(k) = -k\int_0^\infty j_0(kr)2\mu_h V_K(r,r')j_0(kr')^2 r^2 dr r'^2 dr'$$

$$\|$$

$$\tan\delta^B(k) = -k\int_0^\infty j_0(kr)^2 2\mu_h V^B_{loc}(r)r^2 dr$$

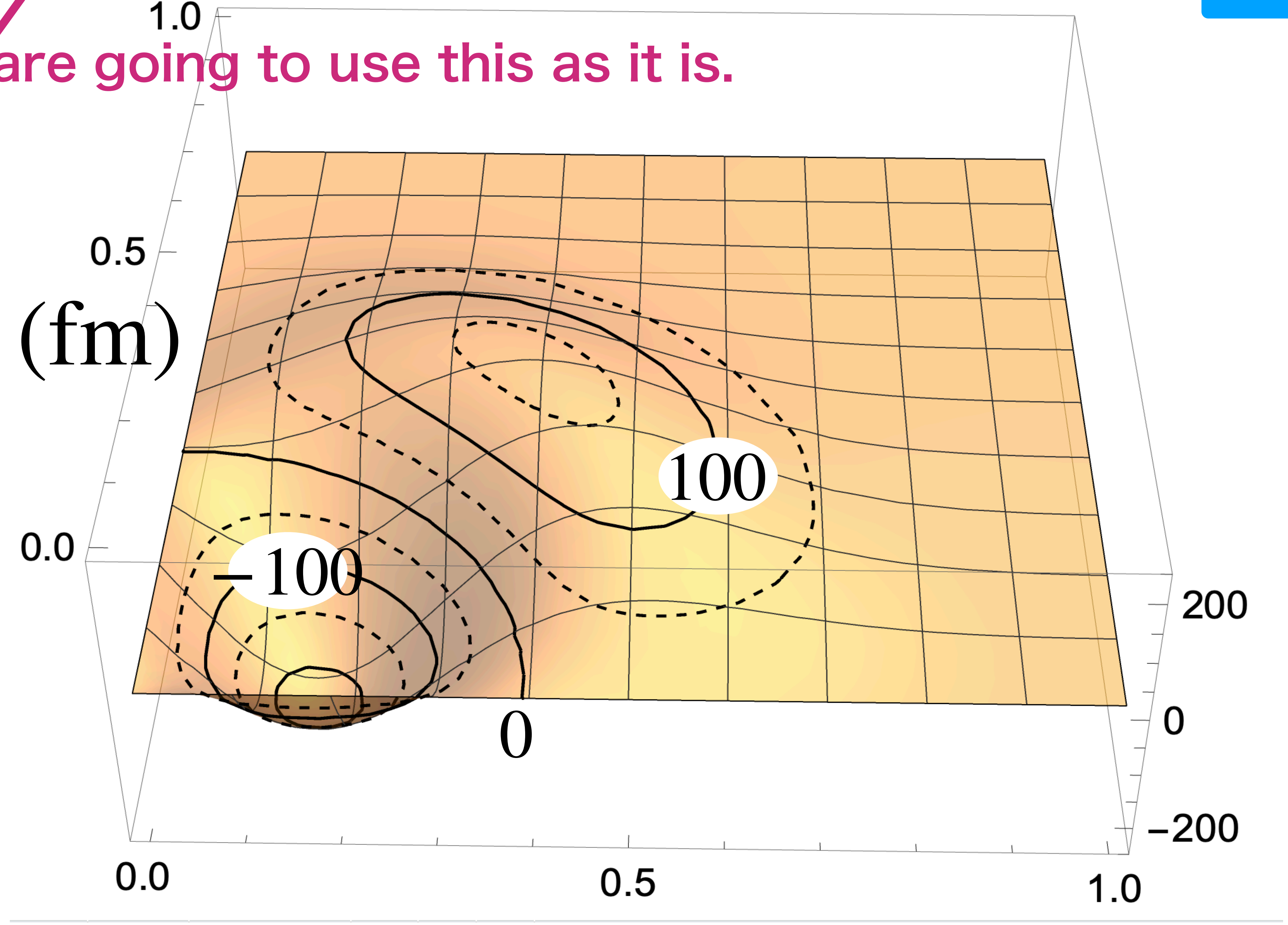
(a little bit) different from the one from the inverse scattering method, but the analytic form can be obtained in this way…

V_K Potential shape

$c\bar{c}c\bar{c}(J = 1, 2)$

$V_K(r, r')\, rr'b$ (MeV)

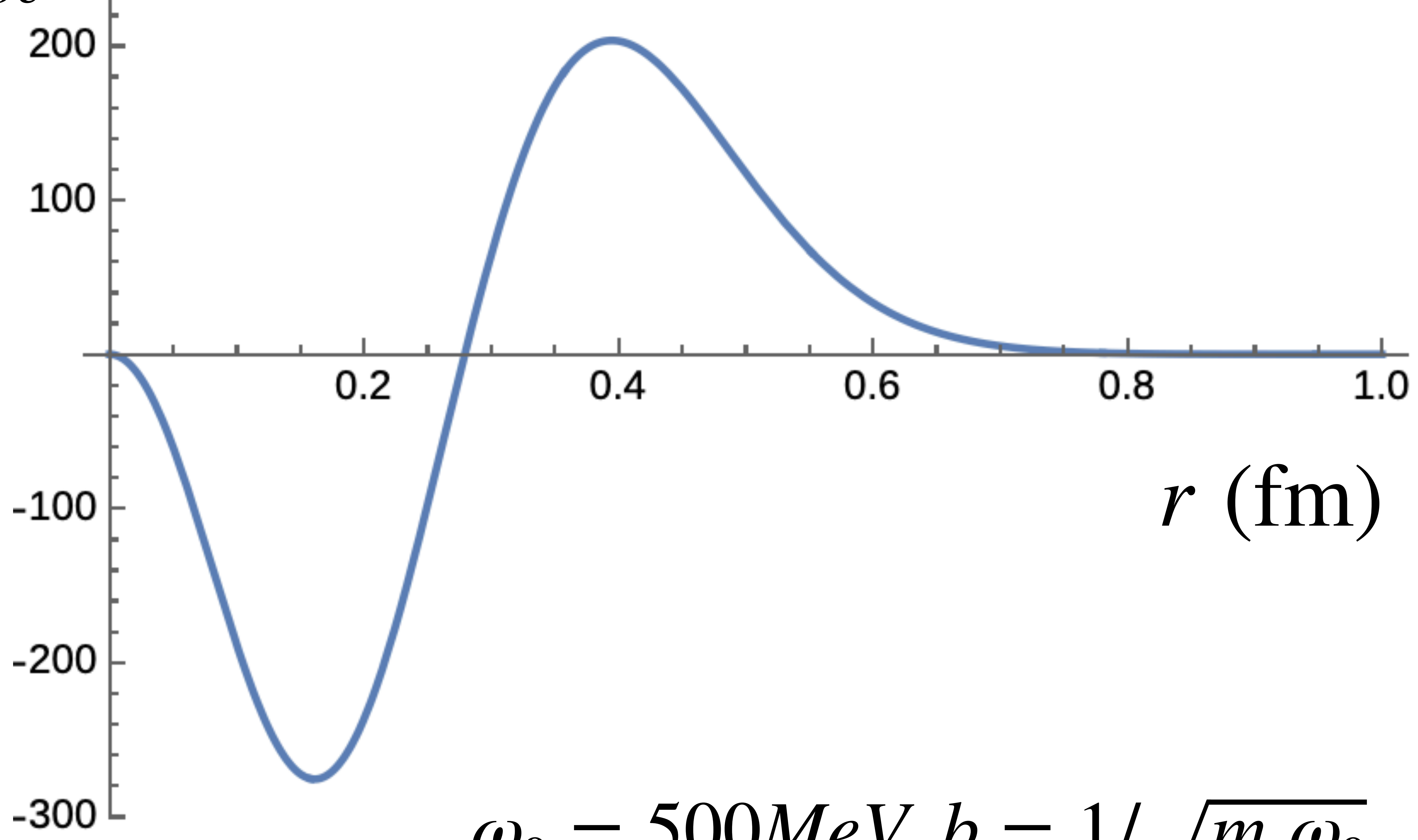
We are going to use this as it is.



$$V^B_{loc} = 12\omega_0(\sqrt{2/3} - 1)\frac{r^2}{b^2}(1 - \frac{2}{3}\frac{r^2}{b^2})\exp[-\frac{r^2}{b^2}]$$

$$b = (m_c\omega_0)^{-1/2}$$

$V^B_{loc}(r)$ (MeV)



$\omega_0 = 500MeV, b = 1/\sqrt{m_c\omega_0}$

V_K Potential, summary

Size of the effects can be evaluated by $\nu_{hh'}^{sfc}$

- $\nu < 1$: (a) Pauli-blocking effect
  - 0s-1s mixing reduces $\rightarrow$ repulsion at low p
- $\nu > 1$: (b) Quark many body effects
  - 0s-1s mixing enhances $\rightarrow$ attraction at low p
- ν is determined by the color-flavor-spin symmetry $\nu = \langle hh'0s \,|\, \mathscr{A} \,|\, hh'0s \rangle = \langle hh' \,|\, \left(1 - n_h n_{h'} P_{14}^{ex,sfc}\right) |\, hh' \rangle_{sfc}$
- Examples:
  - Two baryon systems (q^3-q^3) $\quad 0 \leq \nu_{hh'}^{sfc} \leq 2$
  - Pentaquarks (q^3-$q\bar{q}$) $\quad 0 \leq \nu_{hh'}^{sfc} \leq \frac{4}{3}$
  - Two meson systems ($q\bar{q}$-$q\bar{q}$) $\quad \frac{2}{3} \leq \nu_{hh'}^{sfc} \leq \frac{4}{3}$
- ‘taking one hadron out of 2 hadrons’ is different from ‘taking qqq out of $qqqq$ (+$\bar{q}$) or $q\bar{q}$ out of $q\bar{q}q\bar{q}$.’

Dynamical calculation

Two (not so) free parameters:

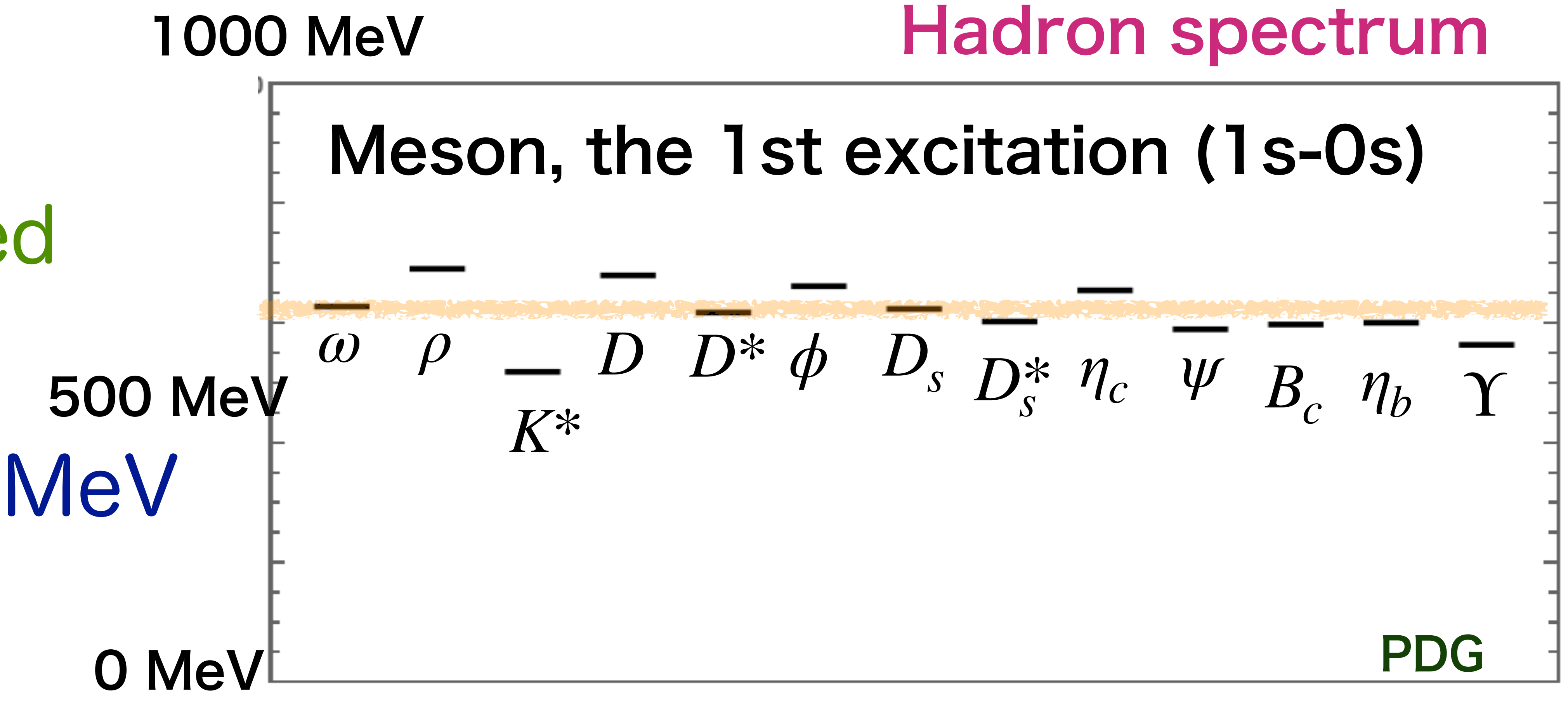
- charm quark mass $m_c = 1500$ MeV.
- mass-scaled meson size, $mb^2 = \omega_0^{-1}$
  - When we assume $mb^2 = \omega_0^{-1}$, we can explain the fact that the 1st excitation energies of hadrons are almost flavor independent.

We employ two values for ω_0

- The excitation energy ∼ 600 MeV, which corresponds to $1.7\hbar\omega_0$ for the linear confinement, $-a(\lambda . \lambda)\ r$.
  - $\hbar\omega_0 \sim 350\text{MeV}$
    corresponding size parameter of nucleon can be obtained by $mb^2 = \omega_0^{-1}$: $m_{u,d} = 313MeV \sim b_{u,d} = 0.6\text{fm}$
- nucleon size $b_{u,d} = 0.5\text{fm}$ with $m_{u,d} = 313MeV$ gives $\omega_0 \sim 500\text{MeV}$

$$\frac{1}{\mu b^2} = \omega_0 \sim 350 - 500\text{MeV}$$

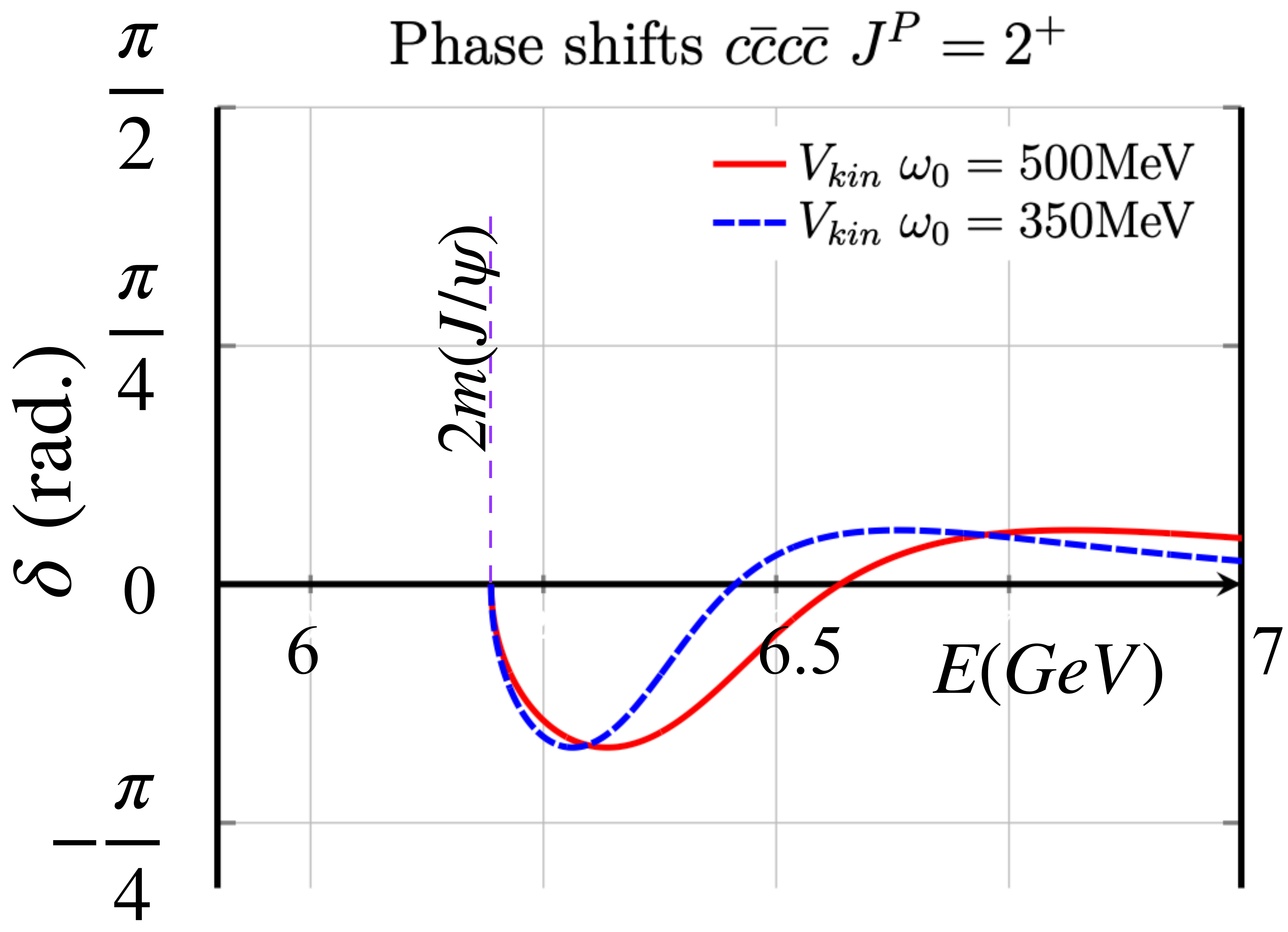
Hadron spectrum



Phase shift of $c\bar{c}$-$c\bar{c}$ scattering by V_K

The simplest channel (J=2)

- Phase shift has a node at $\frac{3}{4}\omega_0$ above the threshold.



S-matrix poles from V_K

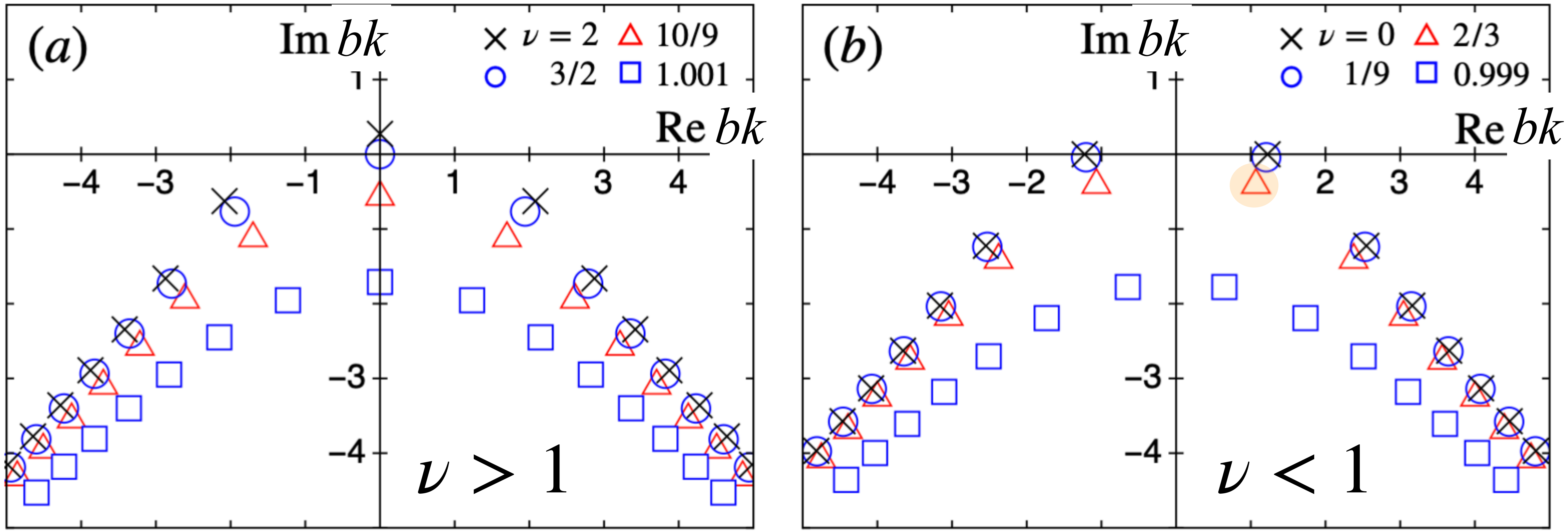


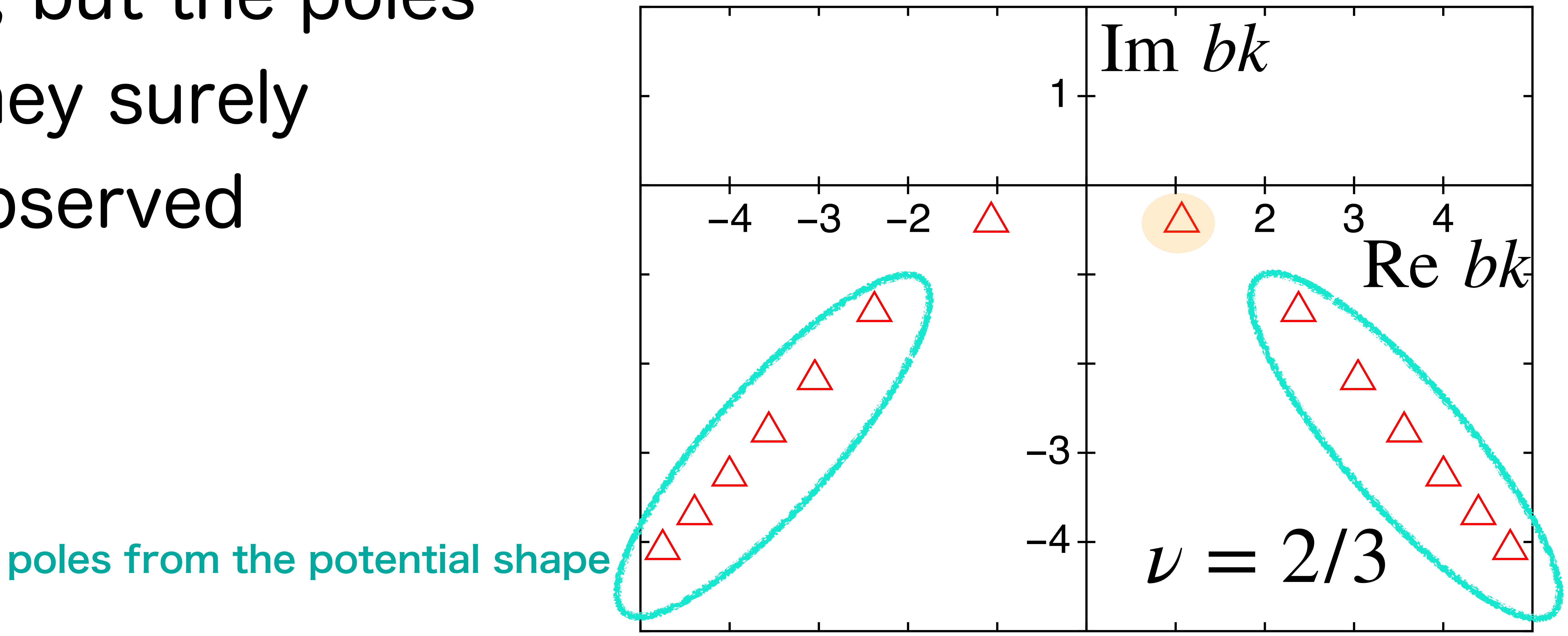
図 2: the S-matrix poles of V_K, for $a_0 = 1$, and (a) ν=2(×), 1.5(◯), 10/9(△), 0.001(□), and (b) ν=0.999(□) ν=2/3(△), 1/9(∘), 0(×). $\bar{k}$ stands for the momentum multiplied by the size parameter between the two clusters, bk.

S-matrix poles from V_K

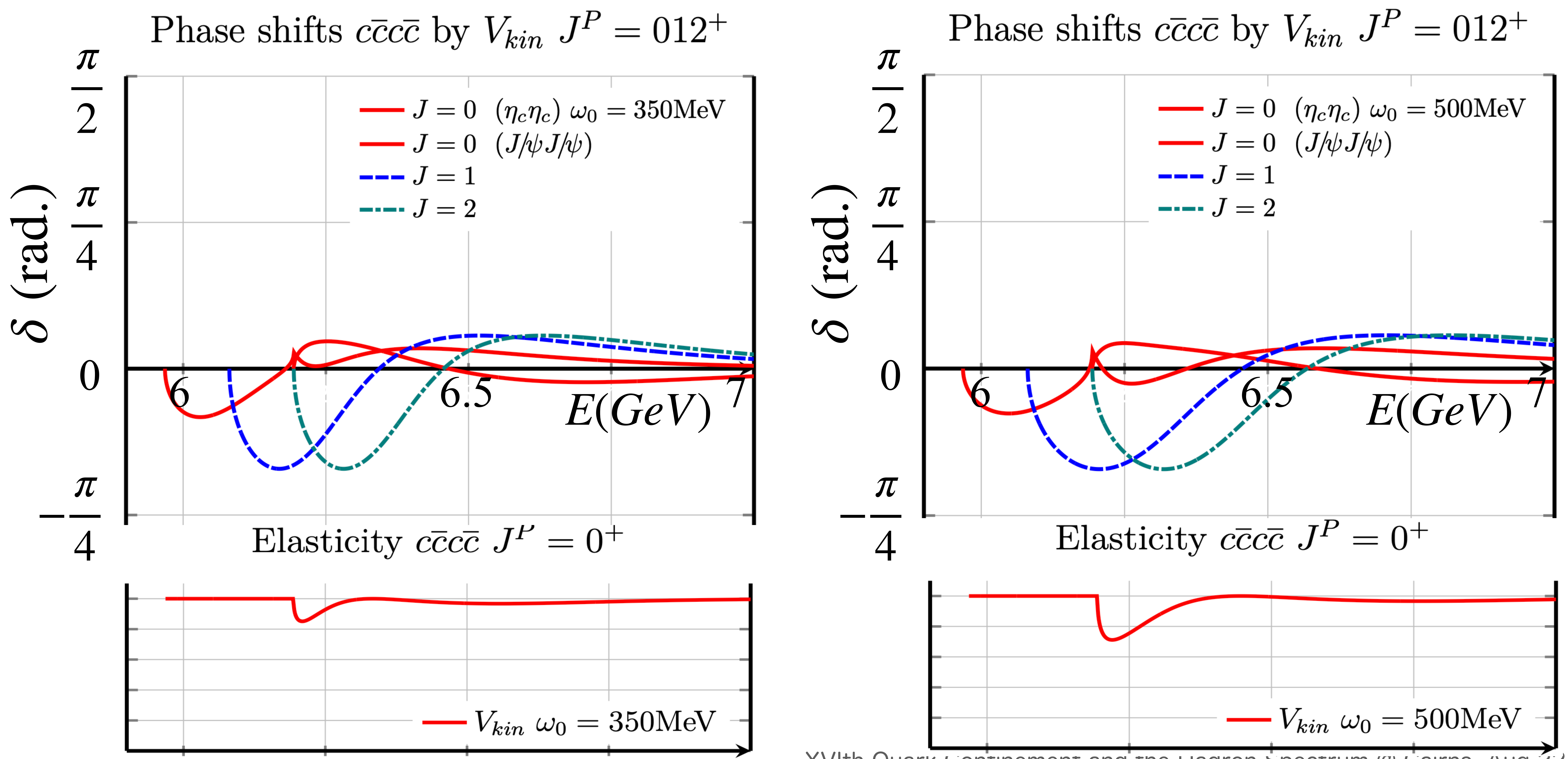
The pole is around $E = 6400 - 300/2\,i$ (J=2)

- $E = 6453 - 191i \quad (\omega_0 = 500MeV),\ E = 6377 - 133i \quad (\omega_0 = 350MeV)$

It has a large width, but the poles are really there. They surely contribute to the observed spectrum.

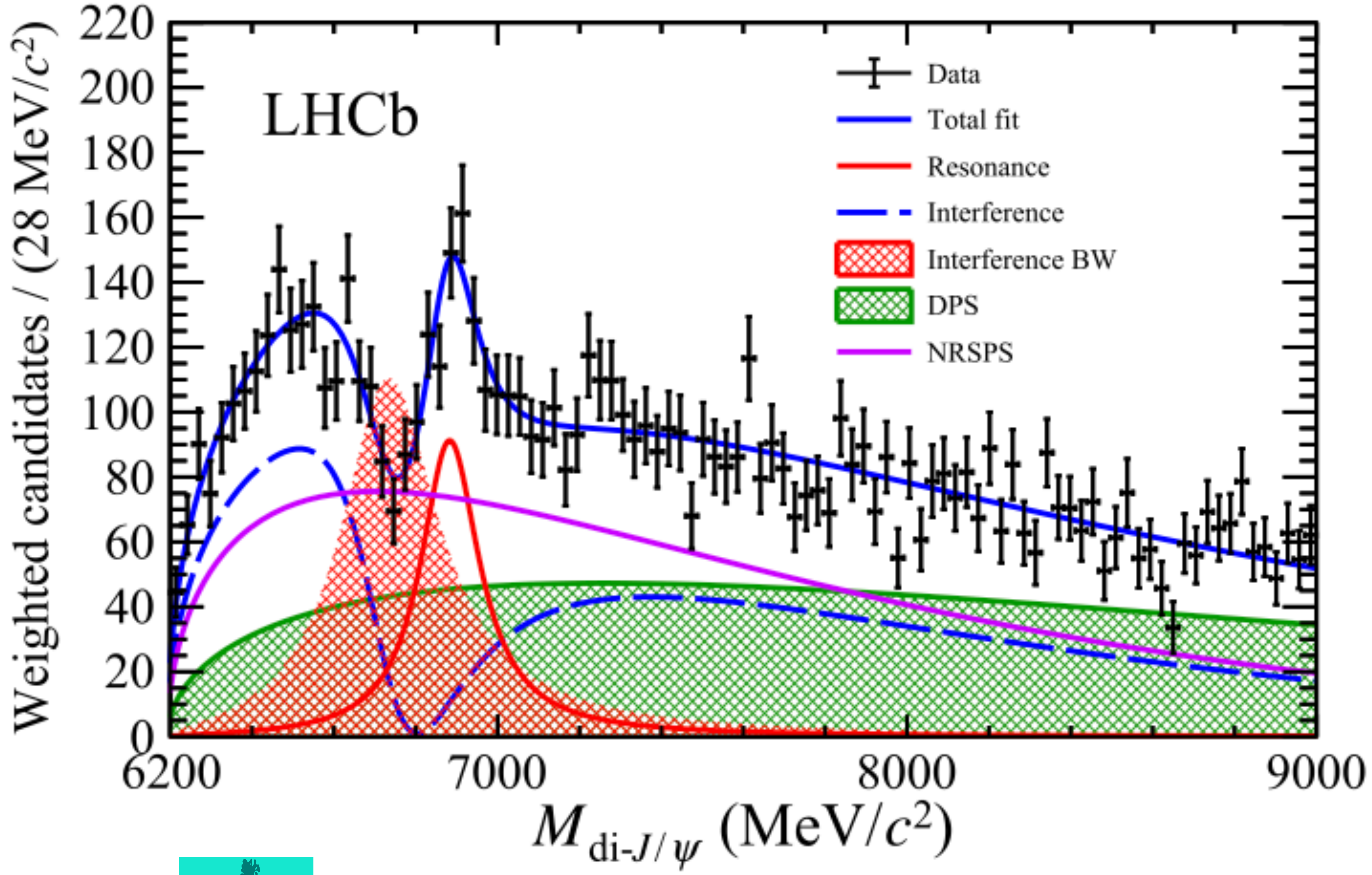


Phase shift of $c\bar{c}$-$c\bar{c}$ scattering by V_K



Do they produce the structure in di-J/ψ spectrum?

They produce a broad resonance, or a broad structure, in the final di-J/ψ states.



$E = 6400 - 300/2\,i$ (J=2)

LHCb, Science Bulletin 65 (2020) 1983–1993

QM

2++

E=6273 Γ=234

(Ortega et al)

corresponds to the present work?

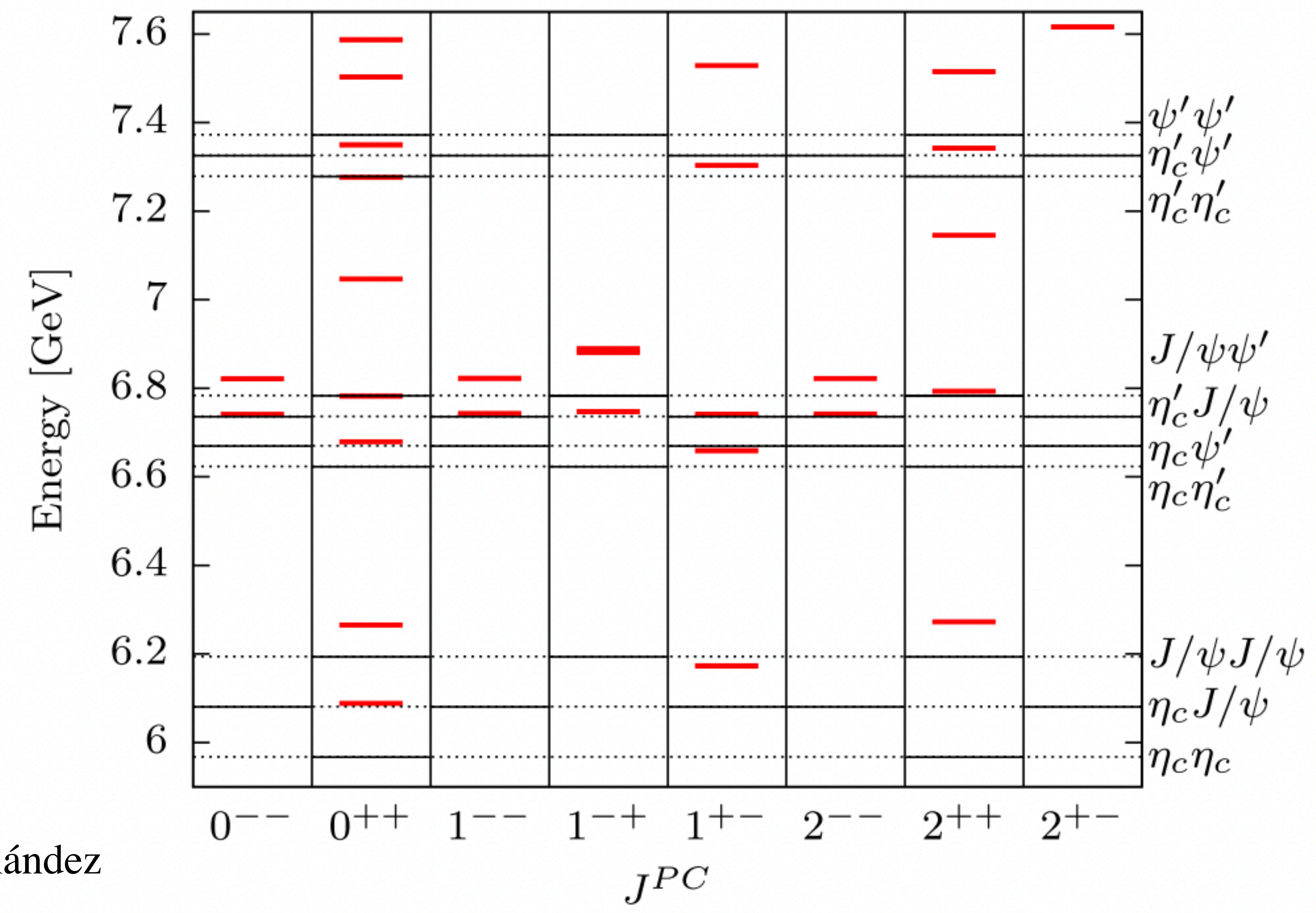


FIG. 1. Summary of the $T_{\psi\psi}$ candidates found in this work (red lines). The opened (closed) $c\bar{c} - c\bar{c}$ thresholds are shown as horizontal solid (dashed) lines. See Table II for more details.

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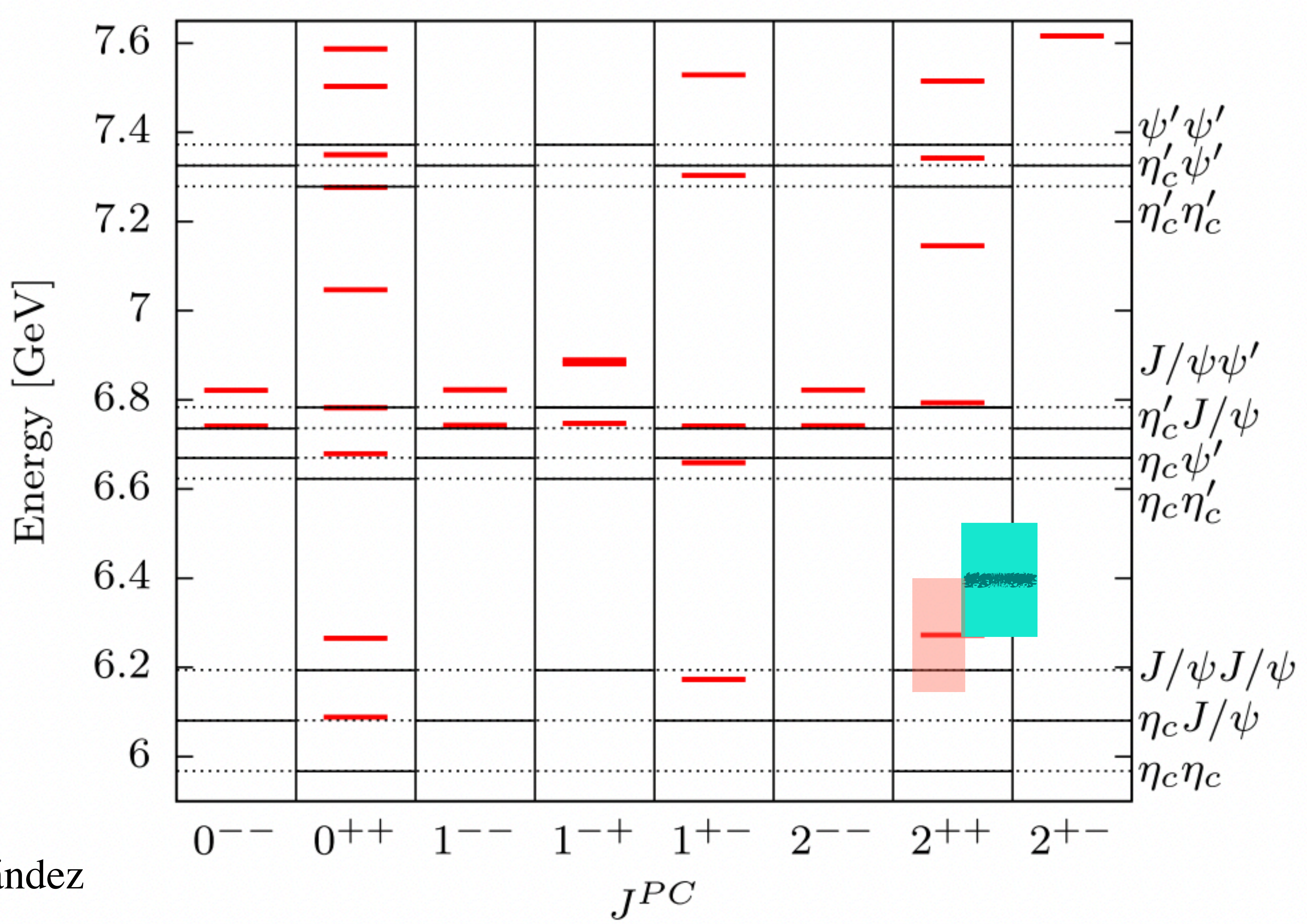

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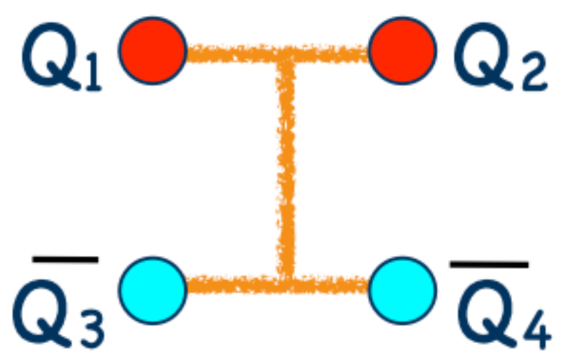
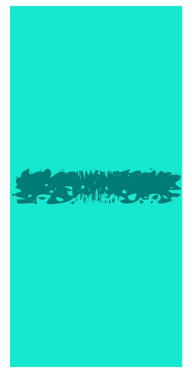
FIG. 1. Summary of the $T_{\psi\psi}$ candidates found in this work (red lines). The opened (closed) $c\bar{c} - c\bar{c}$ thresholds are shown as horizontal solid (dashed) lines. See Table II for more details.

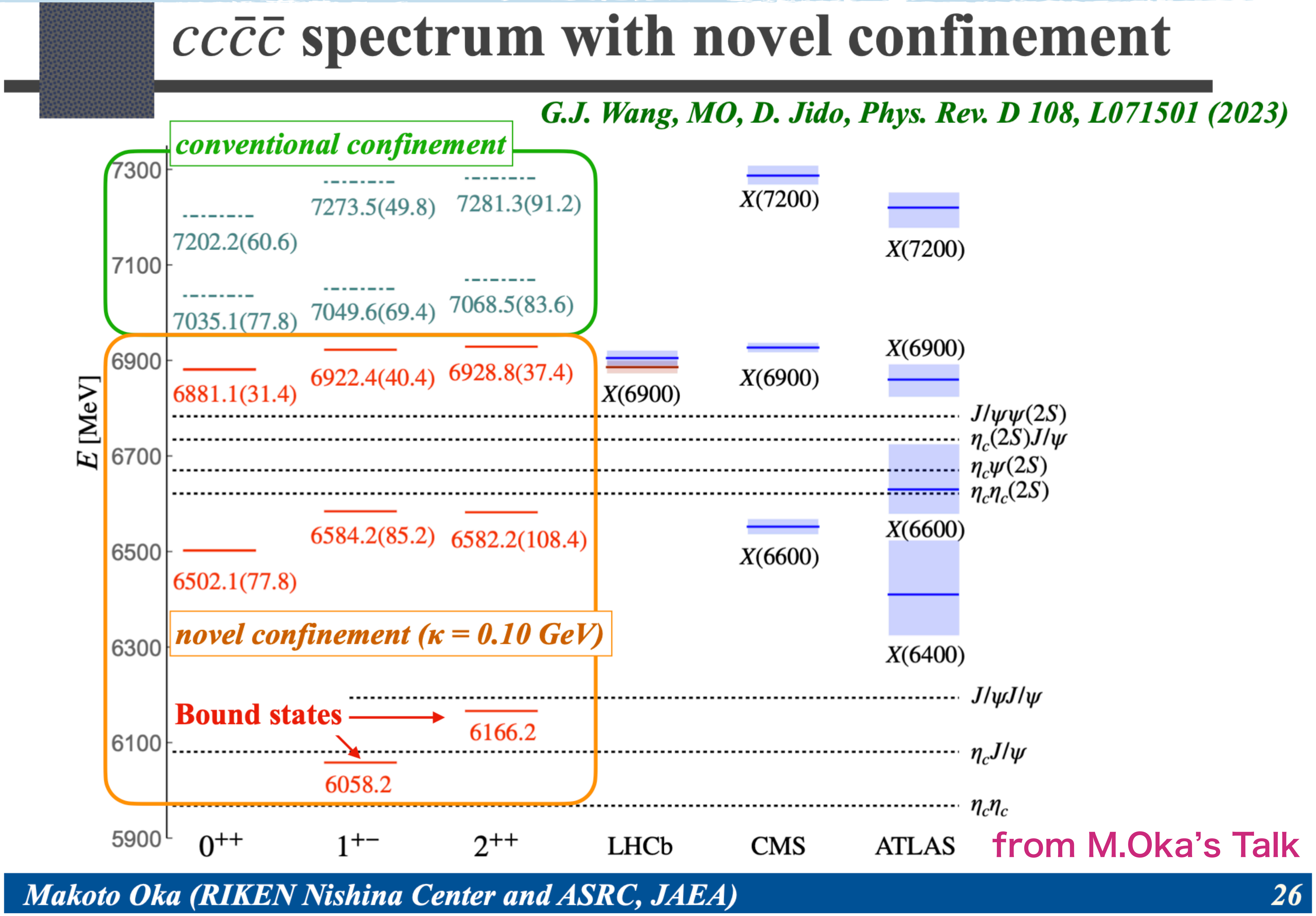
QM

The present structure appears around their ‘non-conventional’ area.



J=2





from M.Oka’s Talk

(c) $q\bar{c}q\bar{c}$ color-spin interaction

Assumptions:

- color-spin potential between quarks V^{cs} affects those only in the relative $(0s)$ configuration.
- V is proportional to $\lambda \cdot \lambda \sigma \cdot \sigma$
- c_{0s}^{α}'s are c-numbers, and obtained from the hadron hyper-splitting,
  - obtained from the observed η_c-J/ψ mass diff.

$$V_{ij}^{cs}(r_{ij}, r'_{ij}) = (\lambda_i \, . \, \lambda_j)(\sigma_i \, . \, \sigma_j) \; c_{0s}^{f_i f_j} \; \langle r_{ij} \,|\, 0s \rangle \langle 0s \,|\, r'_{ij} \rangle$$

Channel dependence of the color-spin interaction

They can be expressed by the mass difference btw J/ψ and η_c as:

We fold the interaction by the hadron clusters.

That also gives the threshold difference.

$c\bar{c}c\bar{c}(J=2)$

$$V_{cs} = 0$$

$c\bar{c}c\bar{c}(J=1)$

$$V_{cs} = M_{hfs}\frac{1}{2}\phi_{0s}(r)\phi^*_{0s}(r')$$

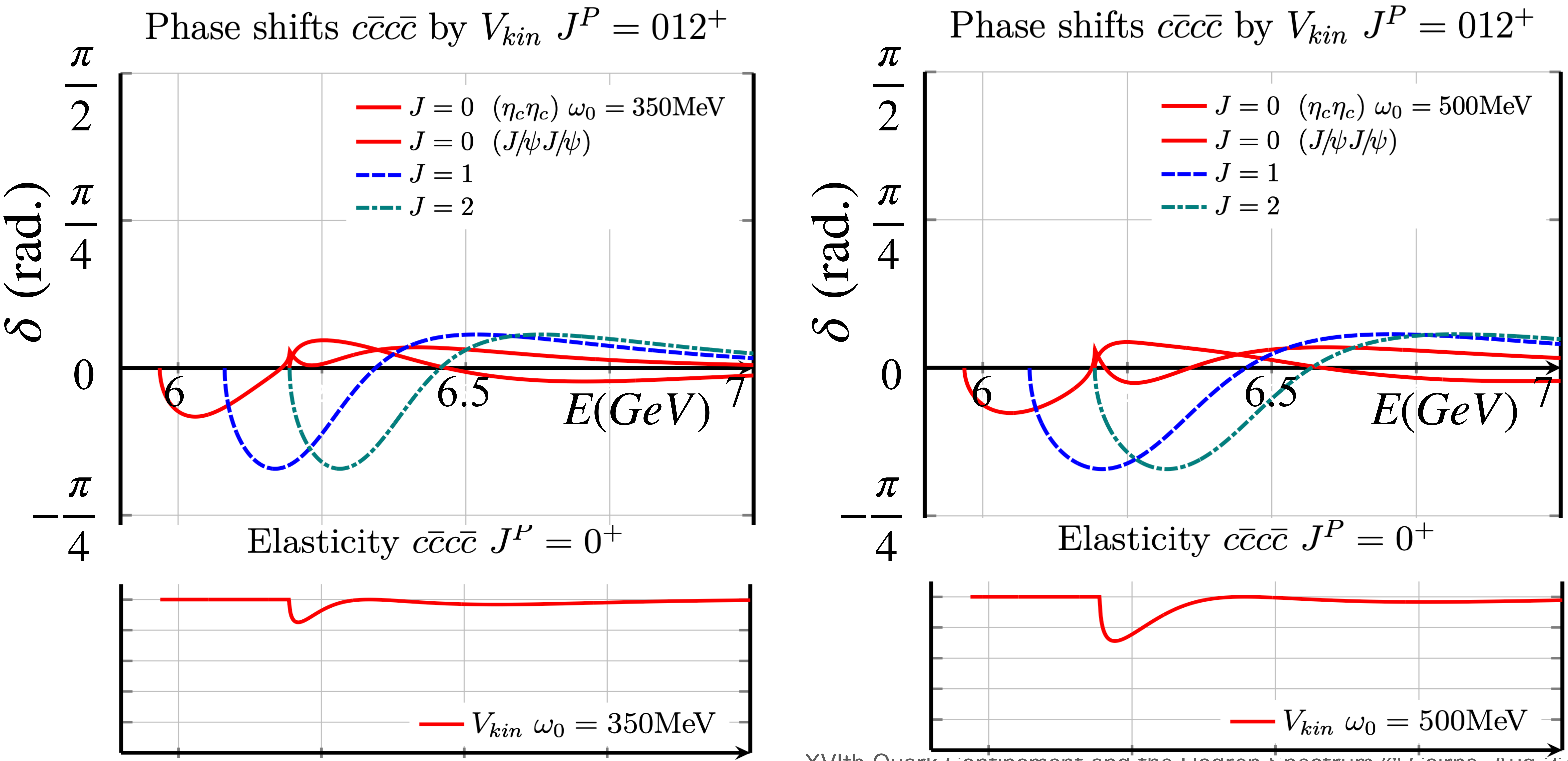
Mostly repulsive.

$c\bar{c}c\bar{c}(J=0)$

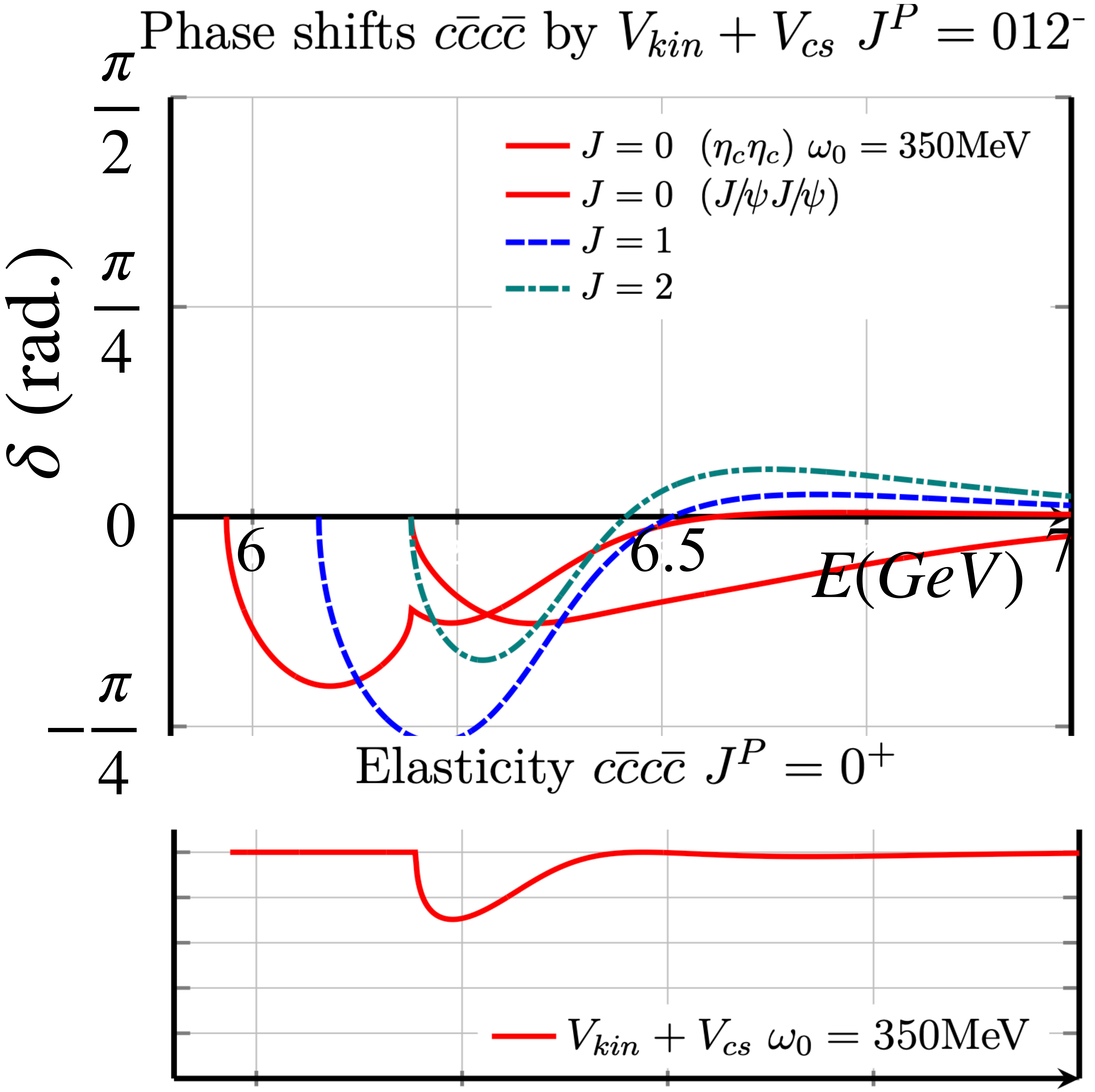
$$V_{cs} = M_{hfs}\begin{pmatrix} 0.61 & -0.19 \\ -0.19 & 0.51 \end{pmatrix}\phi_{0s}(r)\phi^*_{0s}(r')$$

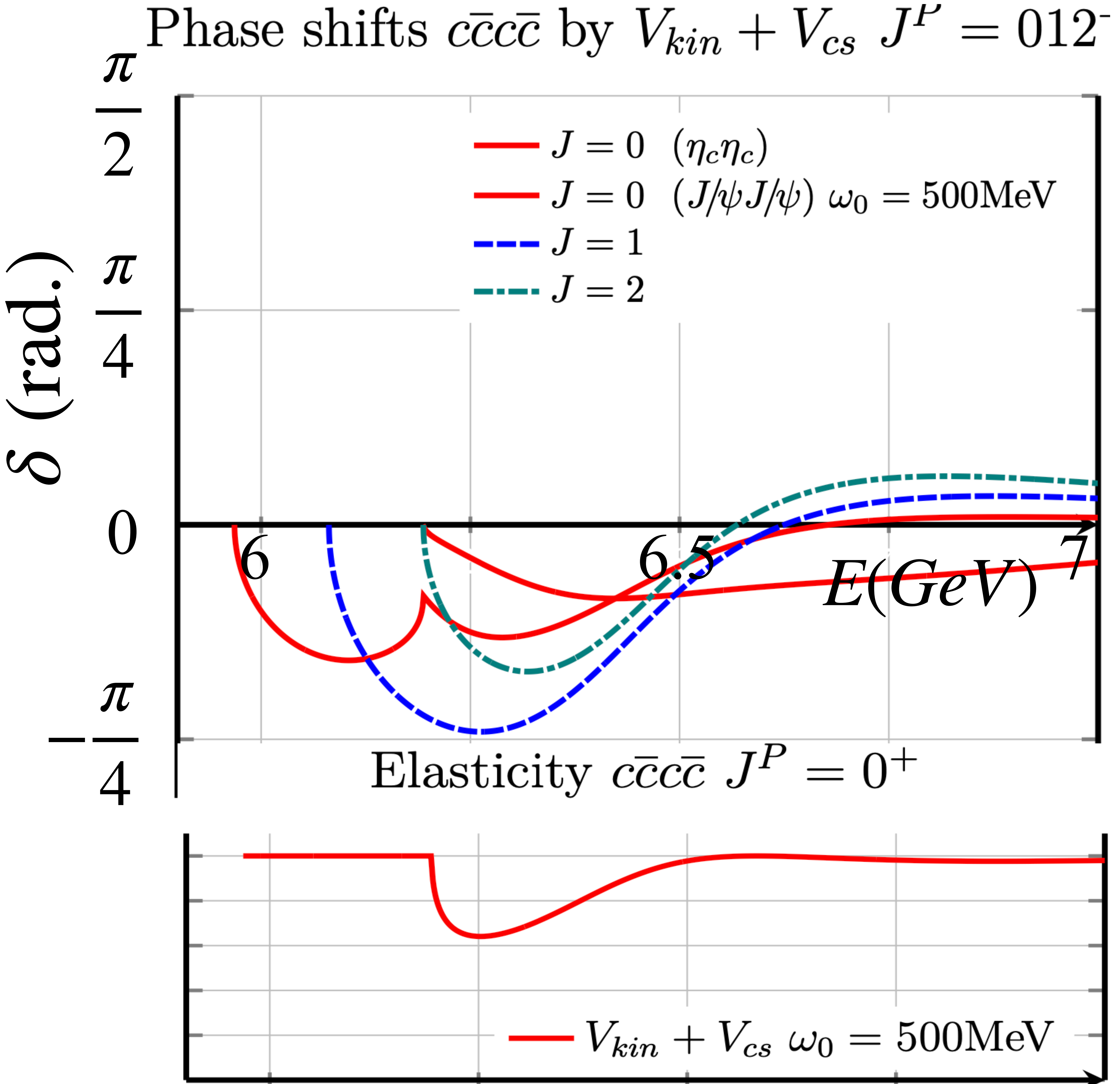
$$M_{hfs} = m(J/\psi) - m(\eta_c) \quad \text{(observed)}$$

Phase shift of $c\bar{c}$-$c\bar{c}$ scattering by V_K



Phase shift of $c\bar{c}$-$c\bar{c}$ scattering by $V_K + V_{cs}$





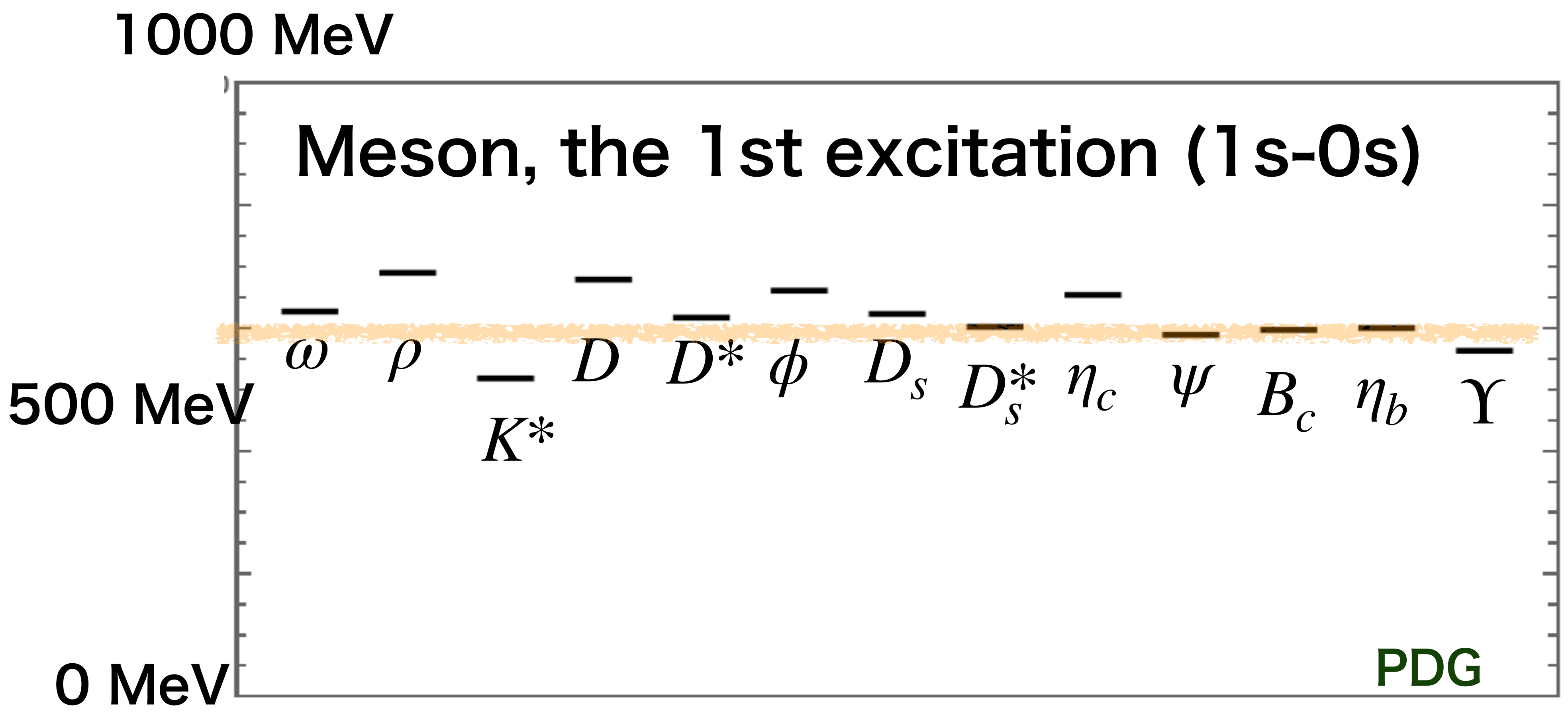
Summary and Outlook

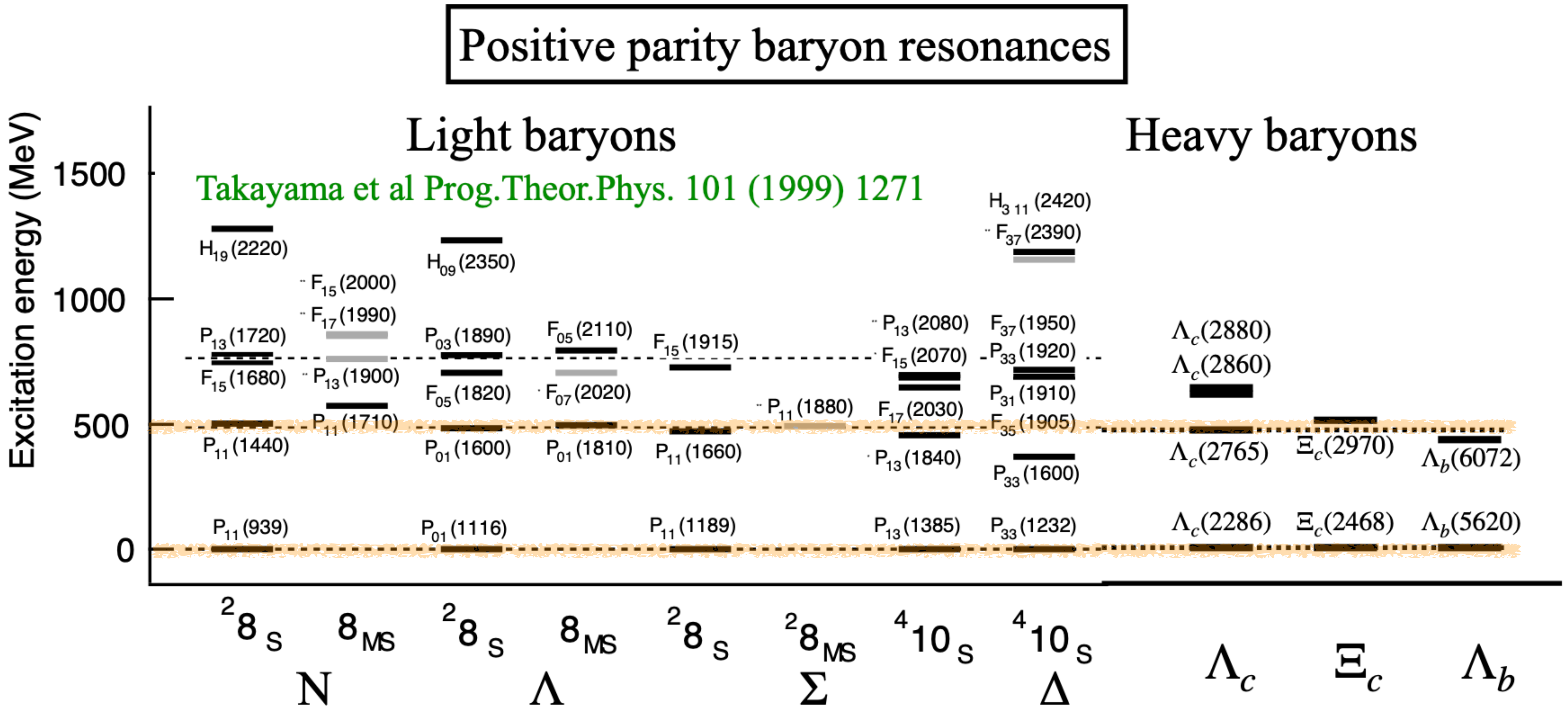
- We study the quark Pauli-blocking effects and the quark interchange effects in the 'simplest' $c\bar{c}c\bar{c}$ multiquark systems.
- They can be expressed by the non-local potential with 0s-1s mixing.
- For the di-J/ψ states, the mixing is mostly repulsive.
- The effects give a structure around 3200+0.75ω_0 $\sim$ 6500 MeV with $\Gamma \sim$ 300 MeV for each total spin J=0, 1, 2. Corresponding poles appear in the S-matrix.
- By investigating the effects in $c\bar{c}c\bar{c}$'s, or in other multiquark states, or their decay or production etc., we can construct comprehensive picture of the exotic states $\cdots$

back up

Quark potential to Hadron potential (How to derive (a) and (b))

orbital excitation does not depend on the flavors.





Quark potential to Hadron potential (How to derive (a) and (b))

What can we learn from these?

- Evaluation of the excited energy by the Gaussian wave function with size parameter b.

$$H_1 = H_0 - \frac{4\alpha_s}{3}\frac{1}{r} + \sigma r$$

$$\langle 1s | H_1 | 1s \rangle - \langle 0s | H_1 | 0s \rangle = \frac{1}{\mu b^2} + \frac{4\alpha_s}{3}\frac{1}{3\sqrt{\pi}}\frac{1}{b} + \sigma\frac{1}{\sqrt{\pi}} b$$

$$\langle 1s | H_1 | 0s \rangle = \frac{1}{\mu b^2}\sqrt{\frac{3}{8}} - \frac{4\alpha_s}{3}\sqrt{\frac{2}{3\pi}}\frac{1}{b} - \sigma\sqrt{\frac{2}{3\pi}}\, b$$

The size parameter becomes small as the mass becomes large. But, if we take $\mu b^2 \sim const.$, the excited energy will depend on the mass only weakly. (we are not claiming that gaussian is good.)

$$\frac{1}{\mu b^2} = \omega_0 \sim 350 - 500\text{MeV}$$

One of the two free parameters in our model