Tetraquark equations

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Introduction

- Tetraquarks consist of 2 quarks and 2 antiquarks $2q2\bar{q}$. We shall number them as:

1	2	3	4
q	q	$\bar{q}$	$\bar{q}$

- Tetraquarks need relativistic QFT for their description

- Goal 1: Formulate practical covariant equations describing the $2q2\bar{q}$ system where interactions are pairwise, and transitions to two-body states, $2q2\bar{q} \to q\bar{q}$ are taken into account

- Goal 2: To provide a unified description of long-standing and seemingly unrelated tetraquark models used for *calculating* properties of tetraquarks

Covariant 4-body equations

A. N. Kvinikhidze and A. M. Khvedelidze (KK), Theor. Math. Phys. **90**, 62 (1992)

- Goal: to solve the covariant 4-body bound-state equation

$$\Phi = K^{(4)} G_0^{(4)} \Phi$$

for $2q2\bar{q}$ system with pairwise interactions only.

- Starting point: No particle annihilation or creation (KK 1992)
- Only 3 possible pairwise channels in a 4-body system:

$$\overbrace{(q\bar{q})(q\bar{q})}^{1} \quad \overbrace{(q\bar{q})(q\bar{q})}^{2} \quad \overbrace{(qq)(\bar{q}\bar{q})}^{3}$$

$$(13)(24) \quad (14)(23) \quad (12)(34)$$

- Correspondingly, the 4-body kernel can be written as

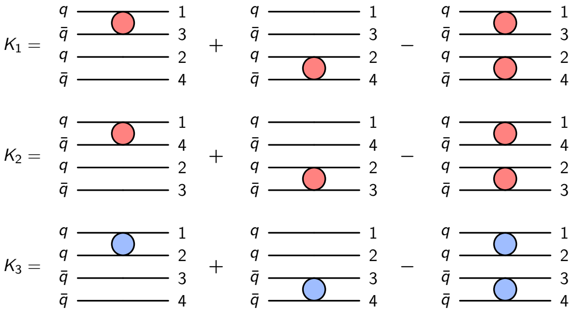
$$K^{(4)} = K_1 + K_2 + K_3$$

Covariant 4-body equations

- Each K_α expressed in terms of 2-body kernels K_{ij}:

$$K_1 = K_{13} + K_{24} - K_{13}K_{24}$$
$$K_2 = K_{14} + K_{23} - K_{14}K_{23}$$
$$K_3 = K_{12} + K_{34} - K_{12}K_{34}$$

- Graphically:

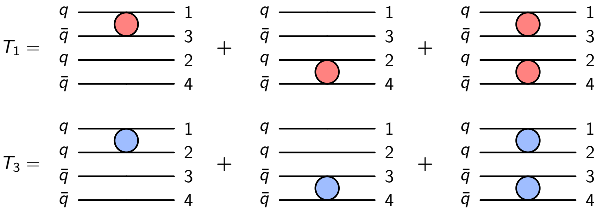


- Define 4-body pairwise-channel t matrices

$$T_\alpha = K_\alpha + K_\alpha G_0^{(4)} T_\alpha$$

- Each T_α expressed in terms of 2-body t matrices T_{ij}:

$$\begin{aligned} T_1 &= T_{13} + T_{24} + T_{13} T_{24} &= T_1^{+} + T_1^{\times} \\ T_2 &= T_{14} + T_{23} + T_{14} T_{23} &= T_2^{+} + T_2^{\times} \\ T_3 &= T_{12} + T_{34} + T_{12} T_{34} &= T_3^{+} + T_3^{\times} \end{aligned}$$

- 

Covariant 4-body equations

- $\Phi = K^{(4)} G_0^{(4)} \Phi$ can then be rearranged similarly to the Faddeev rearrangement of 3-body theory: $\Phi = \Phi_1 + \Phi_2 + \Phi_3$ where

$$\boxed{\Phi_\alpha = T_\alpha \sum_{\beta \neq \alpha} G_0^{(4)} \Phi_\beta}$$

- Application to the $2q2\bar{q}$ system requires antisymmetriation of $2q$ and $2\bar{q}$ states, reducing above to

$$\begin{pmatrix} \Phi_1 \\ \Phi_3 \end{pmatrix} = \left[\begin{pmatrix} \frac{1}{2} T_1^+ & 0 \\ 0 & T_3^+ \end{pmatrix} + \begin{pmatrix} \frac{1}{2} T_1^\times & 0 \\ 0 & T_3^\times \end{pmatrix} \right] 2 \begin{pmatrix} -\mathcal{P}_{12} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_3 \end{pmatrix}$$

- or symbolically

$$\boxed{\tilde{\Phi} = \left(\mathcal{T}^+ + \mathcal{T}^\times \right) \mathcal{R} \tilde{\Phi}}$$

Giessen tetraquark model

W. Heupel, G. Eichmann, C. S. Fischer, Phys.Lett. **B718**, 545 (2012)

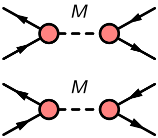
- Model

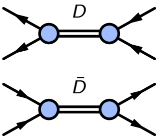
  (i) KK's covariant 4-body equations applied to $2q2\bar{q}$

  (ii) Single scattering terms $\mathcal{T}^+$ neglected:

$$T_1 = \cancel{T_{13} + T_{24}} + T_{13}T_{24}$$
$$T_3 = \cancel{T_{12} + T_{34}} + T_{12}T_{34}$$

  (iii) Meson and diquark pole approximation: $T_{ij} = \Gamma_{ij} D_{ij} \bar{\Gamma}_{ij}$

$T_{13}T_{24} =$ 

$T_{12}T_{34} =$ 

- Bound state equation with $\mathcal{T}^+$ neglected

$$\tilde{\Phi} = \left(\cancel{\mathcal{T}^+} + \mathcal{T}^\times\right) \mathcal{R}\tilde{\Phi}$$

- Meson and diquark pole approximation

$$\mathcal{T}^\times = -\Gamma D \bar{\Gamma}$$

$$= \begin{pmatrix} \Gamma_{13}\Gamma_{24} & 0 \\ 0 & \Gamma_{12}\Gamma_{34} \end{pmatrix} \begin{pmatrix} \frac{1}{2} D_{13} D_{24} & 0 \\ 0 & D_{12} D_{34} \end{pmatrix} \begin{pmatrix} \bar{\Gamma}_{13}\bar{\Gamma}_{24} & 0 \\ 0 & \bar{\Gamma}_{12}\bar{\Gamma}_{34} \end{pmatrix}$$

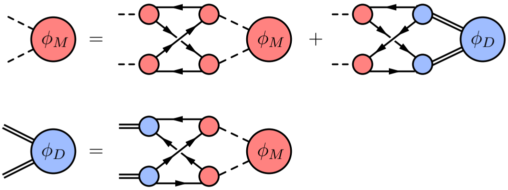
- Define ϕ: *tetraquark* $\rightarrow MM$, $D\bar{D}$ amplitudes

$$\tilde{\Phi} = \mathcal{T}^\times \mathcal{R}\tilde{\Phi} = -\Gamma D \bar{\Gamma} \mathcal{R}\tilde{\Phi} \quad \Rightarrow \quad \boxed{\phi \equiv \bar{\Gamma}\mathcal{R}\tilde{\Phi}}$$

- Tetraquark equations of Heupel *et al.*

$$\phi = VD\phi \qquad V = -\bar{\Gamma}\mathcal{R}\Gamma$$

- Graphic representation



where Φ_M and Φ_D are transition amplitudes for *tetraquark* $\to$ MM and *tetraquark* $\to$ $D\bar{D}$, respectively

Unified tetraquark equations

A. N. Kvinikhidze and B. Blankleider (KB), Phys. Rev. D **107**, 094014 (2023)

- We considered the bound state equation with $\mathcal{T}^+$ retained

$$\tilde{\Phi} = \left(\mathcal{T}^+ + \mathcal{T}^\times\right) \mathcal{R}\tilde{\Phi}$$

- Rearrange so that $\mathcal{T}^+$ can be included perturbatively

$$\tilde{\Phi} = (1 - \mathcal{T}^+\mathcal{R})^{-1}\mathcal{T}^\times\mathcal{R}\tilde{\Phi}$$

- Meson and diquark pole approximation in $\mathcal{T}^\times$ only

$$\mathcal{T}^\times = -\Gamma D\bar{\Gamma}$$

- Define ϕ: *tetraquark* $\rightarrow$ MM, $D\bar{D}$ amplitudes

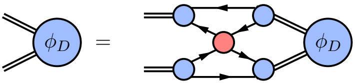
$$\tilde{\Phi} = -(1 - \mathcal{T}^+\mathcal{R})^{-1}\Gamma D\bar{\Gamma}\mathcal{R}\tilde{\Phi} \quad \Rightarrow \quad \boxed{\phi \equiv \bar{\Gamma}\mathcal{R}\tilde{\Phi}}$$

Unified tetraquark equations

- Unified tetraquark equations $\phi = VD\phi$

$$V = -\bar{\Gamma}\mathcal{R}\left[1 + \mathcal{T}^+\mathcal{R} + \left(\mathcal{T}^+\mathcal{R}\right)^2 + \dots\right]\Gamma$$

- Tetraquark model of Heupel *et al*. $V \to V^{(0)} = -\bar{\Gamma}\mathcal{R}\Gamma$
- Tetraquark model of Faustov *et al*. $V \to V^{(1)} = -\bar{\Gamma}\mathcal{R}\mathcal{T}^+\mathcal{R}\Gamma$



- This suggests that the Giessen and Moscow groups have been calculating non-overlapping parts of the same tetraquark equations!

$$\phi = [V^{(0)} + V^{(1)}]D\phi$$

References

- Moscow model:

[6] D. Ebert, R. N. Faustov, and V. O. Galkin, Masses of heavy tetraquarks in the relativistic quark model, Phys. Lett. B **634**, 214 (2006), arXiv:hep-ph/0512230.

[7] R. N. Faustov, V. O. Galkin, and E. M. Savchenko, Masses of the $QQ\bar{Q}\bar{Q}$ tetraquarks in the relativistic diquark–antidiquark picture, Phys. Rev. D **102**, 114030 (2020), arXiv:2009.13237 [hep-ph].

[8] R. N. Faustov, V. O. Galkin, and E. M. Savchenko, Heavy tetraquarks in the relativistic quark model, Universe **7**, 94 (2021), arXiv:2103.01763 [hep-ph].

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- Giessen model:

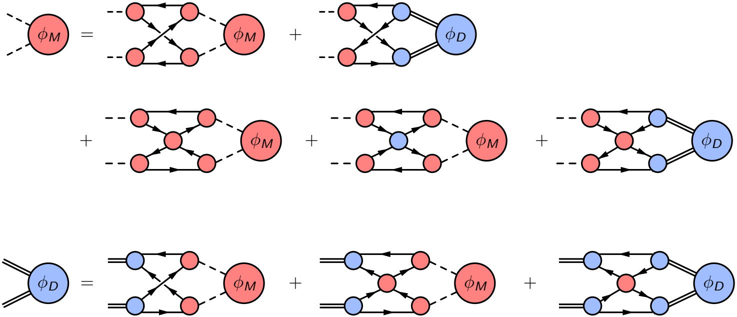
[10] W. Heupel, G. Eichmann, and C. S. Fischer, Tetraquark Bound States in a Bethe-Salpeter Approach, Phys. Lett. **B718**, 545 (2012), arXiv:1206.5129 [hep-ph].

[11] G. Eichmann, C. S. Fischer, and W. Heupel, The light scalar mesons as tetraquarks, Phys. Lett. B **753**, 282 (2016), arXiv:1508.07178 [hep-ph].

[12] G. Eichmann, C. S. Fischer, W. Heupel, N. Santowsky, and P. C. Wallbott, Four-Quark States from Functional Methods, Few Body Syst. **61**, 38 (2020), arXiv:2008.10240 [hep-ph].

[13] N. Santowsky and C. S. Fischer, Four-quark states with charm quarks in a two-body Bethe–Salpeter approach, Eur. Phys. J. C **82**, 313 (2022), arXiv:2111.15310 [hep-ph].

- Unified tetraquark equations: $\phi = [V^{(0)} + V^{(1)}]D\phi$



Unified tetraquark equations with nonperturbative inclusion of all meson and diquark contributions

- $V = V^{(0)} + V^{(1)} + V^{(2)} + \dots$ first 2 terms unify two popular tetraquarks models, but is the series convergent?
- Meson and diquark poles appear in each $2q$ scattering term
- Perhaps ALL pole terms should be taken into account non-perturbatively
- This can be done! The clue lies in the unified approach, where the 4-body equations

$$\tilde{\Phi} = \left(\mathcal{T}^{+} + \mathcal{T}^{\times}\right) \mathcal{R}\tilde{\Phi}$$

are rearranged as

$$\tilde{\Phi} = (1 - \mathcal{T}^{+}\mathcal{R})^{-1}\mathcal{T}^{\times}\mathcal{R}\tilde{\Phi}$$

precisely because $\mathcal{T}^{\times} = -\Gamma D \bar{\Gamma}$ is a pole term!

Unified tetraquark equations with nonperturbative inclusion of all meson and diquark contributions

- Write ALL 2-body t matrices as "pole" + "background":

$$T_{ij} = T_{ij}^P + K_{ij}$$

- which then gives

$$\mathcal{T}^+ + \mathcal{T}^\times = \mathcal{T}_P^+ + \mathcal{T}_P^\times + \mathcal{T}_{PK}^\times + \mathcal{K}^+ + \mathcal{K}^\times$$

$$\equiv \sum_{j=1}^{3} \mathcal{F}_j \mathcal{D}_j \bar{\mathcal{F}}_j + \mathcal{K}$$

- Resulting in the 4-body tetraquark equations

$$\boxed{\phi_i = \sum_{j=1}^{3} \bar{\mathcal{F}}_i \mathcal{R}(1 - \mathcal{K}\mathcal{R})^{-1} \mathcal{F}_j \mathcal{D}_j \phi_j}$$

where $$\phi_j = \bar{\mathcal{F}}_j \mathcal{R} \tilde{\Phi}$$

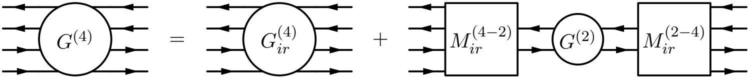
Incorporating $q\bar{q}$ annihilation:

K. & B. Phys. Rev. **D** 90, 04502 (2014); Phys. Rev. **D** 106, 054024 (2022)
B. & K. Few Body Syst. **65** 2, 59 (2024)

- In Quantum Field Theory (QFT) the number of particles is not conserved.
- But HOW to include $2q2\bar{q} \leftrightarrow q\bar{q}$ transitions into a pure 4-body theory?
- We provided a correct but lengthy answer in 2014 involving a disconnected 2-body ($q\bar{q}$) kernel - it was ignored!
- More recently we found a short and simple answer to this question

The way to incorporate $q\bar{q}$ annihilation

- **Step 1:** Express the full $2q2\bar{q}$ Green function $G^{(4)}$ in terms of its $q\bar{q}$-irreducible and $q\bar{q}$-reducible parts



where $G^{(2)}$ is the full $q\bar{q}$ Green function specified by a two-body kernel $K^{(2)}$ as $G^{(2)} = G_0^{(2)} + G_0^{(2)} K^{(2)} G^{(2)}$

- Note that the same tetraquark pole must be present in both $G^{(4)}$ and $G^{(2)}$: as $P^2 \to M^2$

$$G^{(4)} \to i\frac{\Psi\bar{\Psi}}{P^2 - M^2}, \qquad G^{(2)} \to i\frac{G_0^{(2)}\Gamma^*\bar{\Gamma}^* G_0^{(2)}}{P^2 - M^2},$$

- But all poles in $G^{(2)}$ will appear in $G^{(4)}$, suggesting that a tetraquark be *defined* in QFT as a pole in $G^{(4)}_{ir}$!

The way to incorporate $q\bar{q}$ annihilation

- **Step 2:** Express the EXACT two-body ($q\bar{q}$) kernel $K^{(2)}$ as

$$K^{(2)} = \Delta + \bar{N} G_{ir}^{(4)} N$$

(i) $\bar{N}$ and N are $2 \leftarrow 4$ and $2 \rightarrow 4$ $q\bar{q}$-irreducible amplitudes

(ii) Δ defined by ALL contributions missing from last term

- Assume that $G_{ir}^{(4)}$ has a "tetraquark" pole at $P^2 = M_0^2$ so that

$$G_{ir}^{(4)} \rightarrow i \frac{\Psi_0 \bar{\Psi}_0}{P^2 - M_0^2} + B$$

- Then $G^{(2)}$ has a "tetraquark" pole at $P^2 = M^2$ where

$$M^2 = M_0^2 + i\bar{\Psi}_0 N \left[G_0^{(2)-1} - \Delta - \bar{N} B N \right]^{-1} \bar{N} \Psi_0$$

Tetraquark equations of QFT

- Direct use of
  (i) Exact two-body bound state equation: $\Gamma^* = K^{(2)} G_0^{(2)} \Gamma^*$
  (ii) Exact two-body kernel: $K^{(2)} = \Delta + \bar{N} G_{ir}^{(4)} N$
  (ii) Four-body Green function: $G_{ir}^{(4)} = G_0^{(4)} \left(1 - K^{(4)} G_0^{(4)}\right)^{-1}$

- Results in the Exact Tetraquark Equations of QFT ($q\bar{q}$ annihilation included):

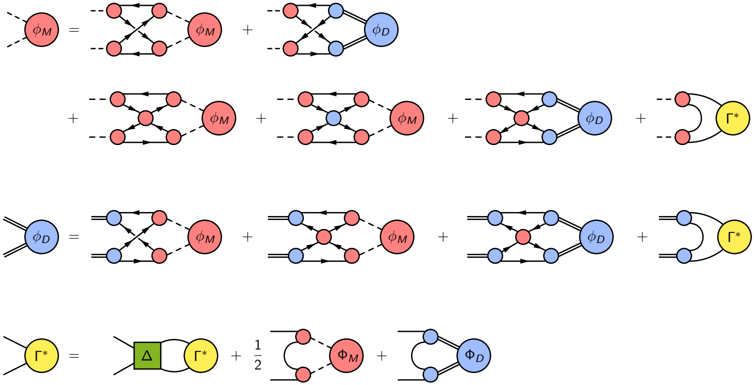
$$\phi = K^{(4)} G_0^{(4)} \phi + N G_0^{(2)} \Gamma^*$$

$$\Gamma^* = \Delta G_0^{(2)} \Gamma^* + \bar{N} G_0^{(0)} \phi$$

Exact unified tetraquark equations

B. Blankleider and A. N. Kvinikhidze, Few-Body Syst. **65**, 59 (2024)

- Graphic representation of the unified tetraquark equations with $q\bar{q}$ annihilation



Summary and Conclusion

- Tetraquarks ($2q2\bar{q}$ bound states) need to be described in QFT, and that means taking into account $q\bar{q}$ annihilation

- We have found a remarkable method for describing tetraquarks *exactly*, by expressing the $q\bar{q}$ kernel as

$$K^{(2)} = \Delta + \bar{N} G_{ir}^{(4)} N$$

(i) $G_{ir}^{(4)}$ is the $2q2\bar{q}$ Green function with $q\bar{q}$ annihilation "switched off"

(ii) Δ is defined as consisting of all contributions not included in $\bar{N} G_{ir}^{(4)} N$

Summary and Conclusion

- We have developed 4-body equations for $G_{ir}^{(4)}$ that:

  (i) Extend the $MM - D\bar{D}$ coupled channels model of Heupel *et al.* to include $2q$ multiple-scattering while the another $2q$ pair is "spectating"

    - The resulting equations, truncated to just one such rescattering, provide a unified description of 2 seemingly unrelated tetraquark models ("Giessen" and "Moscow")

  (ii) Extend the above 4-body model for $G_{ir}^{(4)}$ to include *all* pairwise interactions with *all* pole contributions (corresponding to meson and diquark states) included nonperturbatively

- Our tetraquark equations can provide the rigorous theoretical foundation needed for future calculations.