One Born-Oppenheimer Effective Theory to rule all Exotics

XVIth Quark Confinement and the Hadron Spectrum Conference
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[arXiv 2408.04719](https://arxiv.org/abs/2408.04719)

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DFG Deutsche Forschungsgemeinschaft

QCD spectrum: Hadrons

- QCD spectrum: only **color singlet** states exist in nature that we term "Hadrons".
- Hadrons: **bound states** of quarks and gluons bound by strong interactions.

Conventional Hadrons until 2003

Gell-Mann & Zweig

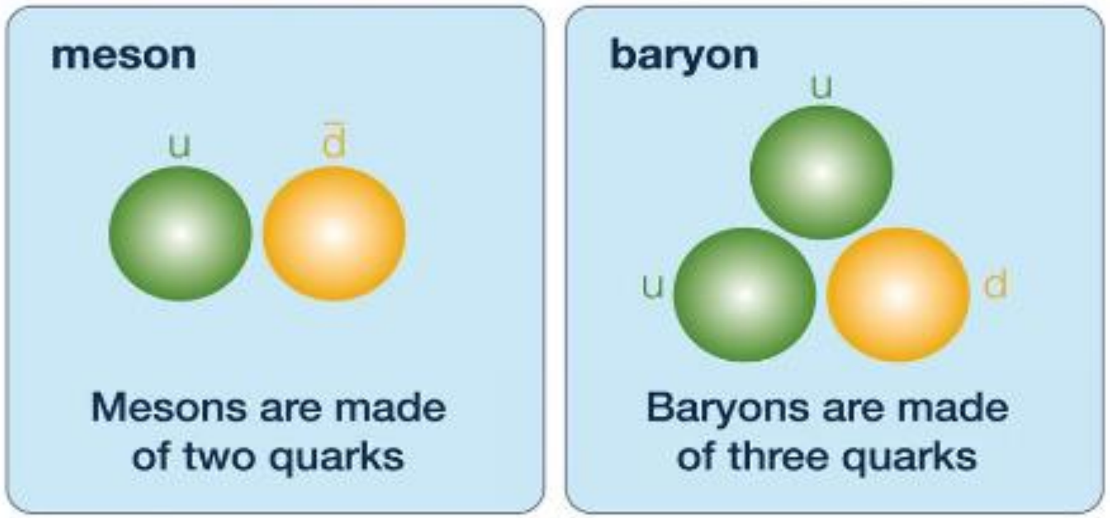


Fig taken from https://www.eurekalert.org/multimedia/911829

New species discovered after 2003 !!!!

New species

XYZ mesons

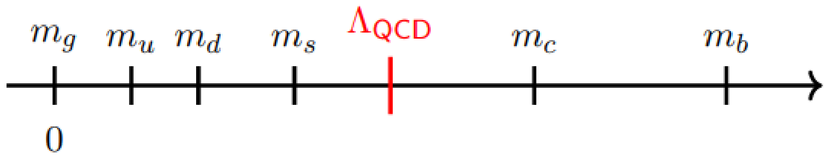
(hybrids, tetraquarks, molecules, pentaquarks, ….)

X(3872): First exotic state discovered in 2003 by Belle.

Phys. Rev. Lett. 91, 262001 (2003)

Exotic Hadron

$m_c \approx 1.5\,\text{GeV} \quad m_b \approx 5\,\text{GeV}$



- Exotics : more complex structures

- Exotic states with at-least 2-heavy quarks : **XYZ mesons**

✓ States that don't fit traditional $Q\bar{Q}$ spectrum.

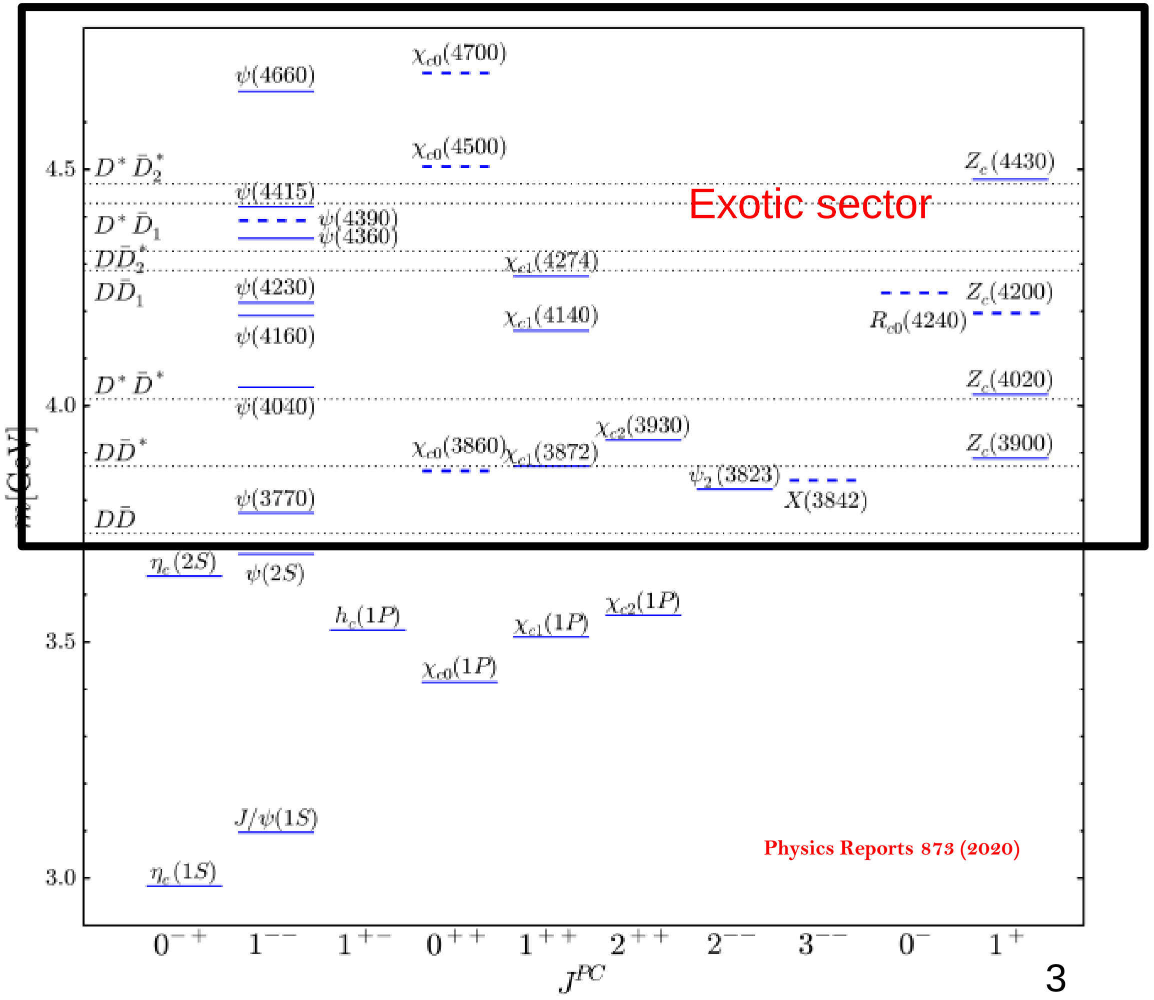
✓ Exotic quantum numbers:

- $J^{PC} = 0^{--}, 0^{+-}, 1^{-+}, 2^{+-}$ etc. are exotic

- Charged: Ex. Z_c and Z_b states: minimal 4-quarks:

$Z_c(4430)^{\pm}$ $Z_b(10650)^{\pm}$

For review see Brambilla et al. Phys. Reports. 873 (2020)



- Dozens of XYZ mesons discovered since 2003.

Exotic Hadron

- Multiple Models for XYZ Exotics:

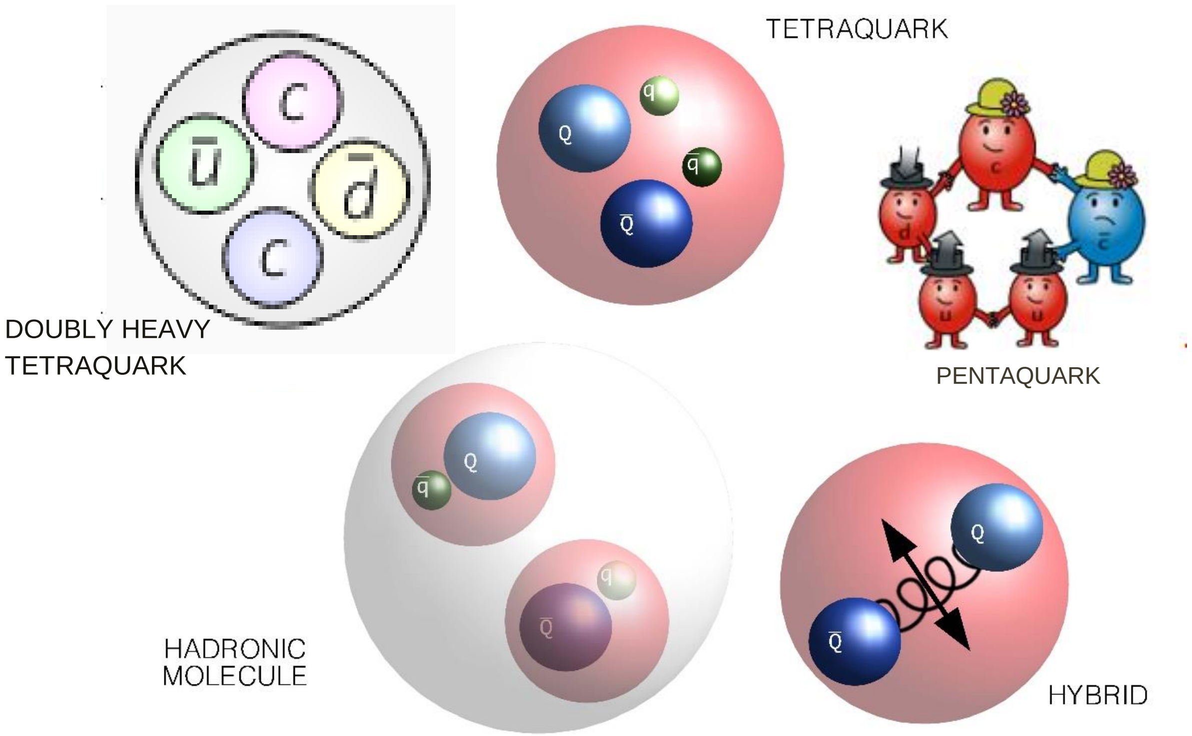
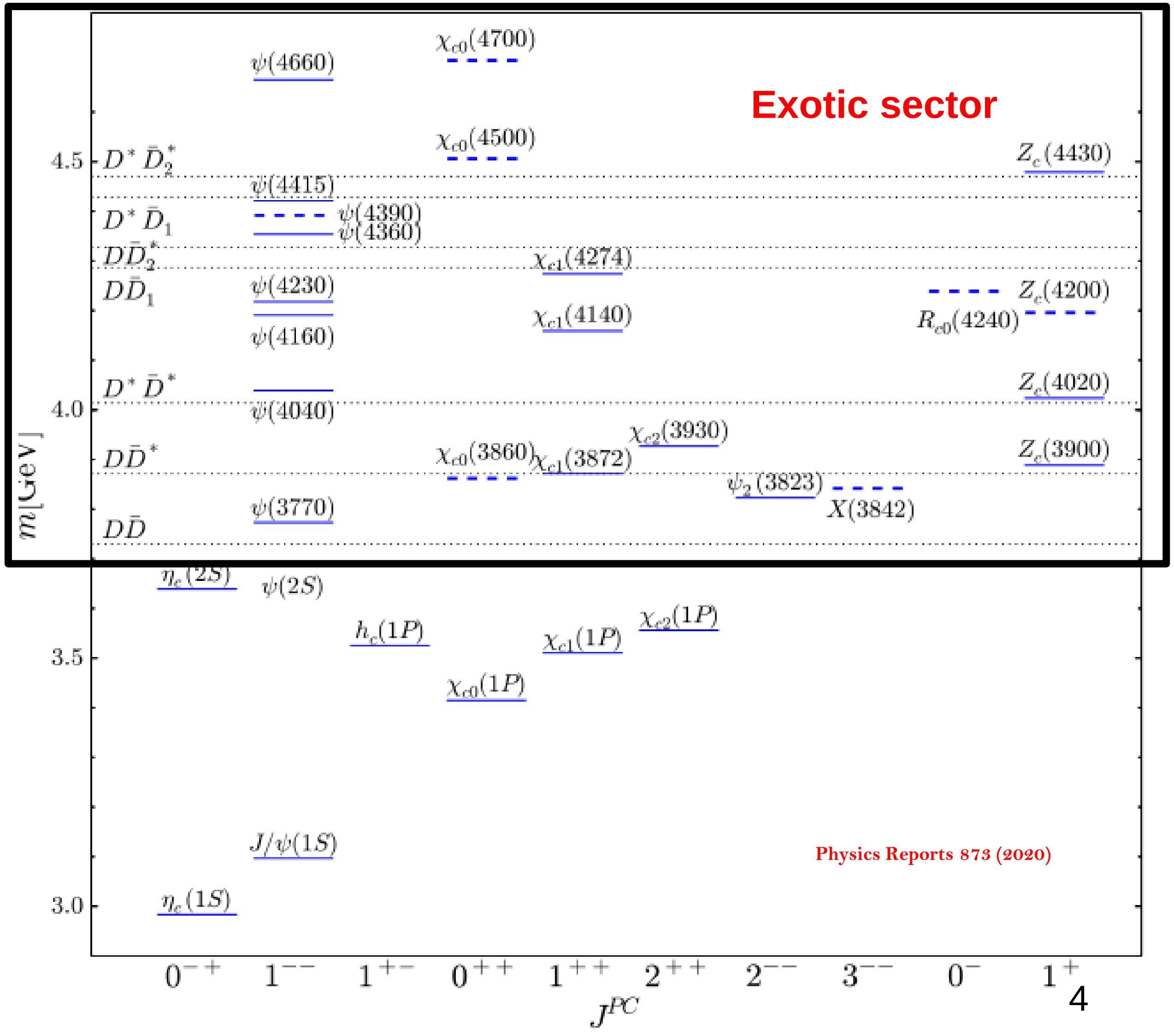


Figure from https://www.fz-juelich.de/en/ias/ias-4/research/exotic-hadrons/exotics_pad.jpg

Figure from Nat Rev Phys 1, 480-494 (2019) Figure from Montesinos Meson 2023 talk



- Individual success in describing some XYZ hadrons. No success in revealing any general pattern.

Exotic Hadron

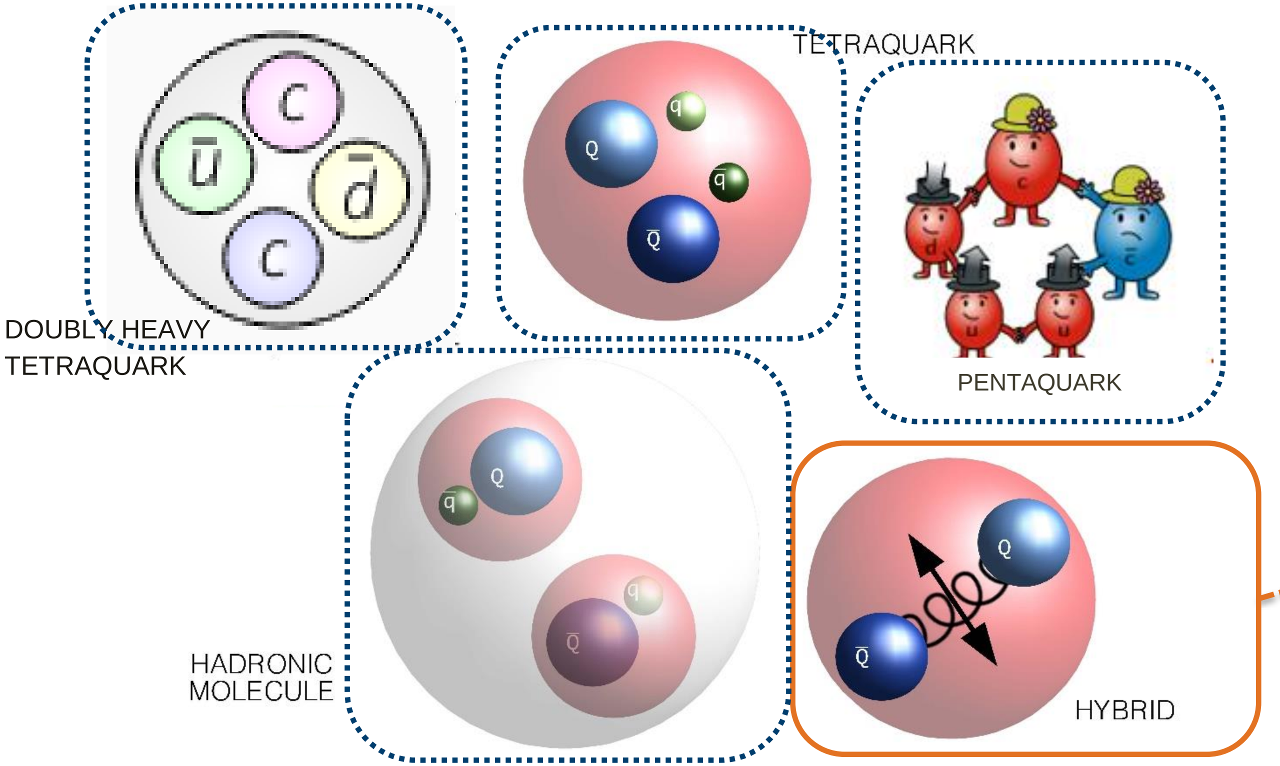


Figure from https://www.fz-juelich.de/en/ias/ias-4/research/exotic-hadrons/exotics_pad.jpg

Figure from Nat Rev Phys 1, 480-494 (2019)

Figure from Montesinos Meson 2023 talk

QUESTION:

Coherent comprehensive framework based on QCD for all X Y Z hadrons ???

Hybrids ($Q\bar{Q}g$): Isospin scalar exotic state.

Use EFT + Lattice

Multiple lattice results on static energies

Brambilla, Lai, AM, Vairo Phys. Rev. D 107, 054034 (2023)

Braaten, Langmack, Smith Phys. Rev. D. 90, 014044 (2014)

Brambilla, Lai, Segovia, Castellà, Phys. Rev. D. 101, 054040 (2020)

Soto, Valls, Phys. Rev. D 108, 014025 (2023)

Brambilla, Krein, Castellà, Vairo Phys. Rev. D. 97, 016016 (2018)

Berwein, Brambilla, Castellà, Vairo Phys. Rev. D. 92, 114019 (2015)

Oncala, Soto, Phys. Rev. D. 96, 014004 (2017)

Brambilla, Lai, Segovia, Castellà, Vairo Phys. Rev. D. 99, 014017 (2019)

Pineda, Castellà, Phys. Rev. D. 100, 054021 (2019)

Non-zero isospin states. Use EFT + Lattice.

However, some lattice results on the static energies are available

Berwein, Brambilla, AM, Vairo arXiv 2408.04719

Soto & Castella Phys. Rev. D. 102, (2020), 014012

Born-Oppenheimer EFT

BOEFT: Exotic Hadron

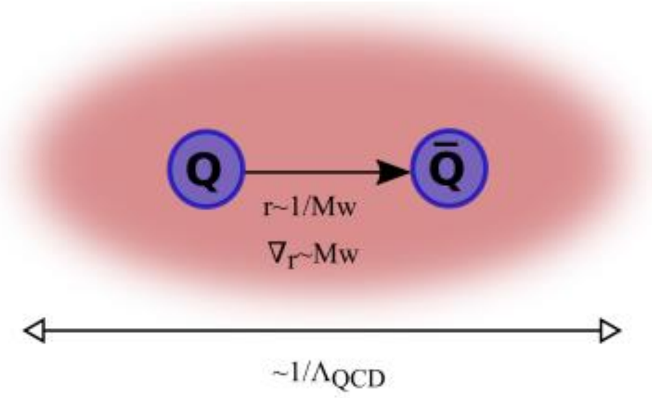
- Exotic hadron ($Q\bar{Q}X, QQX, \ldots\ldots$), X: any combination of light quark and gluons for color singlet.

- Hierarchy of scales in hybrids:

$$m \gg mv \gtrsim \Lambda_{QCD} \gg mv^2$$

  - Mass of heavy quark: $\boldsymbol{m}$
  - Energy scale for light d.o.f: Λ_{QCD}
  - Relative separation between heavy quarks: $r \sim 1/mv$
  - Hybrids are extended objects: $< r > \gtrsim 0.7\ fm$
  - Heavy Quark dynamics scale: mv^2

Extended objects:
$< r > \gtrsim 0.7\ fm$



- Time-scale for dynamics of $Q\bar{Q}$: $\sim \dfrac{1}{mv^2} \gg \dfrac{1}{\Lambda_{QCD}}$

Born-Oppenheimer (BO) Approximation

Juge, Kutti, Morningstar, Phys. Rev. Lett. 90, 161601 (2003)

Braaten, Langmack, Smith Phys. Rev. D. 90, 014044 (2014)

R. Oncala, J. Soto, Phys. Rev. D96 (2017)

Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92 (2015)

Brambilla, Krein, Castellà , Vairo Phys. Rev. D. 97, (2018)

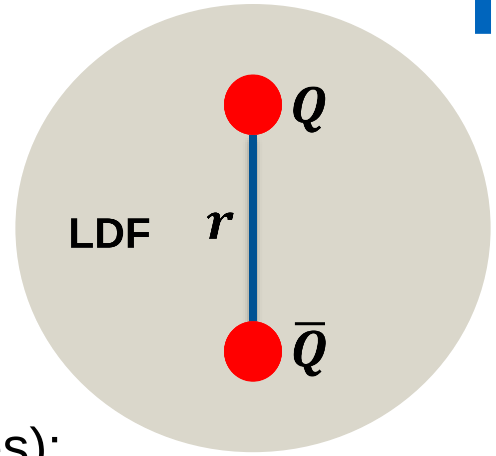
Castellà , Soto Phys. Rev. D. 102 (2020)

Berwein, Brambilla, AM, Vairo, arXiv 2408.04719

BOEFT: Quantum #'s

- **Static limit (m→ ∞):** heavy quarks are fixed in position.

  Cylindrical symmetry $(D_{\infty h})$ due to preferred quark-antiquark axis



- **BO-quantum number Λ_η^σ (r ≠ 0)**: $D_{\infty h}$ representations (diatomic molecules):

  ✓ Absolute value of component of angular momentum of light d.o.f

  $$|\boldsymbol{r}\cdot\boldsymbol{K}_{\text{light}}| \equiv \boldsymbol{\Lambda} = \mathbf{0}, \mathbf{1}, \mathbf{2}, \dots\dots.. (\textbf{or } \boldsymbol{\Sigma}, \boldsymbol{\Pi}, \boldsymbol{\Delta}, \boldsymbol{\Phi}, \dots...)$$

  ✓ Product of charge conjugation and parity ($\boldsymbol{CP}$):

  $$\boldsymbol{\eta} = \mathbf{+1}\ (\mathbf{g}), \mathbf{-1}\ (\boldsymbol{u})$$

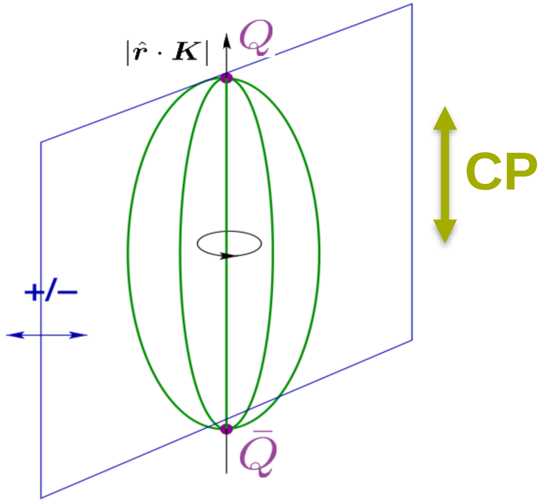
  ✓ $\boldsymbol{\sigma}$: Eigenvalue of reflection about a plane containing static sources.

  $$\sigma = P\,(-1)^{K_{\text{light}}} = \pm 1$$

  Born, Oppenheimer, Annalen der Physik 389 (1927)

  Landau, Lifshitz & Pitaevskii, QM book

$$\Lambda_\eta^\sigma$$



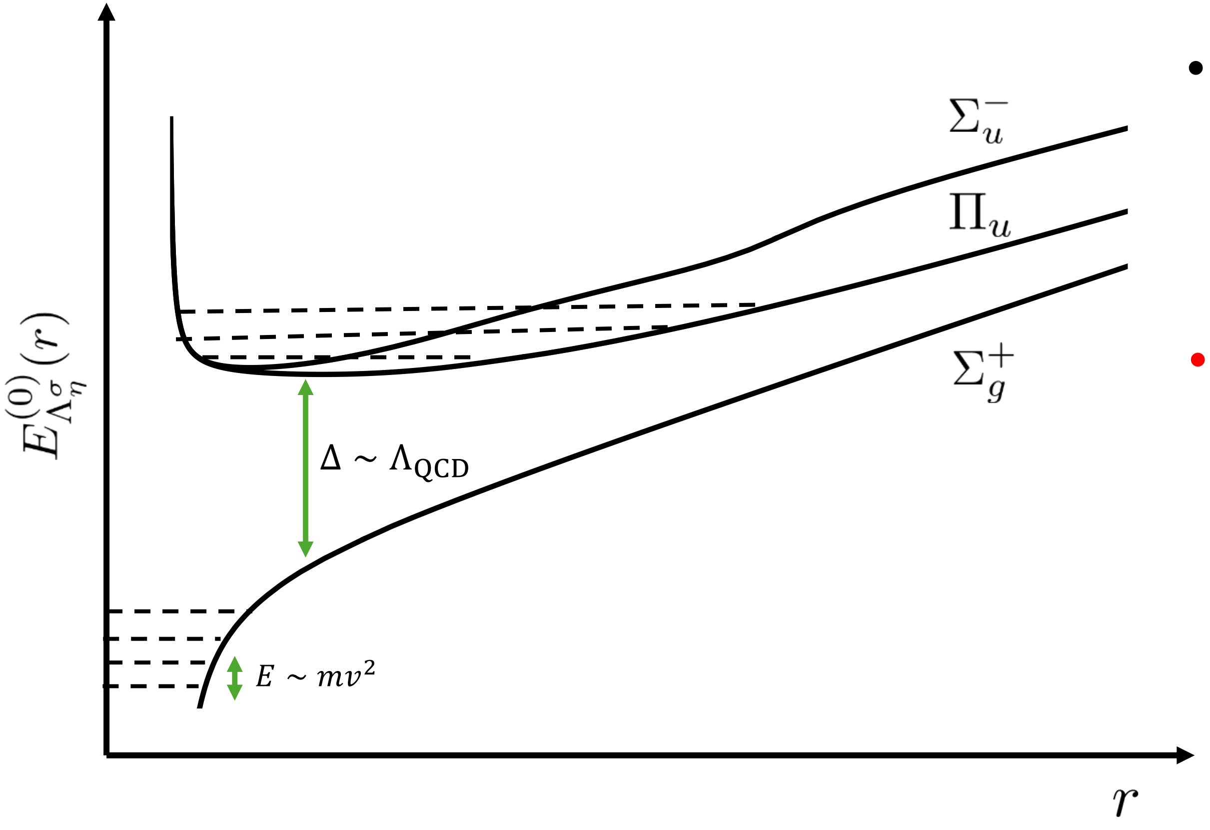
- **Spherical symmetry $O(3)\times C$ (r → 0)**: Labelled by LDF quantum #'s: $\kappa = \{K^{PC}, f\}$

BOEFT

- BOEFT Lagrangian: $$L_{\mathrm{BOEFT}} = L_{Q\bar{Q}} + L_{Q\bar{Q}g} + L_{Q\bar{Q}q\bar{q}} + L_{\mathrm{mixing}} + \cdots$$

Berwein, Brambilla, AM, Vairo, arXiv 2408.04719 Castellà , Soto Phys. Rev. D. 102, 014012 (2020)

Brambilla, Krein, Castellà , Vairo Phys. Rev. D. 97, (2018)



- Gap of order Λ_{QCD} allows us to focus individually on low-lying states corresponding to quarkonium, hybrid, tetraquark etc.

- L_{mixing}: Mixing between different states with similar masses and same quantum-numbers.

Ex: Hybrid-quarkonium mixing, Tetraquark-hybrid & Tetraquark-quarkonium mixing etc.

R. Oncala, J. Soto, Phys. Rev. D96 014004 (2017)

BOEFT

Berwein, Brambilla, AM, Vairo,
arXiv 2408.04719

- BOEFT Lagrangian:

$$L_{\mathrm{BOEFT}} = \int d^3\boldsymbol{R} \int d^3\boldsymbol{r} \sum_{\kappa\lambda\lambda'} \mathrm{Tr}\Big\{ \Psi^{\dagger}_{\kappa\lambda}(\mathbf{r},\,\mathbf{R},\,t) \Big[i\partial_t\, \delta_{\lambda\lambda'} - V_{\kappa\lambda\lambda'}(r) + P^{i\dagger}_{\kappa\lambda}(\theta,\phi) \frac{\boldsymbol{\nabla}^2_r}{m_Q} P^{i}_{\kappa\lambda'}(\theta,\phi) \Big] \Psi_{\kappa\lambda'}(\mathbf{r},\,\mathbf{R},\,t) \Big\}$$

LDF-quantum #: $\kappa = \{K^{PC}, f\}$ BO-quantum #: Λ^{σ}_{η} $\lambda = \pm\Lambda$

Projection vectors : $P^{i}_{K\lambda}(\theta,\varphi) = D^{\lambda *}_{Ki}(0,\theta,\varphi)$

- **BO potentials: Potential between Q & $\bar{Q}$** due to LDF (light quarks, gluons).

Born-Oppenheimer (BO) potential:

$$V_{\kappa\lambda\lambda'}(r) = \boxed{E^{(0)}_{\kappa,|\lambda|}(r)}\,\delta_{\lambda\lambda'} + \boxed{\frac{V^{(1)}_{\kappa\lambda\lambda'}(r)}{m_Q}} + \dots\,,$$

Static Energy Spin-dependent potentials

Brambilla, Lai, Segovia, Castellà, Phys. Rev. D. 101, (2020)

Brambilla, Lai, Segovia, Castellà, Vairo Phys. Rev. D. 99, (2019)

Soto, Valls, Phys. Rev. D 108 (2023)

Castellà , Soto Phys. Rev. D. 102, (2020)

BOEFT

Berwein, Brambilla, AM, Vairo,
arXiv 2408.04719

Wave-function for Exotic State:

$$|X_N\rangle = \sum_\lambda \int d^3r |\boldsymbol{r}\rangle \otimes |k,\lambda\rangle \phi^{(N)}_{\kappa\lambda}(\boldsymbol{r})$$

$|\boldsymbol{r}\rangle$: Heavy quark pair state separated by position r

$|k,\lambda\rangle$: Light quark or gluon state: Parametrically depends on r

Total orbital momentum for Exotic State:

$$\boldsymbol{L} = \boldsymbol{L}_Q + \boldsymbol{K}$$

$\boldsymbol{K}$: angular-momentum of light d.o.f

$\boldsymbol{L_Q}$: orbital-angular momentum of QQ or $Q\bar{Q}$ pair.

Angular wave-function:

$$|l,m;k,\lambda\rangle = \int \frac{d\Omega}{\sqrt{2\pi}} |\theta,\phi\rangle\, |k,\lambda\rangle\, D^{\lambda}_{lm}(\psi,\theta,\varphi)$$

BOEFT

Berwein, Brambilla, AM, Vairo,
arXiv 2408.04719

- Adiabatic Radial Schrödinger equation:

  Mixing different static energies with same LDF-quantum #: $\kappa = \{K^{PC}, f\}$

$$\sum_{\lambda}\left[-\frac{1}{m_Q r^2}\partial_r\, r^2\partial_r + \frac{1}{m_Q r^2} M_{\lambda'\lambda} + E^{(0)}_{\kappa,|\lambda|}(r)\delta_{\lambda\lambda'}\right]\psi^{(N)}_{\kappa\lambda}(r) = \mathcal{E}_N\, \psi^{(N)}_{\kappa\lambda'}(r)$$

Mixing term from angular momentum piece:

Coupling static energies with different BO-quantum numbers Λ^{σ}_{η}

- General expression of $M_{\lambda'\lambda}$ (matrix in $\lambda' - \lambda$ basis) :

$$\lambda, \lambda' = \pm\Lambda$$

$$M_{\lambda'\lambda} = \langle l, m; k, \lambda'|\boldsymbol{L}_Q^2|l, m; k, \lambda\rangle$$
$$= \left(l(l+1) - 2\lambda^2 + k(k+1)\right)\delta^{\lambda'\lambda} - \sqrt{k(k+1) - \lambda(\lambda+1)}\sqrt{l(l+1) - \lambda(\lambda+1)}\,\delta^{\lambda'\lambda+1}$$
$$- \sqrt{k(k+1) - \lambda(\lambda-1)}\sqrt{l(l+1) - \lambda(\lambda-1)}\,\delta^{\lambda'\lambda-1}$$

BOEFT

Berwein, Brambilla, AM, Vairo,
arXiv 2408.04719

Coupled Equations for lowest Hybrids ($Q\bar{Q}g$) and Tetraquarks ($QQ\bar{q}\bar{q}$ or $Q\bar{Q}q\bar{q}$):

LDF quantum # K=1

$$\left[-\frac{1}{m_Q r^2}\partial_r r^2 \partial_r + \frac{1}{m_Q r^2}\begin{pmatrix} l(l+1)+2 & -2\sqrt{l(l+1)} \\ -2\sqrt{l(l+1)} & l(l+1)\end{pmatrix} + \begin{pmatrix} E_\Sigma & 0 \\ 0 & E_\Pi \end{pmatrix}\right]\begin{pmatrix}\psi^{(N)}_{\Sigma,\sigma_p} \\ \psi^{(N)}_{\Pi,\sigma_P}\end{pmatrix} = \mathcal{E}_N \begin{pmatrix}\psi^{(N)}_{\Sigma,\sigma_P} \\ \psi^{(N)}_{\Pi,\sigma_P}\end{pmatrix}$$

$$\left[-\frac{1}{m_Q r^2}\,\partial_r\, r^2\, \partial_r + \frac{l(l+1)}{m_Q r^2} + E_\Pi\right]\psi^{(N)}_{\Pi,-\sigma_P} = \mathcal{E}_N\,\psi^{(N)}_{\Pi,-\sigma_P}$$

Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92 (2015)

LDF quantum # K=2

$$\left[-\frac{1}{m_Q r^2}\partial_r r^2 \partial_r + \frac{1}{m_Q r^2}\begin{pmatrix} l(l+1)+6 & -2\sqrt{3l(l+1)} & 0 \\ -2\sqrt{3l(l+1)} & l(l+1)+4 & -2\sqrt{(l-1)(l+2)} \\ 0 & -2\sqrt{(l-1)(l+2)} & (l-1)(l+2)\end{pmatrix} + \begin{pmatrix} E_\Sigma & 0 & 0 \\ 0 & E_\Pi & 0 \\ 0 & 0 & E_\Delta \end{pmatrix}\right]\begin{pmatrix}\psi^{(N)}_{\Sigma,\sigma_p} \\ \psi^{(N)}_{\Pi,\sigma_P} \\ \psi^{(N)}_{\Delta,\sigma_p}\end{pmatrix} = \mathcal{E}_N \begin{pmatrix}\psi^{(N)}_{\Sigma,\sigma_P} \\ \psi^{(N)}_{\Pi,\sigma_P} \\ \psi^{(N)}_{\Delta,\sigma_p}\end{pmatrix}$$

$$\left[-\frac{1}{m_Q r^2}\partial_r r^2 \partial_r + \frac{1}{m_Q r^2}\begin{pmatrix} l(l+1)+4 & -2\sqrt{(l-1)(l+2)} \\ -2\sqrt{(l-1)(l+2)} & (l-1)(l+2)\end{pmatrix} + \begin{pmatrix} E_\Pi & 0 \\ 0 & E_\Delta \end{pmatrix}\right]\begin{pmatrix}\psi^{(N)}_{\Pi,-\sigma_P} \\ \psi^{(N)}_{\Delta,-\sigma_p}\end{pmatrix} = \mathcal{E}_N \begin{pmatrix}\psi^{(N)}_{\Pi,-\sigma_P} \\ \psi^{(N)}_{\Delta,-\sigma_p}\end{pmatrix}$$

BOEFT

Berwein, Brambilla, AM, Vairo,
arXiv 2408.04719

Coupled Equations for Doubly Heavy Baryons (QQq) and Pentaquarks (QQq̄qq or QQ̄qqq):

LDF quantum # K=1/2

$$\left[-\frac{1}{m_Q r^2}\partial_r\, r^2\,\partial_r+\frac{(l-1/2)(l+1/2)}{m_Q r^2}+E_{K_\eta}\right]\psi^{(N)}_{K_\eta,\sigma_P}=\mathcal{E}_N\,\psi^{(N)}_{K_\eta,\sigma_P}$$

LDF quantum # K=3/2

$$\left[-\frac{1}{m_Q r^2}\partial_r r^2\partial_r+\frac{1}{m_Q r^2}\begin{pmatrix} l(l-1)-\frac{9}{4} & -\sqrt{3l(l+1)-\frac{9}{4}} \\ -\sqrt{3l(l+1)-\frac{9}{4}} & l(l+1)-\frac{3}{4}\end{pmatrix}+\begin{pmatrix}E_{(1/2)_u} & 0 \\ 0 & E_{(3/2)_u}\end{pmatrix}\right]\begin{pmatrix}\psi^{(N)}_{1/2,\sigma_P} \\ \psi^{(N)}_{3/2,\sigma_P}\end{pmatrix}=\mathcal{E}_n\begin{pmatrix}\psi^{(N)}_{1/2,\sigma_P} \\ \psi^{(N)}_{3/2,\sigma_P}\end{pmatrix}$$

$$\left[-\frac{1}{m_Q r^2}\partial_r r^2\partial_r+\frac{1}{m_Q r^2}\begin{pmatrix} l(l+3)+\frac{17}{4} & -\sqrt{3l(l+1)-\frac{9}{4}} \\ -\sqrt{3l(l+1)-\frac{9}{4}} & l(l+1)-\frac{3}{4}\end{pmatrix}+\begin{pmatrix}E_{(1/2)_u} & 0 \\ 0 & E_{(3/2)_u}(r)\end{pmatrix}\right]\begin{pmatrix}\psi^{(N)}_{1/2,-\sigma_P} \\ \psi^{(N)}_{3/2,-\sigma_P}\end{pmatrix}=\mathcal{E}_n\begin{pmatrix}\psi^{(N)}_{1/2,-\sigma_P} \\ \psi^{(N)}_{3/2,-\sigma_P}\end{pmatrix}$$

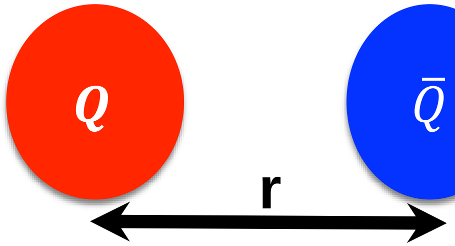
Castellà , Soto Phys. Rev. D. 104, 074027 (2021)
Castellà , Soto Phys. Rev. D. 102, 014013 (2020)

BO-Potentials (Static energy)

Exotic Hadron

Berwein, Brambilla, AM, Vairo,
arXiv 2408.04719

$Q\bar{Q}X$



color: $3 \otimes \bar{3} = 1 \oplus 8$

X_8= gluon → **Hybrid**

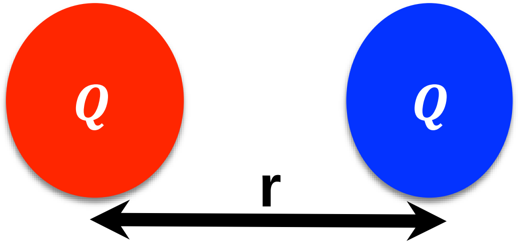
X_8 = $q\bar{q}$ → **Tetraquark / Molecule**

X_8 = qqq → **Pentaquark / Molecule** and so on

Total angular momentum of **$Q\bar{Q}X$** or **QQX** :

$$\mathbf{J} = \mathbf{L}_{Q\bar{Q}} + \mathbf{K} + \mathbf{S}_{Q\bar{Q}}$$

QQX



color: $3 \otimes \bar{3} = \bar{3} \oplus 6$

X= q → **Double heavy baryon**

X= $\bar{q}\bar{q}$ → **Tetraquark**

X= $q\bar{q}q$ → **Pentaquark** and so on

BOEFT potentials $E^{(0)}_{\kappa,|\lambda|}(\mathbf{r})$: LDF (light quarks, gluons) static energies.

Potential between 2 heavy quarks

BOEFT can address all these states with inputs from Lattice QCD on $E^{(0)}_{\kappa,|\lambda|}$

BOEFT: Lattice Operators

Berwein, Brambilla, AM, Vairo,
arXiv 2408.04719

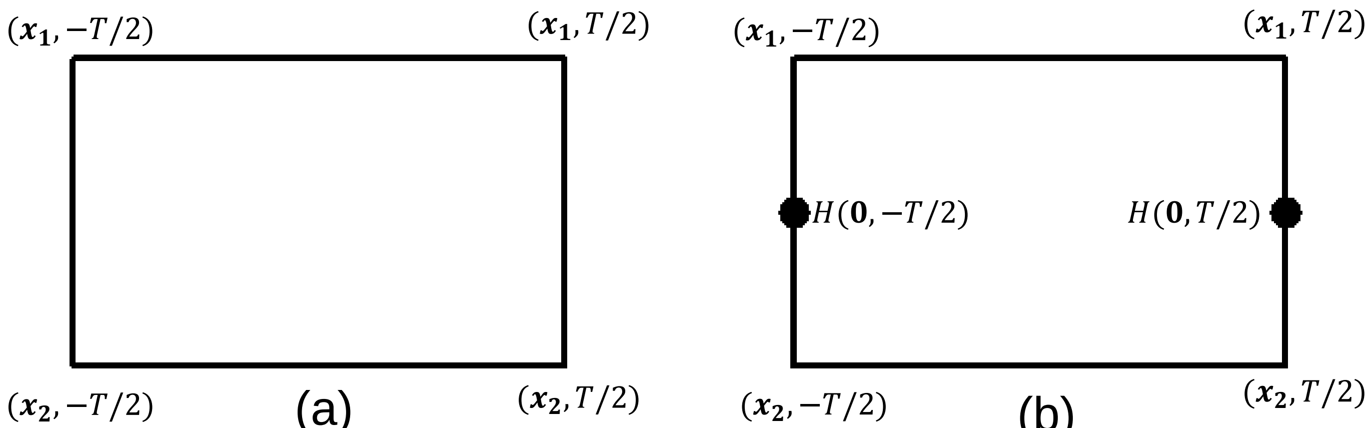
NRQCD operator (gauge invariant) for exotic hadron $Q\bar{Q}X$ or QQX :

$$\mathcal{O}_{\kappa,\lambda}(t,\boldsymbol{r}) = \chi^\dagger(t,\boldsymbol{r}/2)\,\phi(t;\boldsymbol{r}/2,\boldsymbol{0})\,P^{\alpha\dagger}_{\kappa,\lambda}H^{\alpha}_{\kappa}(t,\boldsymbol{0})\,\phi(t;\boldsymbol{0},-\boldsymbol{r}/2)\,\psi(t,-\boldsymbol{r}/2)$$

H^{α}_{κ} : LDF (gluon or light-quarks) operator characterizing X based on quantum # κ (isospin, color etc..)

$P^{\alpha}_{\kappa,\lambda}$: Projection vectors for projecting onto cyclindrical symmetry $D_{\infty h}$ representations.

$$E^{(0)}_{\kappa,|\lambda|}(r) = \lim_{T\to\infty}\frac{i}{T}\log\left[\langle\text{vac}|\mathcal{O}_{\kappa,\lambda}(T/2,\,\boldsymbol{r},\,\boldsymbol{R})\,\mathcal{O}^{\dagger}_{\kappa,\lambda}(-T/2,\,\boldsymbol{r},\,\boldsymbol{R})|\text{vac}\rangle\right]$$



(a) Quarkonium

(b) Wilson loop for exotics

BOEFT: Lattice Operators

Berwein, Brambilla, AM, Vairo,
arXiv 2408.04719

Hybrids $Q\bar{Q}g$

Λ_η^σ	k^{PC}	Representation	Operator Examples $H_{8,\kappa}^{\alpha,a}T^a$	Projectors $P_{\kappa\lambda}^\alpha$
Σ_g^+	0^{++}	scalar	$\mathbb{1}^a$	1
Σ_u^+	0^{+-}	scalar	$\boldsymbol{D}\cdot\boldsymbol{E}$	1
Σ_g^-	0^{--}	pseudoscalar	$[\boldsymbol{E}\cdot,\boldsymbol{B}]$	1
Σ_u^-	0^{-+}	pseudoscalar	$\{\boldsymbol{E}\cdot,\boldsymbol{B}\}$	1
$\{\Sigma_g^+,\Pi_g\}$	1^{--}	vector	E^i	$\{\hat{r}^i,\hat{r}_\pm^i\}$
$\{\Sigma_u^+,\Pi_u\}$	1^{-+}	vector	$([\boldsymbol{E}\times,\boldsymbol{B}])^i$	$\{\hat{r}^i,\hat{r}_\pm^i\}$
$\{\Sigma_g^-,\Pi_g\}$	1^{++}	pseudovector	$(\boldsymbol{D}\times[\boldsymbol{E}\times,\boldsymbol{B}])^i$	$\{\hat{r}^i,\hat{r}_\pm^i\}$
$\{\Sigma_u^-,\Pi_u\}$	1^{+-}	pseudovector	B^i	$\{\hat{r}^i,\hat{r}_\pm^i\}$

Quarkonium tetraquarks $Q\bar{Q}q\bar{q}$ (I=0)

Λ_η^σ	k^{PC}	Representation	Operator Examples $H_{8,\kappa}^{\alpha,a}\,(I=0)$	Projectors $P_{\kappa\lambda}^\alpha$
Σ_g^+	0^{++}	scalar	$\bar{q}T^aq$	1
Σ_u^-	0^{-+}	pseudoscalar	$\bar{q}\gamma^5T^aq$	1
$\{\Sigma_g^+,\Pi_g\}$	1^{--}	vector	$\bar{q}\,\gamma^i\,T^aq$	$\{\hat{r}^i,\hat{r}_\pm^i\}$
$\{\Sigma_g^-,\Pi_g\}$	1^{++}	pseudovector	$\bar{q}\,\gamma^i\,\gamma^5\,T^aq$	$\{\hat{r}^i,\hat{r}_\pm^i\}$
$\{\Sigma_u^-,\Pi_u\}$	1^{+-}	pseudovector	$\bar{q}\,(\boldsymbol{\gamma}\times\boldsymbol{\gamma})^i\,\gamma^5\,T^aq$	$\{\hat{r}^i,\hat{r}_\pm^i\}$

I=1 operator: Insert $\boldsymbol{e}_{I_3}\cdot\boldsymbol{\tau}$ between light quarks

Quarkonium pentaquarks $Q\bar{Q}qqq$

$$H^{\alpha,a}_{8,I_3=\pm1/2,(1/2)^+}(t,\boldsymbol{x}) = \Big[\left(\delta_{\alpha\beta_1}\sigma^2_{\beta_2\beta_3}+\delta_{\alpha\beta_2}\sigma^2_{\beta_1\beta_3}+\delta_{\alpha\beta_3}\sigma^2_{\beta_1\beta_2}\right)\left(\delta_{I_3f_1}\tau^2_{f_2f_3}+\delta_{I_3f_2}\tau^2_{f_1f_3}+\delta_{I_3f_3}\tau^2_{f_1f_2}\right)(T_2)^a_{l_1,l_2,l_3}$$
$$+\left(\delta_{\alpha\beta_1}\sigma^2_{\beta_2\beta_3}+\delta_{\alpha\beta_2}\sigma^2_{\beta_3\beta_1}+\delta_{\alpha\beta_3}\sigma^2_{\beta_2\beta_1}\right)\left(\delta_{I_3f_1}\tau^2_{f_2f_3}+\delta_{I_3f_2}\tau^2_{f_3f_1}+\delta_{I_3f_3}\tau^2_{f_2f_1}\right)(T_3)^a_{l_1,l_2,l_3}$$
$$+\left(\delta_{\alpha\beta_1}\sigma^2_{\beta_3\beta_2}+\delta_{\alpha\beta_2}\sigma^2_{\beta_3\beta_1}+\delta_{\alpha\beta_3}\sigma^2_{\beta_1\beta_2}\right)\left(\delta_{I_3f_1}\tau^2_{f_3f_2}+\delta_{I_3f_2}\tau^2_{f_3f_1}+\delta_{I_3f_3}\tau^2_{f_1f_2}\right)(T_1)^a_{l_1,l_2,l_3}\Big]$$
$$(P_+q_{l_1f_1}(t,\boldsymbol{x}))^{\beta_1}\,(P_+q_{l_2f_2}(t,\boldsymbol{x}))^{\beta_2}\,(P_+q_{l_3f_3}(t,\boldsymbol{x}))^{\beta_3}$$

Castellà , Soto Phys. Rev. D. 102, 014012 (2020)

BOEFT: Lattice Operators

Berwein, Brambilla, AM, Vairo,

arXiv 2408.04719

Doubly heavy baryons **QQq**

BO quantum # $D_{\infty h}$	k^P	$(k-1/2)$ Representation	Operator Examples $H^{\alpha,\ell}_{3,\kappa}$	Projectors $P^{\alpha}_{\kappa\lambda}$
$(1/2)_g$	$(1/2)^+$	scalar	$[P_+ q^a]^\alpha$	$P^\alpha_{1/2,\pm 1/2}$
$(1/2)'_u$	$(1/2)^-$	pseudoscalar	$[P_+\gamma^5 q^a]^\alpha$	$P^\alpha_{1/2,\pm 1/2}$
$\{(1/2)_u, (3/2)_u\}$	$(3/2)^-$	vector	$\mathcal{C}^{3/2\,\alpha}_{1\,m\,1/2\,\beta}\left[(\boldsymbol{e}_m\cdot\boldsymbol{D})\,(P_+q^a)^\beta\right]$	$\{P^\alpha_{3/2,\pm 1/2}, P^\alpha_{3/2,\pm 3/2}\}$

Castellà , Soto

Phys. Rev. D. 102, 014012 (2020)

Doubly heavy tetraquarks **$QQ\bar{q}\bar{q}$** (I=0)

Λ^σ_η	k^P	Representation	Operator Examples $H^{\alpha,\ell}_{3,\kappa}\,(I=0)$	Operator Examples $H^{\alpha,\sigma}_{\bar{6},\kappa}\,(I=0)$	Projectors $P^{\alpha}_{\kappa\lambda}$
Σ^+_g	0^+	scalar	$\bar{q}\gamma^5\gamma^2\tau^2\,\underline{T}^a\,q^*$	–	1
Σ^-_u	0^-	pseudoscalar	–	$\bar{q}\gamma^2\tau^2\,\underline{\Sigma}^a\,q^*$	1
$\{\Sigma^+_u, \Pi_u\}$	1^-	vector	$\bar{q}\gamma^i\,\gamma^5\gamma^2\tau^2\,\underline{T}^a\,q^*$	$\bar{q}\gamma^i\,\gamma^5\gamma^2\tau^2\,\underline{\Sigma}^a\,q^*$	$\{\hat{r}^i, \hat{r}^i_\pm\}$
$\{\Sigma^-_g, \Pi_g\}$	1^+	pseudovector	–	$\bar{q}\gamma^i\,\gamma^2\tau^2\,\underline{\Sigma}^a\,q^*$	$\{\hat{r}^i, \hat{r}^i_\pm\}$

Doubly heavy tetraquarks **$QQ\bar{q}\bar{q}$** (I=1)

Λ^σ_η	k^P	Representation	Operator Examples $H^{\alpha,\ell}_{3,\kappa}\,(I=1)$	Operator Examples $H^{\alpha,\sigma}_{\bar{6},\kappa}\,(I=1)$	Projectors $P^{\alpha}_{\kappa\lambda}$
Σ^+_g	0^+	scalar	–	$\bar{q}\gamma^5\gamma^2\boldsymbol{e}_{I_3}\cdot(\tau^2\boldsymbol{\tau})\,\underline{\Sigma}^a\,q^*$	1
Σ^-_u	0^-	pseudoscalar	$\bar{q}\gamma^2\boldsymbol{e}_{I_3}\cdot(\tau^2\boldsymbol{\tau})\,\underline{T}^a\,q^*$	–	1
$\{\Sigma^+_u, \Pi_u\}$	1^-	vector	$\bar{q}\gamma^i\,\gamma^5\gamma^2\boldsymbol{e}_{I_3}\cdot(\tau^2\boldsymbol{\tau})\,\underline{T}^a\,q^*$	$\bar{q}\gamma^i\,\gamma^5\gamma^2\boldsymbol{e}_{I_3}\cdot(\tau^2\boldsymbol{\tau})\,\underline{\Sigma}^a\,q^*$	$\{\hat{r}^i, \hat{r}^i_\pm\}$
$\{\Sigma^-_g, \Pi_g\}$	1^+	pseudovector	$\bar{q}\gamma^i\,\gamma^2\boldsymbol{e}_{I_3}\cdot(\tau^2\boldsymbol{\tau})\,\underline{T}^a\,q^*$	–	$\{\hat{r}^i, \hat{r}^i_\pm\}$

Doubly heavy pentaquark **$QQqq\bar{q}$**

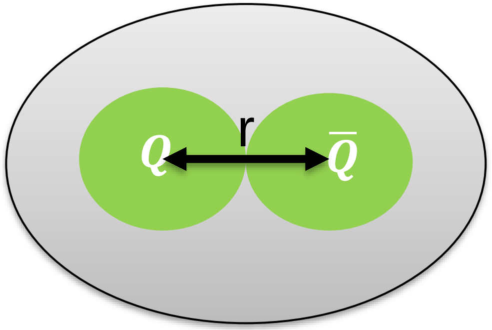
$$H^{\alpha,\ell}_{3,I_3=\pm 1/2,(1/2)^+}(t,\boldsymbol{x}) = \Big[\left(\delta_{\alpha\beta_1}\sigma^2_{\beta_2\beta_3} + \delta_{\alpha\beta_2}\sigma^2_{\beta_3\beta_1} + \delta_{\alpha\beta_3}\sigma^2_{\beta_2\beta_1}\right)\left(\delta_{I_3 f_1}\tau^2_{f_2 f_3} + \delta_{I_3 f_2}\tau^2_{f_3 f_1} + \delta_{I_3 f_3}\tau^2_{f_2 f_1}\right)\underline{T}^i_{l_1,l_2}\underline{T}^\ell_{i,l_3}\Big]$$
$$\left(P_+ q_{l_1 f_1}(t,\boldsymbol{x})\right)^{\beta_1}\left(P_+ q_{l_2 f_2}(t,\boldsymbol{x})\right)^{\beta_2}\left(\bar{q}_{l_3 f_3}(t,\boldsymbol{x})P_-\right)^{\beta_3},$$

BOEFT: Potentials

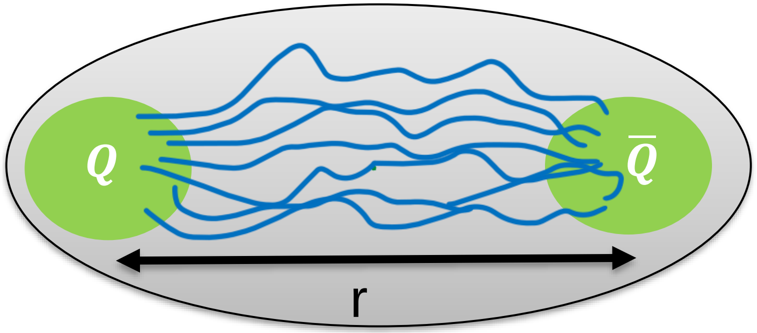
Berwein, Brambilla, AM, Vairo,
arXiv 2408.04719

LDF-quantum #: $\kappa = \{K^{PC}, f\}$

BO-quantum #: Λ_η^σ



Short-distance ($r \to 0$)



Large-distance ($r \to \infty$)

$Q\bar{Q}$: $$E^{(0)}_{\Sigma_g^+}(r) = V_s(r) + b_{\Sigma_g^+} r^2 + \cdots$$

$Q\bar{Q}X$: $$E^{(0)}_{\Lambda_\eta^\sigma}(r) = V_o(r) + \Lambda_{H_\kappa} + b_{\Lambda_\eta^\sigma} r^2 + \cdots$$

QQX: $$E^{(0)}_{\Lambda_\eta^\sigma}(r) = V_l(r) + \Lambda_{H_\kappa,l} + b_{\kappa\lambda,l}\, r^2 + \cdots \quad (l = T, \Sigma)$$

$$V_s(r) = -\frac{4\alpha_s}{3r},\ V_o(r) = \frac{\alpha_s}{6r}$$
$$V_T(r) = -\frac{2\alpha_s}{3r},\ V_\Sigma(r) = \frac{\alpha_s}{3r}$$

- String behavior (pure SU(3) gauge)

$$E_N(r) = \sqrt{\sigma^2 r^2 + 2\pi\sigma\,(N - 1/12)}$$

K. Juge, J. Kuti, C. Morningstar, Phys. Rev. Lett. 90 (2003)

- Mixing with pair of heavy-light states based on BO-quantum number Λ_η^σ representations

Static Energies: Quenched

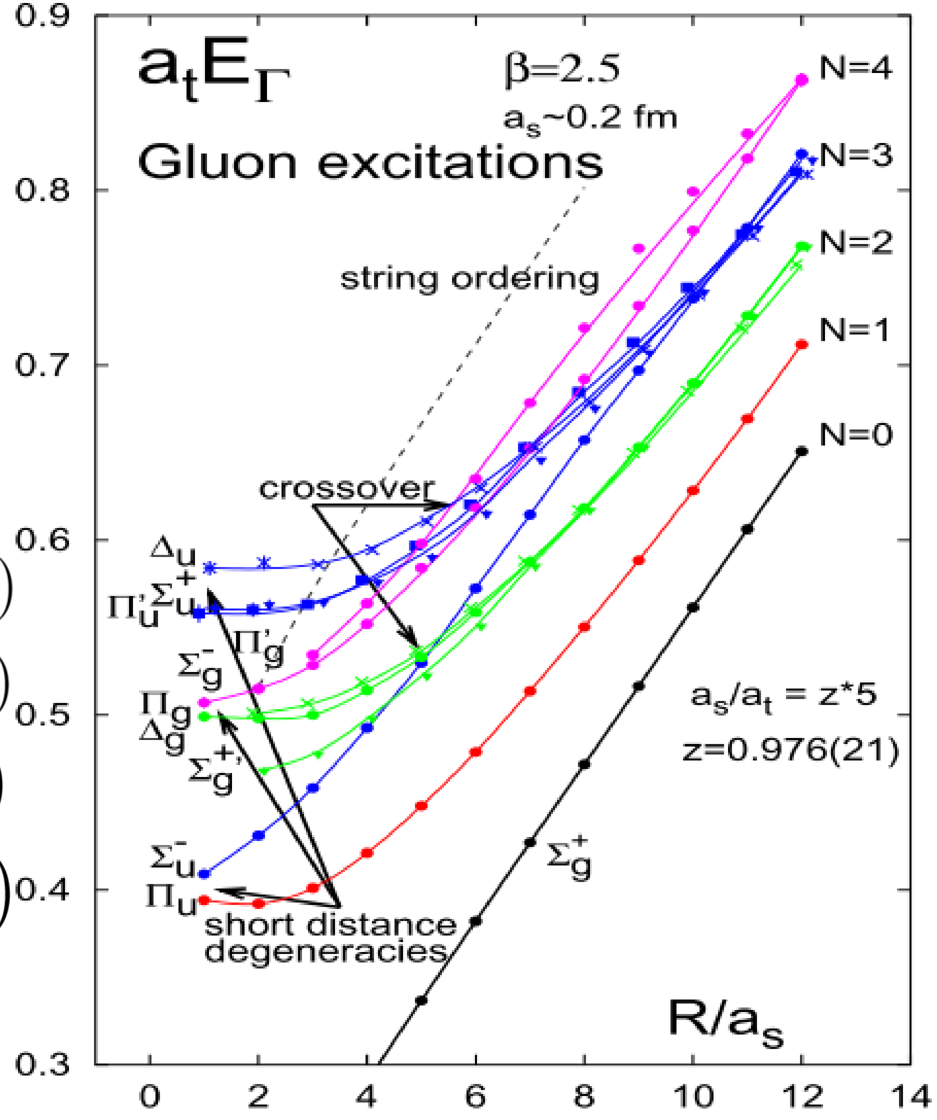
Λ_η^σ corresponding to **gluelump** quantum # K^{PC}

$2^{+-}\ (\Sigma_u^+, \Pi_u', \Delta_u)$

$2^{--}\ (\Delta_g, \Sigma_g^-, \Pi_g')$

$1^{--}\ (\Sigma_g^{+\prime}, \Pi_g)$

$1^{+-}\ (\Pi_u, \Sigma_u^-)$



0 ← r → ∞

$N = 3\ (\Sigma_u^-, \Sigma_u^+, \Pi_u', \Delta_u, \cdots)$

$N = 2\ (\Sigma_g^{+\prime}, \Pi_g, \Delta_g)$

$N = 1\ (\Pi_u)$

$N = 0\ (\Sigma_g^+)$

Observation:

BO-quantum # Λ_η^σ **conserved** at all values of r

K. Juge, J. Kuti, C. Morningstar, Phys. Rev. Lett. 90 (2003)

Bali, Neff, Duessel, Lippert, Schilling
(SESAM collaboration)
Phys. Rev. D. 71, (2005)

Static Energies: Avoided crossing

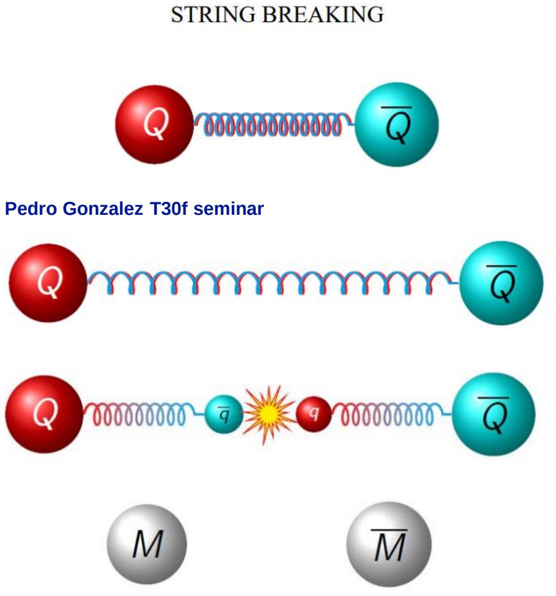
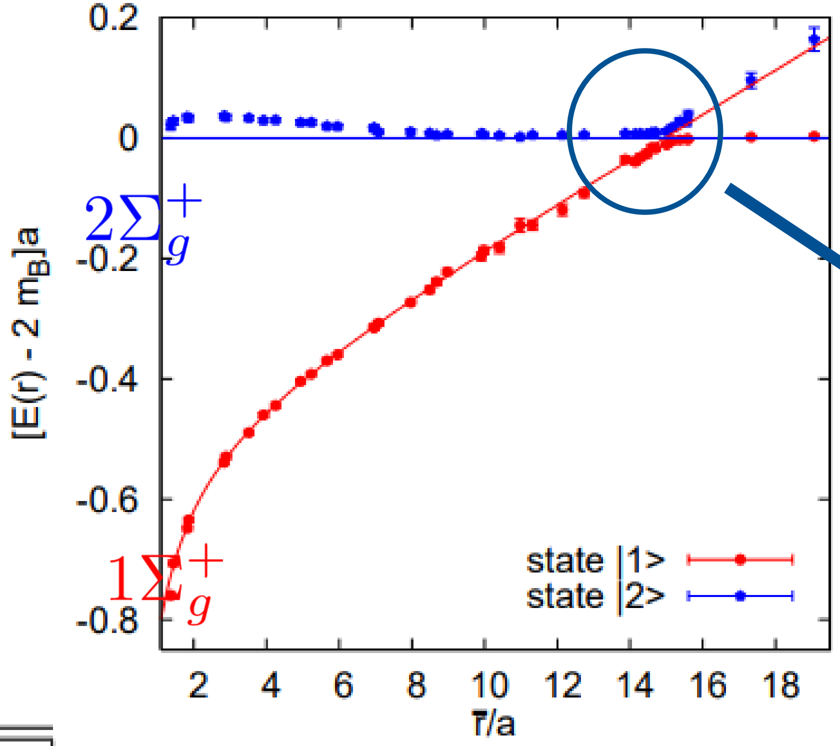


Figure from Pedro Gonzalez T30f seminar

Meson-antimeson threshold

$K_{\bar{q}}^P \otimes K_q^P$	K^{PC}	Static energies $D_{\infty h}$
$(1/2)^- \otimes (1/2)^+$	0^{-+}	$\{\Sigma_u^-\}$
	1^{--}	$\{\Sigma_g^+, \Pi_g\}$
$(1/2)^- \otimes (1/2)^-$	0^{++}	$\{\Sigma_g^+\}$
	1^{+-}	$\{\Sigma_u^-, \Pi_u\}$
$(1/2)^- \otimes (3/2)^-$	1^{+-}	$\{\Sigma_u^-, \Pi_u\}$
	2^{++}	$\{\Sigma_g^+, \Pi_g, \Delta_g\}$

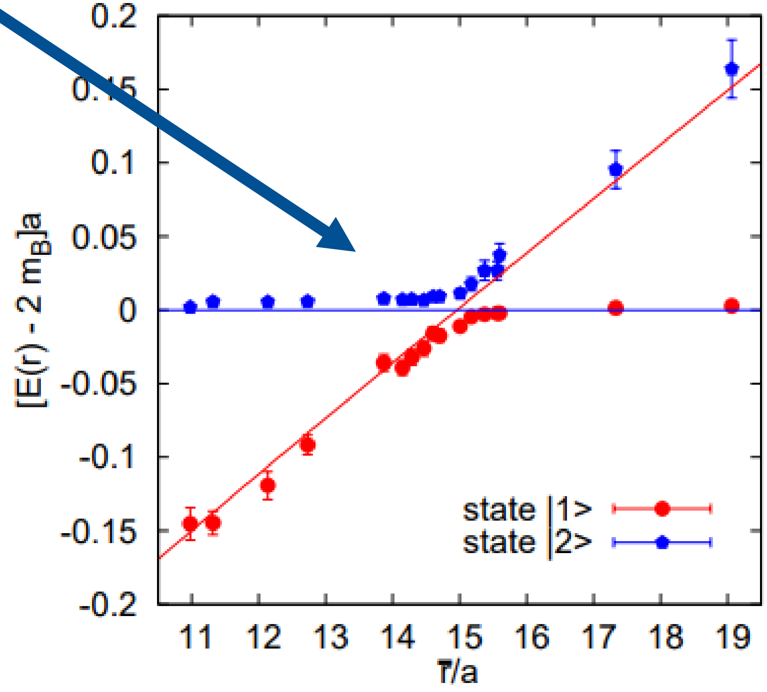
s-wave+s-wave
Ex. $D\bar{D}$ threshold



$$V_\Psi(r) = -\frac{4}{3}\frac{\alpha_s}{r} + \sigma r$$

$$m_M + m_{\bar{M}}$$

$$m_\pi \approx 650 \text{ MeV}$$



String breaking radius $\approx$ 1.25 fm

a $\approx$ 0.083 fm

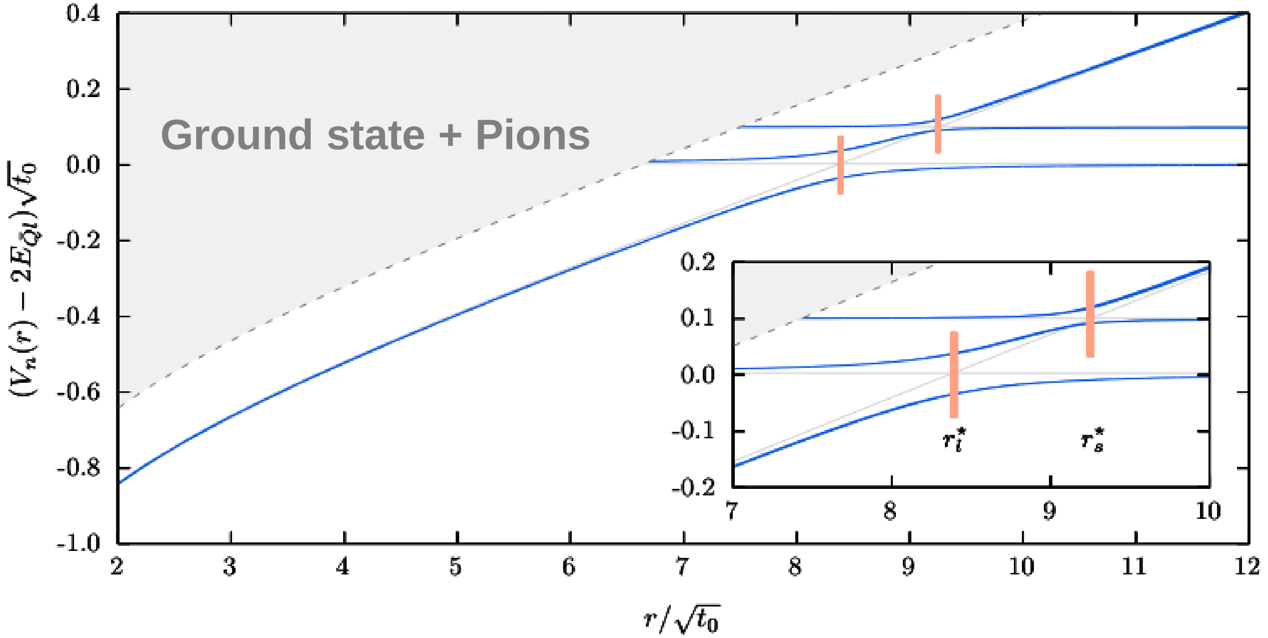
BO-quantum # Σ_g^+ mix: avoided crossing between $Q\bar{Q}$ & $M\bar{M}$

Static Energies: Avoided crossing

More recent computation of string breaking:

Bulava, Hoerz, Knechtli, Koch, moir Morningstar, Peardon, Phys. Lett. B. 793 (2019)

Bulava, Knechtli, Koch, Morningstar, Peardon, Phys. Lett. B. 854 (2024)



String breaking radius ≈ 1.22 fm **a** ≈ 0.063 fm

Model Hamiltonian for determining parameters:

$$H(r) = \begin{pmatrix} \hat{V}(r) & \sqrt{2}g_l & g_s \\ \sqrt{2}g_l & \hat{E}_1 & 0 \\ g_s & 0 & \hat{E}_2 \end{pmatrix}, \hat{V}(r) = \hat{V}_0 + \sigma r + \gamma/r$$

$$m_\pi \approx 200 - 340 \text{ MeV} \quad m_K \approx 440 - 480 \text{ MeV}$$

Hybrid static energies: (Σ_u^-, Π_u)

1) **Avoided crossing** with **s-wave + p-wave threshold.** No lattice results available on this till now !!

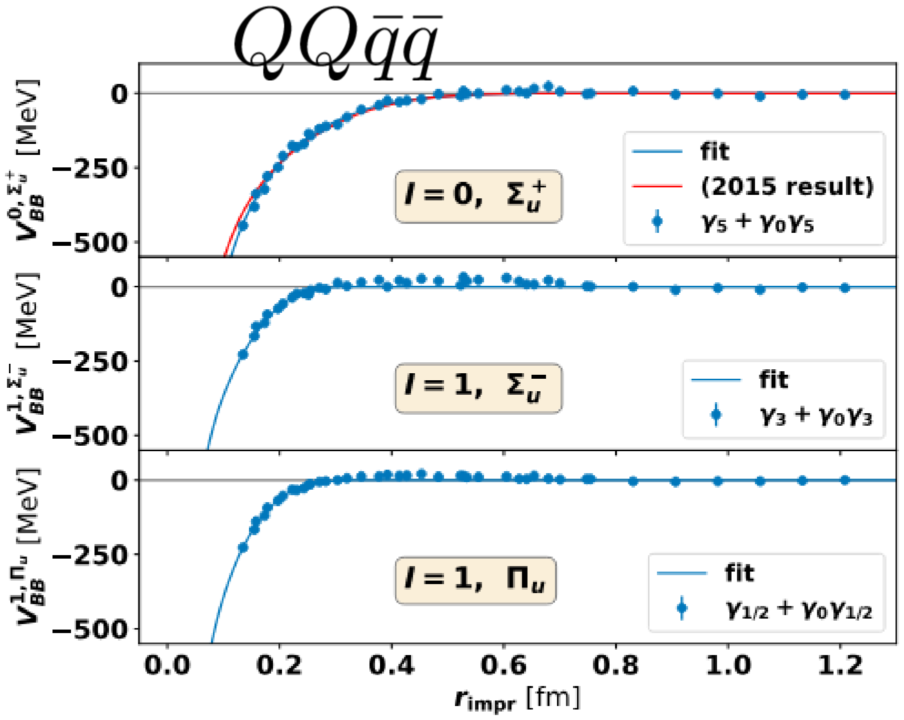
2) Σ_u^- **component mixing** with **s-wave + s-wave threshold** (significant effects only if the energy gap less than Λ_{QCD} scale).

Bruschini Phys. Rev. D 109 L031501 (2024) J. Castella JHEP 06, 107 (2024)

Important observations till now based on lattice results for $Q\bar{Q}$ & $Q\bar{Q}g$ static energies:

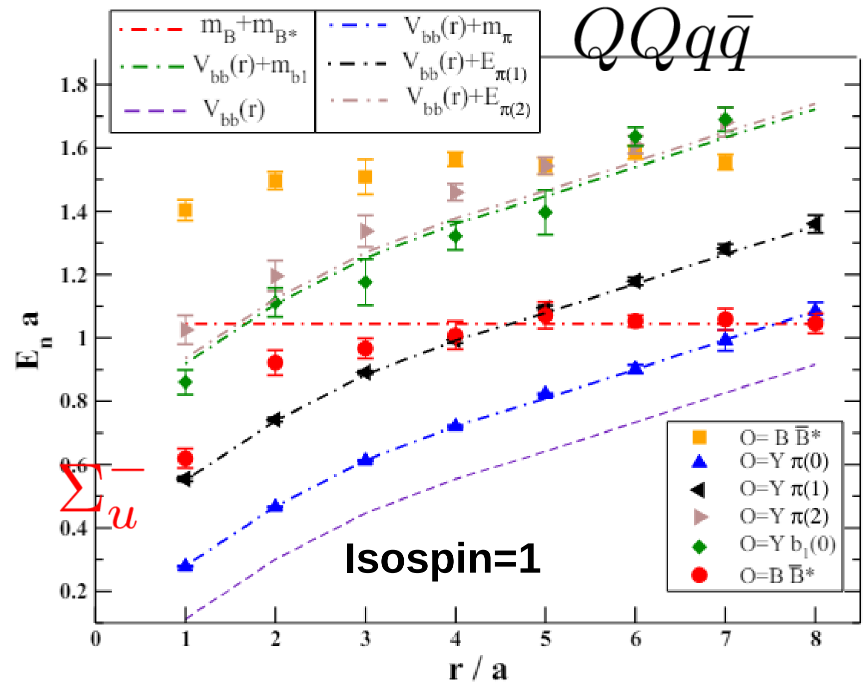
- BO-quantum # Λ_η^σ **conserved** at all values of r
- Different BO-quantum # Λ_η^σ can intersect each other
- In $Q\bar{Q}$ & $Q\bar{Q}g$ avoided crossing between same BO-quantum # Λ_η^σ.

Static Energies: Tetraquark



Mueller et al, PoS LATTICE2023, 64 (2024)

Bicudo, Cichy, Peters, Wagner, Phys. Rev. D. 93, (2016)



Prelovsek, Bahtiyar, Petkovic, Phys. Lett. B. 805, (2020)

See also Tetsuo Hatsuda talk (Monday 10:30)

Tetraquark / pentaquark static energies: Is there any meaning to avoided crossing ?

No, quark configurations can be rearranged to have two meson state.

Similarity with molecular physics. Molecules going to constituent atoms when internuclear separation very large.

Berwein, Brambilla, AM, Vairo, arXiv 2408.04719

$Q\bar{Q}q\bar{q}$: Operator Overlap

Berwein, Brambilla, AM, Vairo,
arXiv 2408.04719

NRQCD operator (gauge invariant) for exotic hadron: $Q\bar{Q}$ pair in **octet** color

$$\mathcal{O}_K(t,\, \mathbf{r},\, \mathbf{0}) = \chi^\dagger(\mathbf{t},\, \mathbf{r}/\mathbf{2})\phi(\mathbf{t},\, \mathbf{r}/\mathbf{2}, \mathbf{0})\mathbf{H_K}(\mathbf{t},\, \mathbf{0})\phi(\mathbf{t},\, \mathbf{0},\, -\mathbf{r}/\mathbf{2})\psi(\mathbf{t},\, -\mathbf{r}/\mathbf{2})$$

$$\mathbf{H}_{\boldsymbol{K}}(t,\boldsymbol{x}) = \left[\bar{q}(t,\boldsymbol{x})\tilde{\boldsymbol{\Gamma}}T^a q(t,\boldsymbol{x})\right] T^a$$

$\tilde{\boldsymbol{\Gamma}}$: Dirac matrices based on quantum #'s

Quarkonium + Pions

Quarkonium state:

$$|\mathcal{Q}\rangle = \mathcal{N}\int d^3\boldsymbol{r}\,\Psi^{(n)}(\boldsymbol{r})\,\psi_b^\dagger(t,-\boldsymbol{r}/2)\,\phi_{bc}(t;-\boldsymbol{r}/2,\boldsymbol{r}/2)\,\chi_c(t,\boldsymbol{r}/2)\,|\Omega\rangle$$

Overlap of our operator on quarkonium + pion:

$$\langle\mathcal{Q}|\mathcal{O}_K^{Q\bar{Q}}(t,\boldsymbol{r})\,|\mathcal{Q}\rangle = 0$$

Meson-antimeson

Meson-antimeson state:

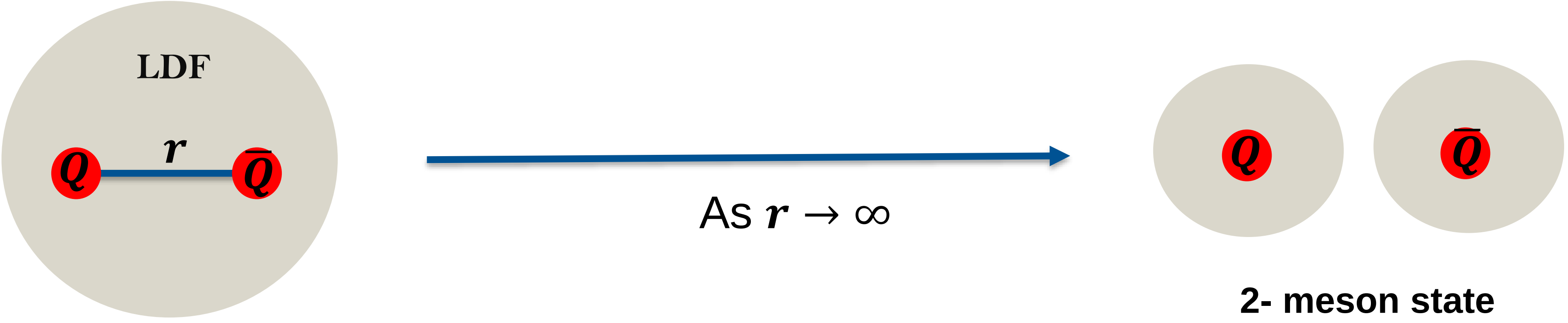
$$|M\bar{M}\rangle = \Bigg[\mathcal{N}\int d^3\boldsymbol{x}\,\Psi_J(\boldsymbol{x}) \times\int d^3\boldsymbol{y}\,\varphi_{J_1}(\boldsymbol{y}+\boldsymbol{x}/2)\,\psi_c^\dagger(t,-\boldsymbol{x}/2)\,\phi_{cd}(t;-\boldsymbol{x}/2,\boldsymbol{y}+\boldsymbol{x}/2)\,[P_+\Gamma_1 q_d(t,\boldsymbol{y}+\boldsymbol{x}/2)] \times\int d^3\boldsymbol{z}\,\varphi_{J_2}(\boldsymbol{z}-\boldsymbol{x}/2)\,[\bar{q}_b(t,\boldsymbol{z}-\boldsymbol{x}/2)\,\Gamma_2 P_-]\,\phi_{be}(t;\boldsymbol{z}-\boldsymbol{x}/2,\boldsymbol{x}/2)\,\chi_e(t,\boldsymbol{x}/2)\Bigg]|\mathrm{vac}\rangle$$

Overlap of our operator on meson-antimeson:

$$\langle M\bar{M}|\mathcal{O}_K^{Q\bar{Q}}(t,\boldsymbol{r})\,|M\bar{M}\rangle \neq 0$$

Adjoint operators are good operators for lattice computation for $Q\bar{Q}q\bar{q}$ potentials !!!

Static Energies: Tetraquark



Consider $Q\bar{Q}q\bar{q}$ system:

BO-quantum # Λ_η^σ for adjoint meson:

$Q\bar{Q}$ (color)	Light Spin K^{PC}	Λ_η^σ $(D_{\infty h})$
Octet	0^{-+}	Σ_u^-
	1^{--}	$\{\Sigma_g^+, \Pi_g\}$

Berwein, Brambilla, AM, Vairo, arXiv 2408.04719

BO-quantum # Λ_η^σ for meson-antimeson

$K_{\bar{q}}^P \otimes K_q^P$	K^{PC}	Static energies $D_{\infty h}$
$(1/2)^- \otimes (1/2)^+$	0^{-+}	$\{\Sigma_u^-\}$
	1^{--}	$\{\Sigma_g^+, \Pi_g\}$

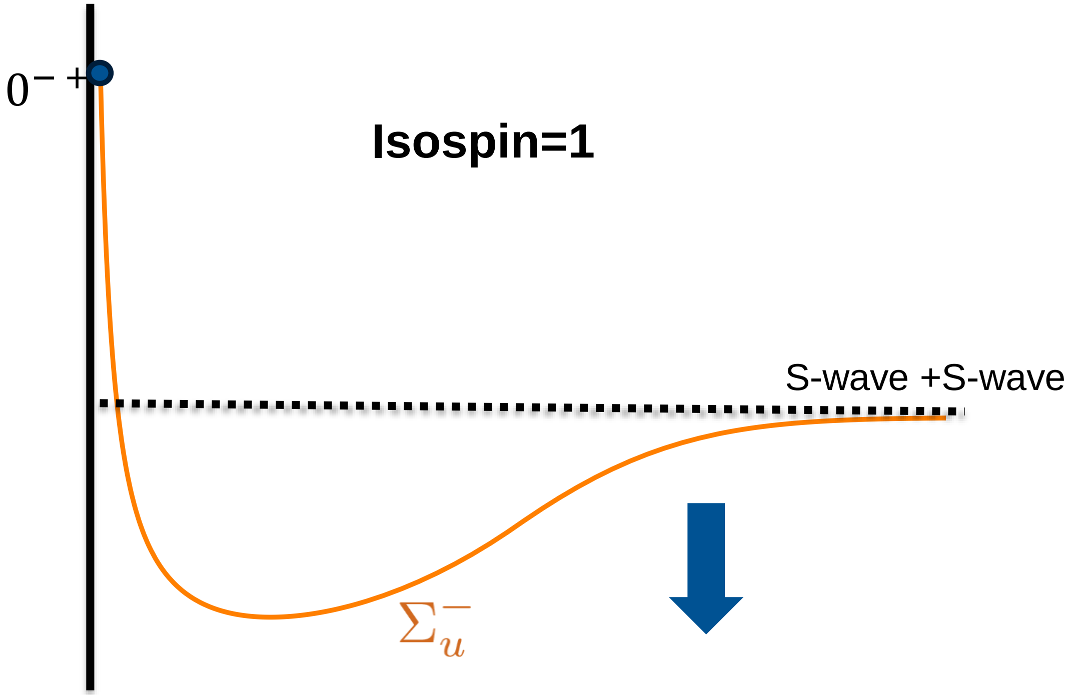
s-wave+s-wave
Ex. $\mathbf{D\bar{D}}$ threshold

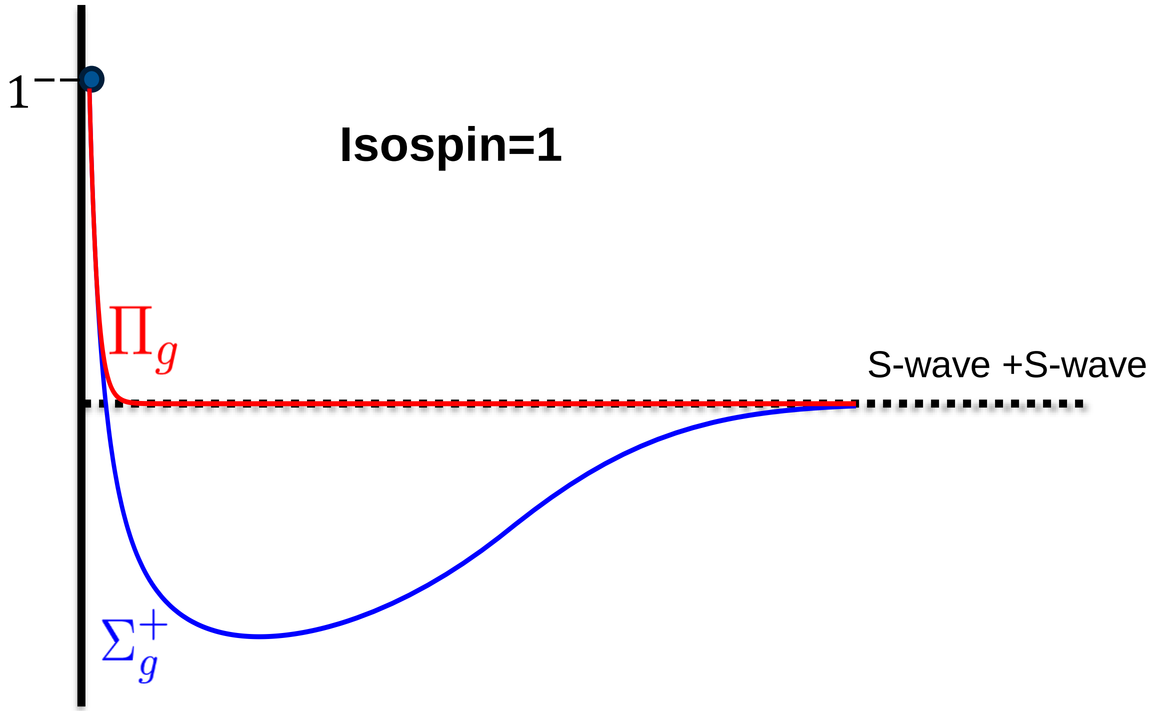
Meson-antimeson have same BO-quantum # Λ_η^σ as of adjoint meson !!!

Static Energies: Tetraquark

Berwein, Brambilla, AM, Vairo,
arXiv 2408.04719

- BO-quantum # Λ_η^σ **conserved** at all values of r





Consistent with Σ_u^- data (r/a > 1) in Prelovsek et al (2020)

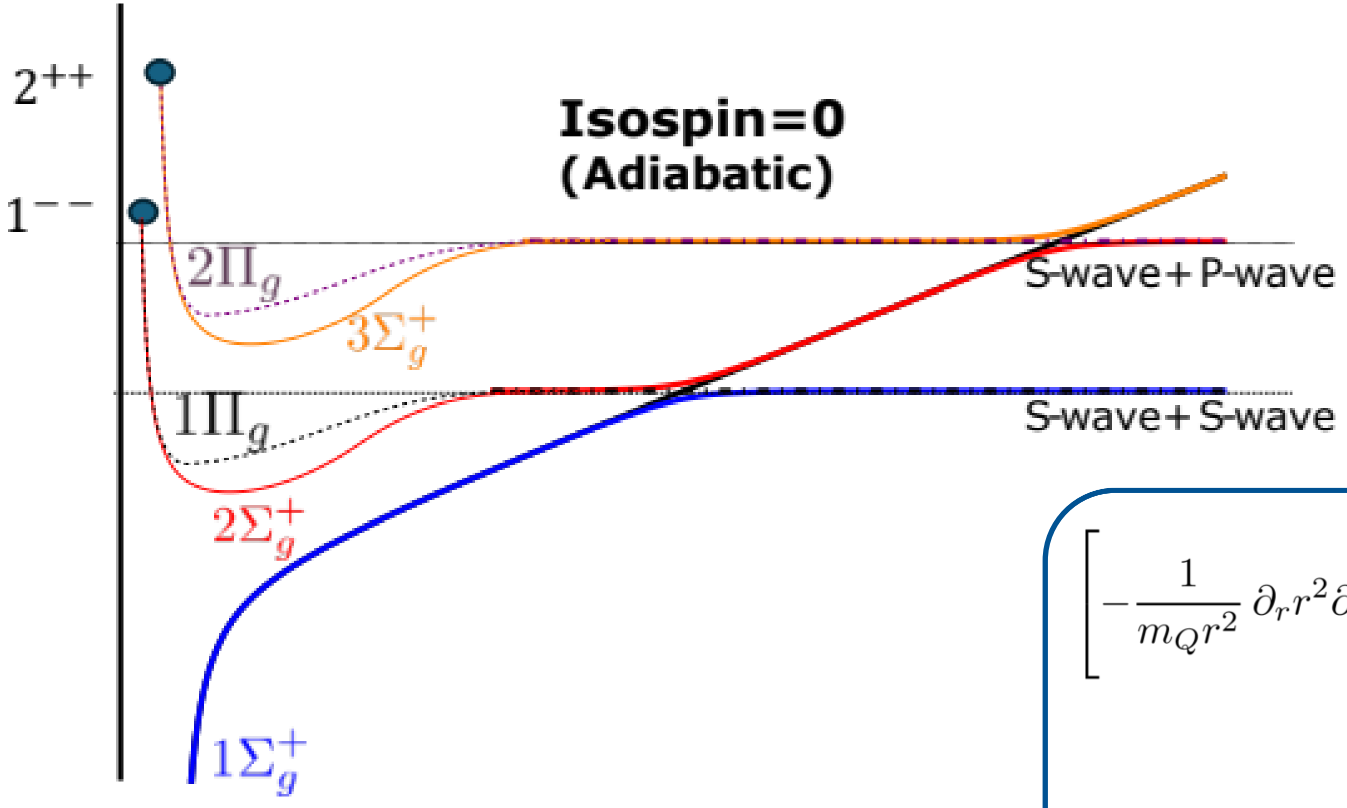
Behavior similar to molecular potentials such as Leonard Jones potential !!!

Meson-antimeson threshold at short distance is connected with adjoint meson !!!

Static Energies: Tetraquark

Berwein, Brambilla, AM, Vairo,
arXiv 2408.04719

- BO-quantum # Λ_η^σ **conserved** at all values of r
- Different BO-quantum # Λ_η^σ can intersect each other
- Avoided crossing between same BO-quantum # Λ_η^σ.



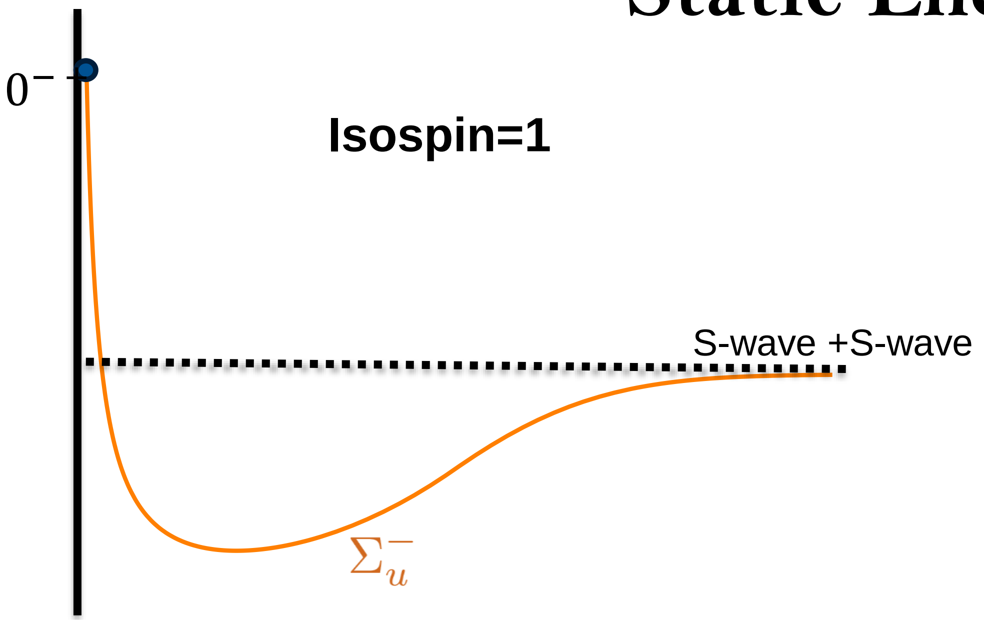
Meson-antimeson threshold at short distance is connected with adjoint meson !!!

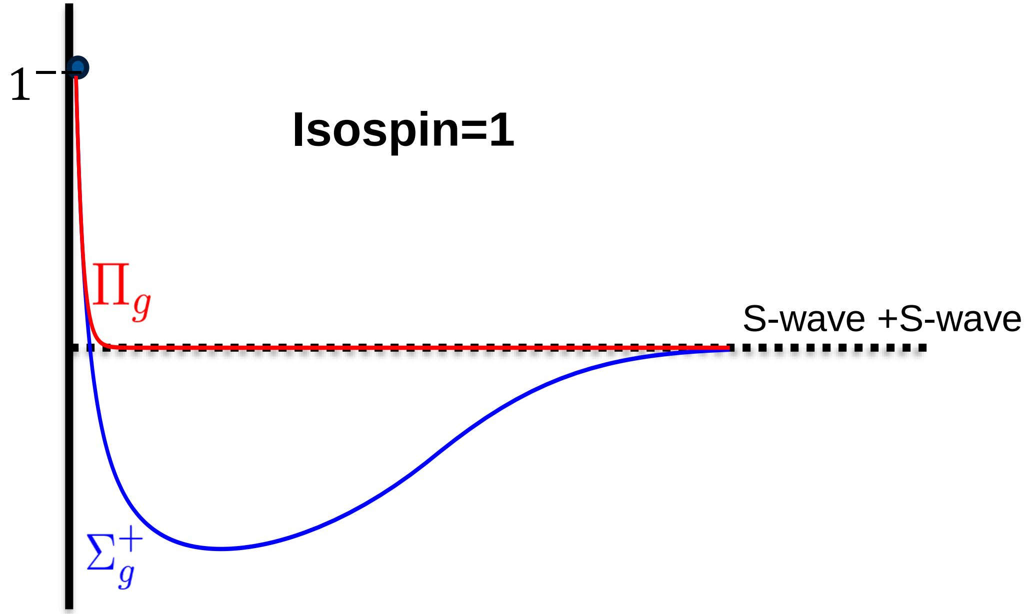
Coupled-channel Equations:

$$\left[-\frac{1}{m_Q r^2}\partial_r r^2 \partial_r + \frac{1}{m_Q r^2}\begin{pmatrix} l\,(l+1) & 0 & 0 \\ 0 & l(l+1)+2 & -2\sqrt{l(l+1)} \\ 0 & -2\sqrt{l(l+1)} & l(l+1) \end{pmatrix} + \begin{pmatrix} E_{\Sigma_g^+}(r) & g(r) & 0 \\ g(r) & E_{\Sigma_g^{+\prime}}(r) & 0 \\ 0 & 0 & E_{\Pi_g}(r) \end{pmatrix}\right]\begin{pmatrix}\psi_\Sigma \\ \psi_{\Sigma'} \\ \psi_\Pi\end{pmatrix} = \mathcal{E}\begin{pmatrix}\psi_\Sigma \\ \psi_{\Sigma'} \\ \psi_\Pi\end{pmatrix}$$

Static Energies: Tetraquark

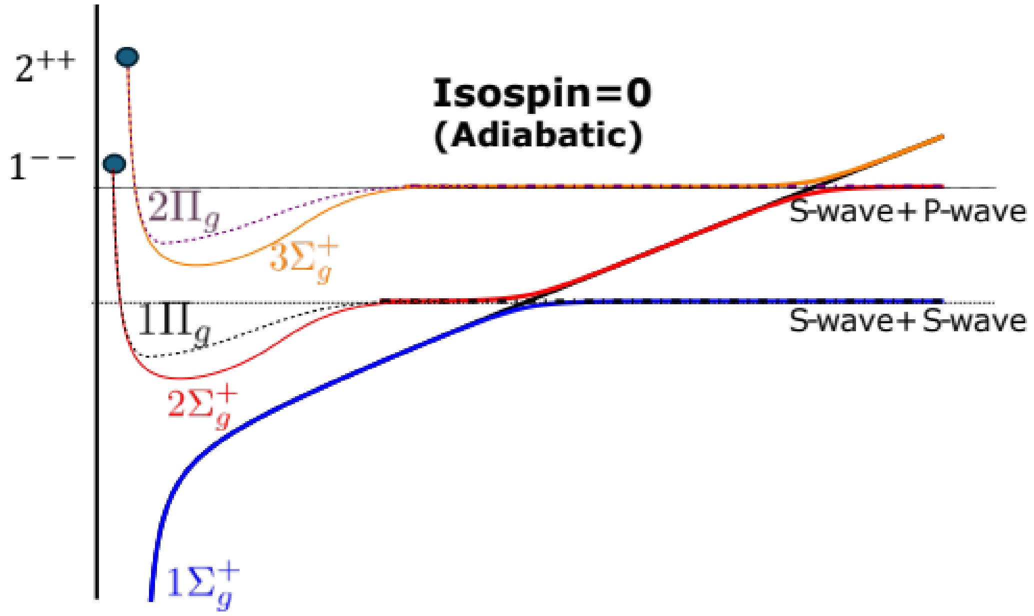
Berwein, Brambilla, AM, Vairo,
arXiv 2408.04719





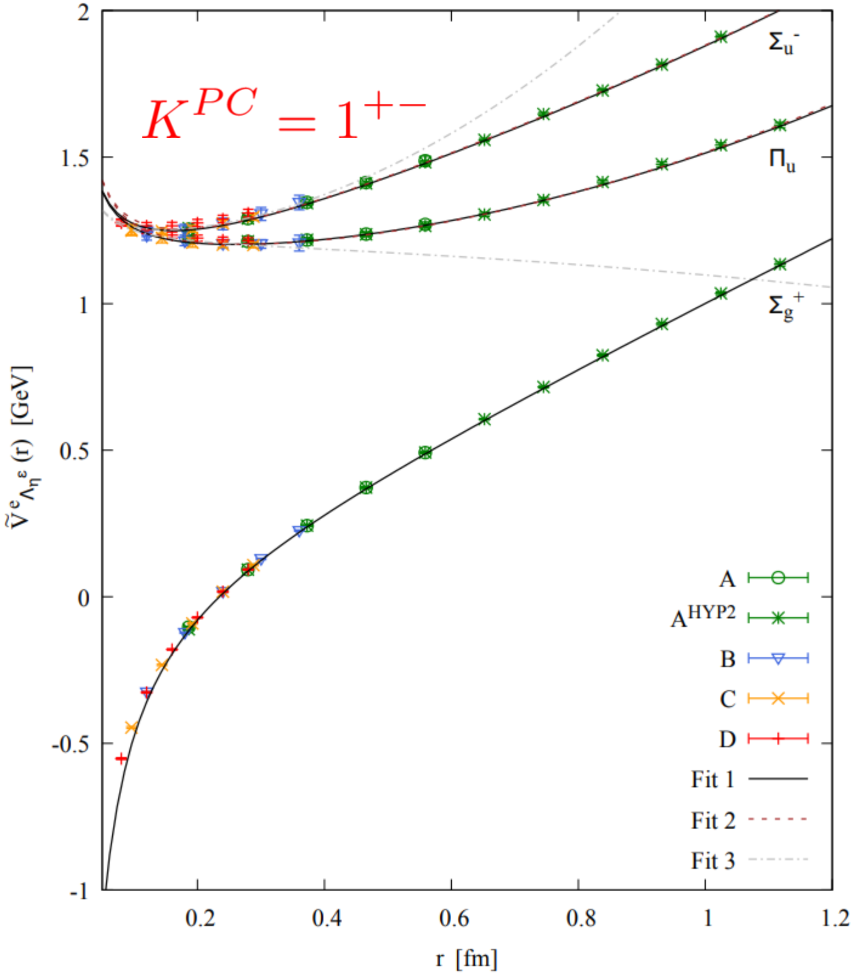
Behavior of tetraquark static energy:

- Adjoint meson behavior at small r **($r \to 0$)**
- Heavy meson pair threshold at large r **($r \to \infty$)**
- Avoided crossing with quarkonium static energy (Isopsin=0)



Hybrids

BOEFT: Hybrids



- Coupled Schrödinger Eq:

$$\left[-\frac{1}{m_Q r^2}\partial_r r^2 \partial_r + \frac{1}{m_Q r^2}\begin{pmatrix} l(l+1)+2 & 2\sqrt{l(l+1)} \\ 2\sqrt{l(l+1)} & l(l+1)\end{pmatrix} + \begin{pmatrix} E_{\Sigma_u^-} & 0 \\ 0 & E_{\Pi_u}\end{pmatrix}\right]\begin{pmatrix}\psi_\Sigma^{(m)} \\ \psi_{-\Pi}^{(m)}\end{pmatrix} = E_m^{Q\bar{Q}g}\begin{pmatrix}\psi_\Sigma^{(m)} \\ \psi_{-\Pi}^{(m)}\end{pmatrix}$$

$$\left[-\frac{1}{m_Q r^2}\partial_r\, r^2\, \partial_r + \frac{l(l+1)}{m_Q r^2} + E_{\Pi_u}\right]\psi_{+\Pi}^{(m)} = E_m^{Q\bar{Q}g}\,\psi_{+\Pi}^{(m)}$$

$$\lambda = 0, \pm 1$$

Hybrid Spectrum:

Multiplet	J^{PC}	$M_{c\bar{c}g}$	$M_{b\bar{b}g}$
H_1	$\{1^{--}, (0,1,2)^{-+}\}$	4155	10786
H_1'		4507	10976
H_1''		4812	11172
H_2	$\{1^{++}, (0,1,2)^{+-}\}$	4286	10846
H_2'		4667	11060
H_2''		5035	11270
H_3	$\{0^{++}, 1^{+-}\}$	4590	11065
H_3'		5054	11352
H_3''		5473	11616
H_4	$\{2^{++}, (1,2,3)^{+-}\}$	4367	10897
H_5	$\{2^{--}, (1,2,3)^{-+}\}$	4476	10948

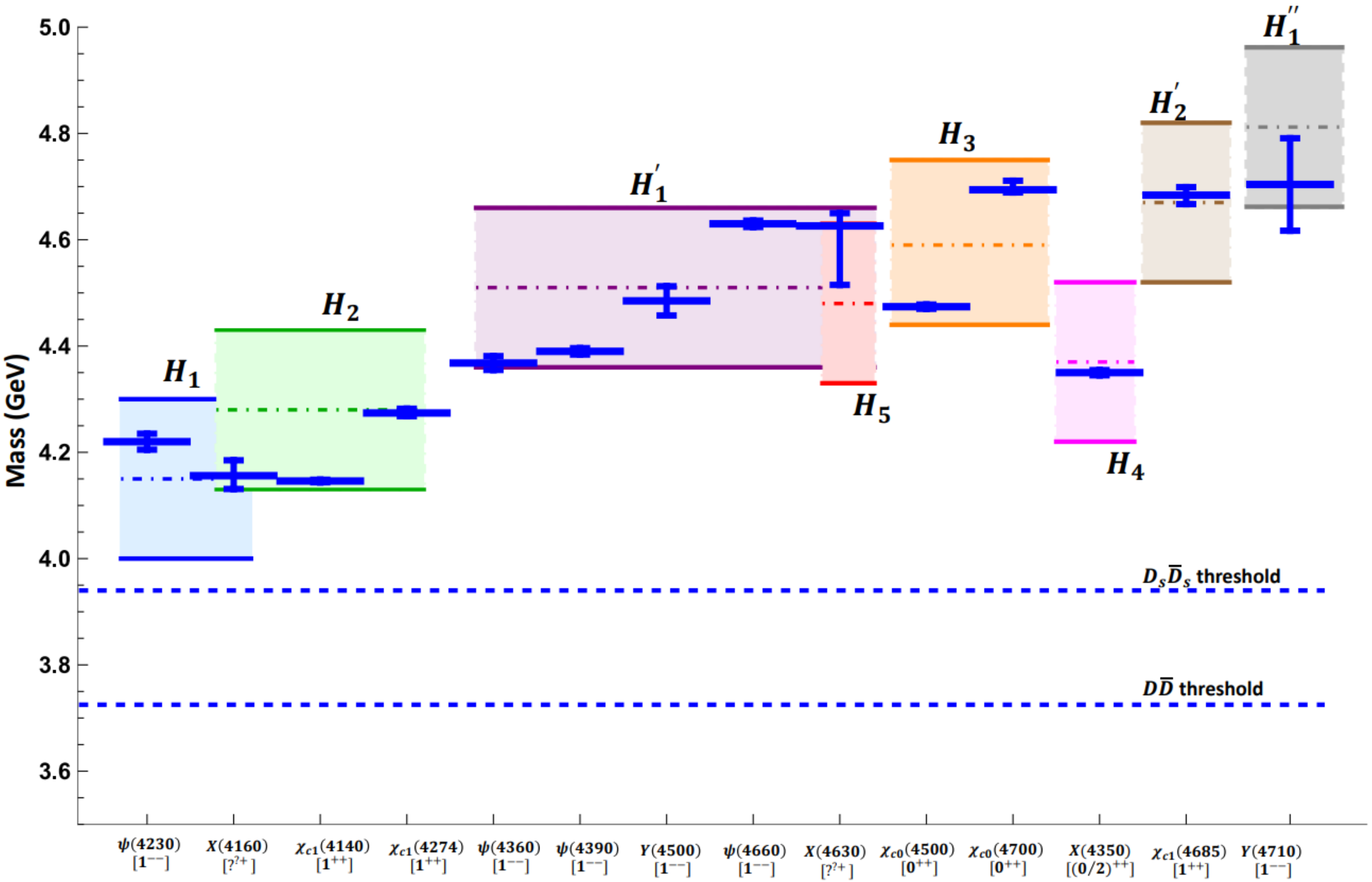
Λ- doubling: opposite parity states non-degenerate.

Brambilla, Lai, AM, Vairo Phys. Rev. D 107, 054034 (2023)

Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)

BOEFT: Hybrids

- **Charmonium hybrids**: comparison with experimental results:



	l	$J^{PC}\{s=0, s=1\}$	$E_n^{(0)}$
H_1	1	$\{1^{--}, (0,1,2)^{-+}\}$	Σ_u^-, Π_u
H_2	1	$\{1^{++}, (0,1,2)^{+-}\}$	Π_u
H_3	0	$\{0^{++}, 1^{+-}\}$	Σ_u^-
H_4	2	$\{2^{++}, (1,2,3)^{+-}\}$	Σ_u^-, Π_u
H_5	2	$\{2^{--}, (1,2,3)^{-+}\}$	Π_u

PDG 2022

Brambilla, Lai, AM, Vairo

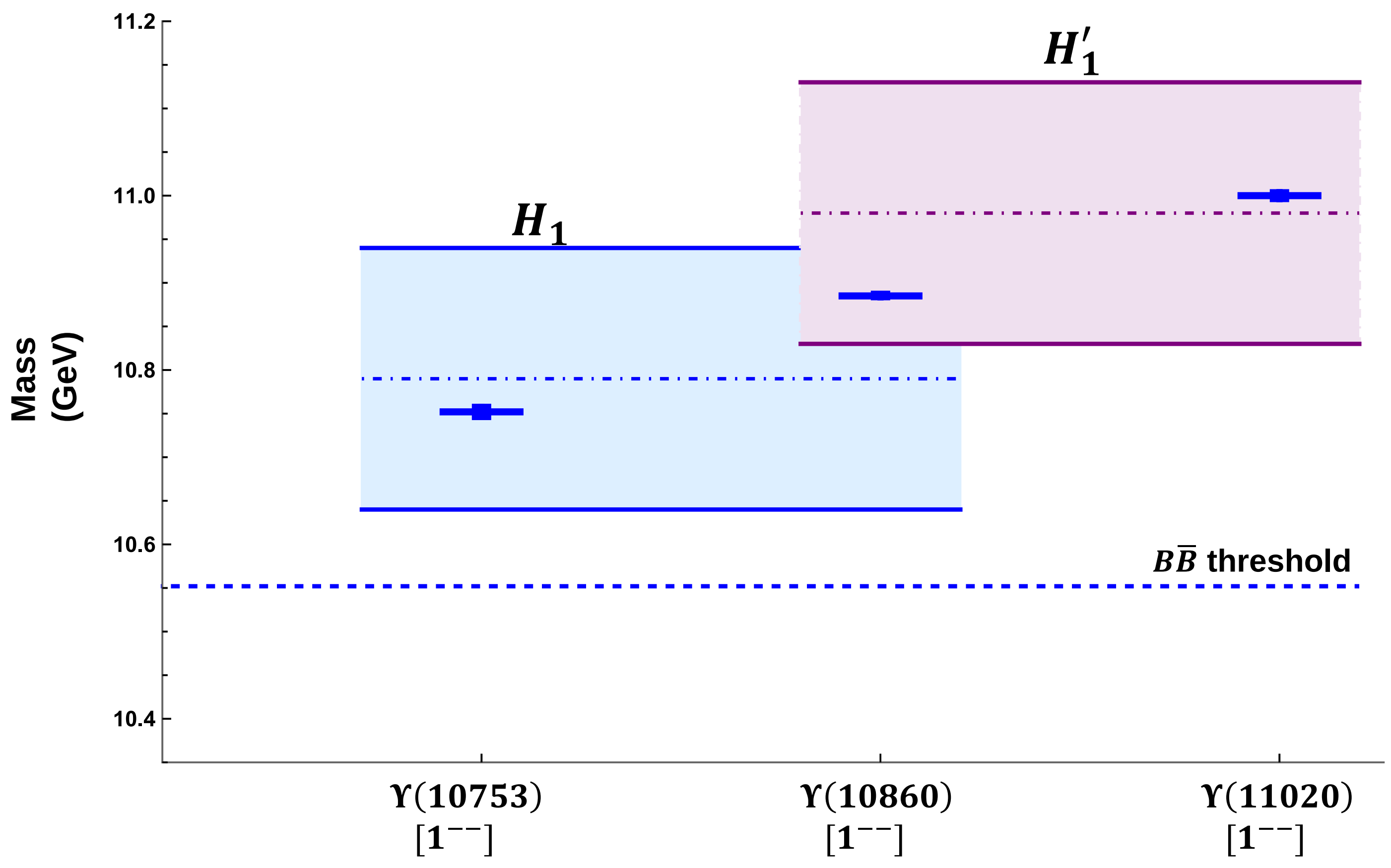
Phys. Rev. D 107, 054034 (2023)

Berwein, Brambilla, Castellà , Vairo

Phys. Rev. D. 92, 114019 (2015)

BOEFT: Hybrids

- **Bottomonium hybrids**: comparison with experimental results:



PDG 2022

	l	$J^{PC}\{s=0, s=1\}$	$E_n^{(0)}$
H_1	1	$\{1^{--}, (0,1,2)^{-+}\}$	Σ_u^-, Π_u
H_2	1	$\{1^{++}, (0,1,2)^{+-}\}$	Π_u
H_3	0	$\{0^{++}, 1^{+-}\}$	Σ_u^-
H_4	2	$\{2^{++}, (1,2,3)^{+-}\}$	Σ_u^-, Π_u
H_5	2	$\{2^{--}, (1,2,3)^{-+}\}$	Π_u

Brambilla, Lai, AM, Vairo
Phys. Rev. D 107, 054034 (2023)

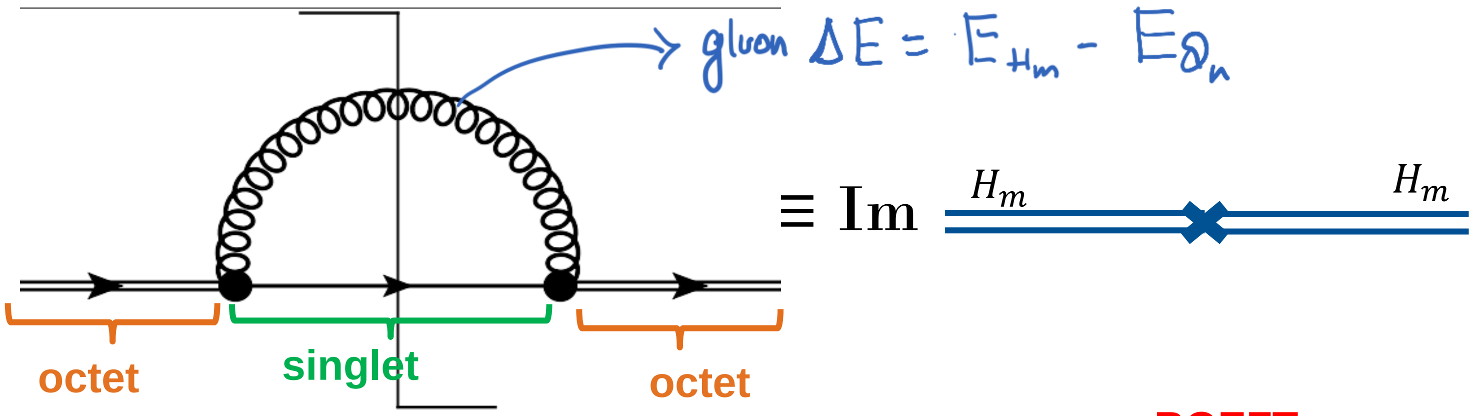
Berwein, Brambilla, Castellà , Vairo
Phys. Rev. D. 92, 114019 (2015)

Hybrid Decays

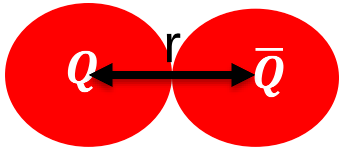
Brambilla, Lai, AM, Vairo Phys. Rev. D 107, 054034 (2023)

- BOEFT can describe decays of hybrids to quarkonium.
- Semi-inclusive process: $H_m \to Q_n + X$; Q_n: low-lying quarkonium (states below threshold) & X: light hadrons.
  - ✓ ΔE: Large energy difference ⇒ $\Delta E \equiv E_{H_m} - E_{Q_n} \gtrsim 1$ GeV.
  - ✓ Constituent gluon of the hybrid is a spectator.
  - ✓ Hierarchy of scales: $\Delta E \gg \Lambda_{QCD} \gg mv^2$ (ΔE ≫ Λ_QCD: Perturbative computation)

matching **pNRQCD** and **BOEFT:**



Virtual gluon resolves color structure of $Q\bar{Q}$ pair ($\mathbf{r} \to \mathbf{0}$) in quarkonium and hybrid in short-distance limit



Quarkonium ---> Singlet

Hybrid ---> Octet

- Decays are computed from local imaginary terms in the hybrid potential (BOEFT potential).

Optical theorem: $\sum_n \Gamma(H_m \to Q_n) = -2\,\mathrm{Im}\,\langle H_m|V|H_m\rangle$

DISCLAIMER!!!
Decay to open-flavor threshold states not accounted here.

Hybrid Decays

Brambilla, Lai, AM, Vairo Phys. Rev. D 107, 054034 (2023)

- Spin-conserving decay due to $\boldsymbol{r} \cdot \boldsymbol{E}$ term :

$|S_H = 1 > \dashrightarrow |S_Q = 1 >$
$|S_H = 0 > \dashrightarrow |S_Q = 0 >$

$$\Gamma(H_m \to Q_n) = \frac{4\alpha_s (\Delta E) T_F}{3N_c} T^{ij} (T^{ij})^\dagger \Delta E^3$$

DISCLAIMER!!!
Decay to open-flavor threshold states not accounted here.

$$T^{ij} \equiv \langle H_m | r^j | Q_n \rangle = \int d^3\mathbf{r}\, \Psi^{i\dagger}_{(m)}(\mathbf{r})\, r^j\, \Phi^{Q\bar{Q}}_{(n)}(\mathbf{r})$$

$\Psi^i_{(m)}$: Hybrid wf

Φ^Q_n : Quarkonium wf

R. Oncala, J. Soto, Phys. Rev. D96, 014004 (2017).

J. Castellà, E. Passemar, Phys. Rev. D104, 034019 (2021)

$$\langle H_m | \boldsymbol{r} | Q_n \rangle = \sqrt{T^{ij} (T^{ij})^\dagger}$$

- Spin-flipping decay due to $\boldsymbol{S}.\boldsymbol{B}$ term:

$|\chi_H\rangle$: Hybrid spin wf

$|\chi_Q\rangle$: Quarkonium spin wf

$|S_H = 1 > \dashrightarrow |S_Q = 0 >$
$|S_H = 0 > \dashrightarrow |S_Q = 1 >$

$$T^{ij} \equiv \langle H_m | \left(S_1^j - S_2^j\right) | Q_n \rangle = \left[\int d^3\mathbf{r}\, \Psi^{i\dagger}_{(m)}(\mathbf{r})\, \Phi^Q_{(n)}(\mathbf{r}) \right] \langle \chi_H | \left(S_1^j - S_2^j\right) | \chi_Q \rangle$$

Depends on overlap of quarkonium and hybrid wavefunctions.

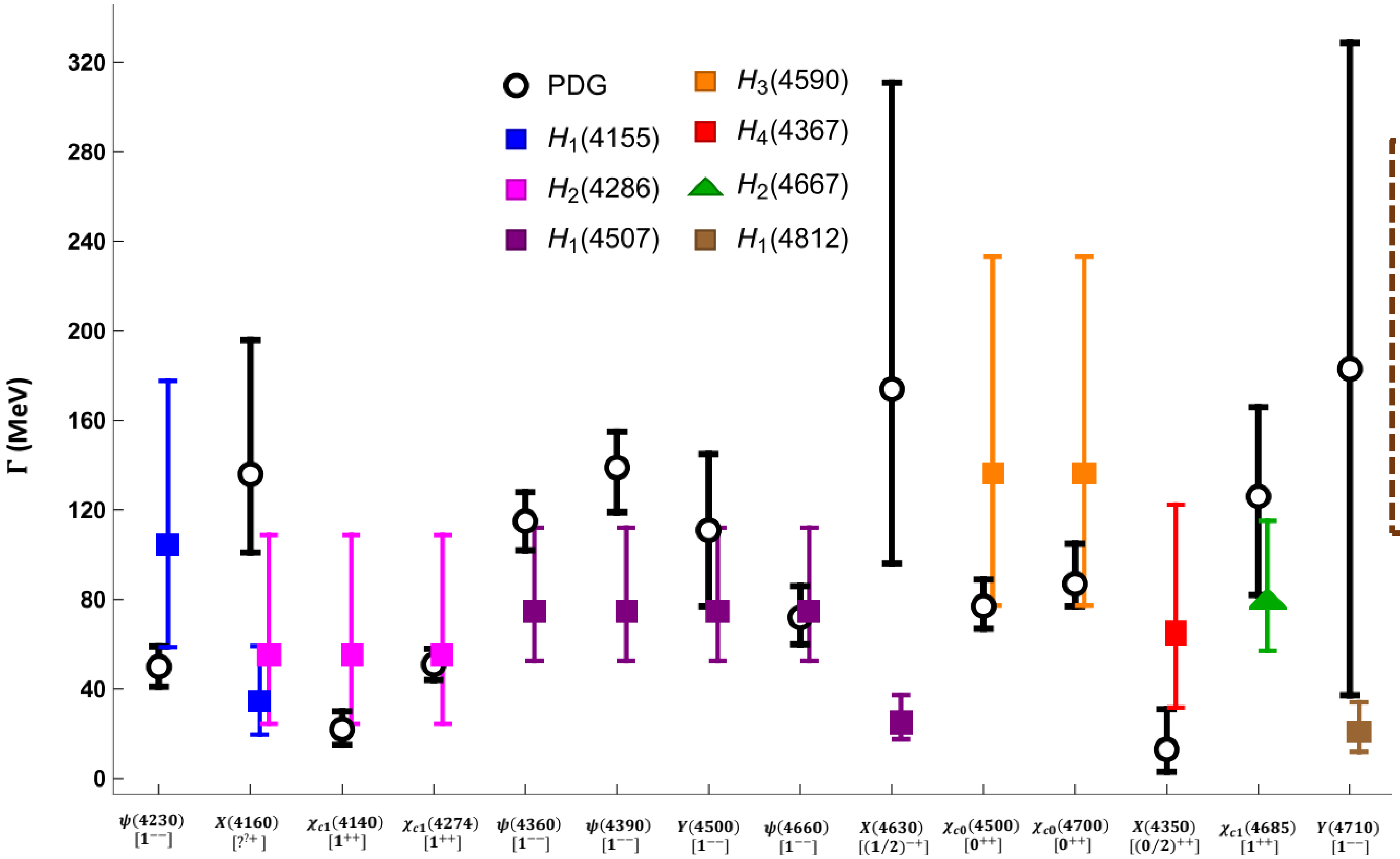
Hybrid-to-Quarkonium transition decay rate
= **spin-conserving** + **spin-flipping** decay rates.

Our estimate of decay rate are lower-bounds for the **total width of hybrids**

Results

Brambilla, Lai, AM, Vairo Phys. Rev. D 107, 054034 (2023)

- Comparison: charm exotic states with corresponding charmonium hybrid state:



$$H_m \to Q_n + X$$

spin-conserving + spin-flipping decays

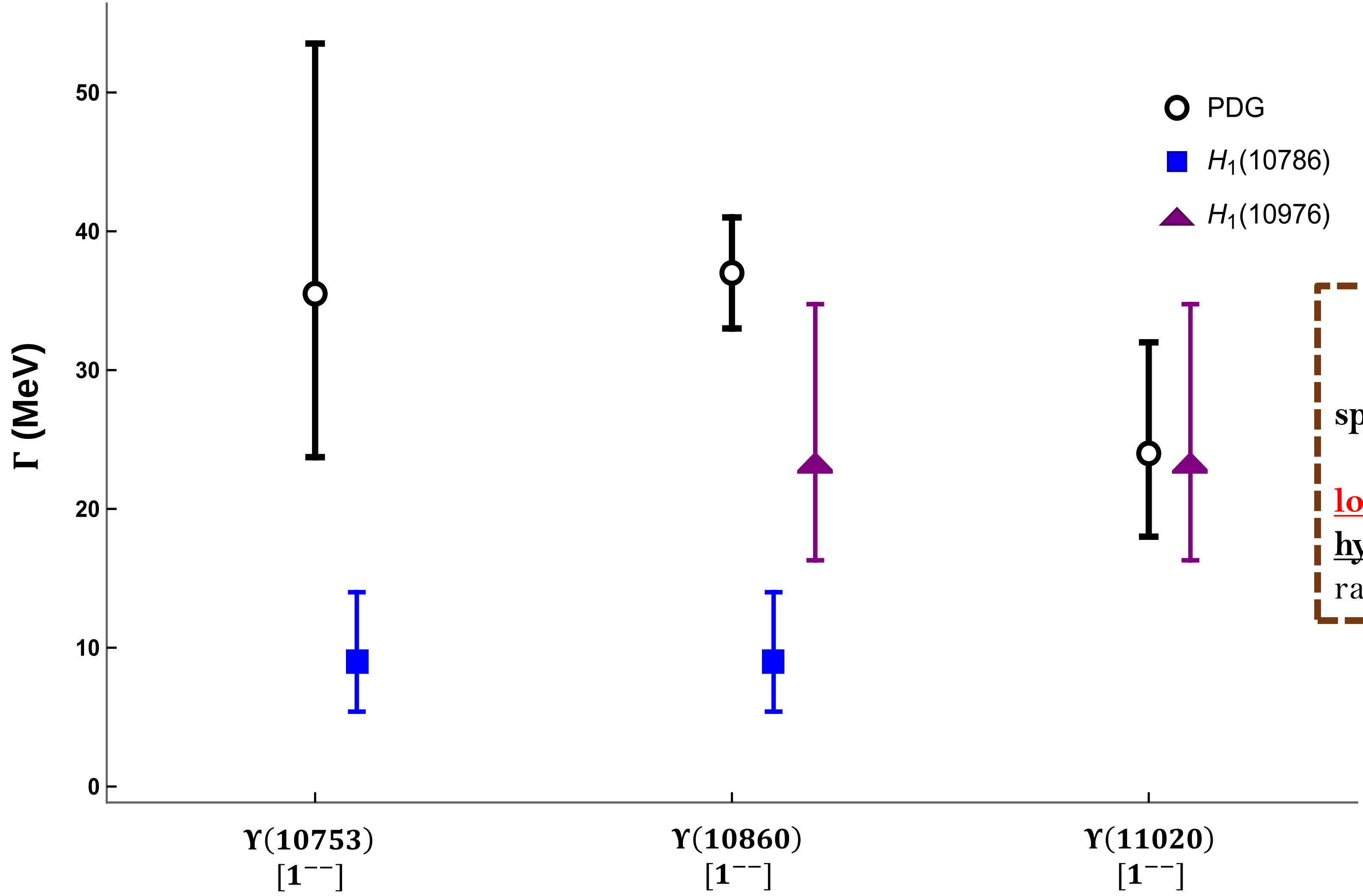
=

lower bound on the total decay widths of hybrids which is compared with inclusive rate of physical states in PDG.

Results

Brambilla, Lai, AM, Vairo Phys. Rev. D 107, 054034 (2023)

- Comparison: bottom exotic states with corresponding bottomonium hybrid state:



$$H_m \to Q_n + X$$

spin-conserving + **spin-flipping decays**

=

lower bound on the total decay widths **of hybrids** which is compared with inclusive rate of physical states in PDG.

Hybrid: Mixing with heavy-light

- **Hybrid decays to s-wave + s-wave meson pairs:**

Conventional Wisdom: Hybrid decays to two S-wave mesons forbidden!

Kou & Pene, Phys Lett B 631 (2005) Page, Phys Lett B 407 (1997) Farina, Tecocoatzi, Giachino, Santopinto & Swanson, Phys Rev D 102 (2020)

Decay allowed based on BO-quantum #

Bruschini Phys. Rev. D 109 L031501 (2024)
J. Castella JHEP 06, 107 (2024)

Hybrid

Light spin K^{PC}	Static energies $D_{\infty h}$	l	J^{PC} $\{S_Q=0, S_Q=1\}$	Multiplets
1^{+-}	$\{\Sigma_u^-, \Pi_u\}$	1	$\{1^{--}, (0,1,2)^{-+}\}$	H_1
	$\{\Pi_u\}$	1	$\{1^{++}, (0,1,2)^{+-}\}$	H_2
	$\{\Sigma_u^-\}$	0	$\{0^{++}, 1^{+-}\}$	H_3
	$\{\Sigma_u^-, \Pi_u\}$	2	$\{2^{++}, (1,2,3)^{+-}\}$	H_4
	$\{\Pi_u\}$	2	$\{2^{--}, (1,2,3)^{-+}\}$	H_5

BO-quantum # Λ_η^σ for threshold

$K_{\bar{q}}^P \otimes K_q^P$	K^{PC}	Static energies $D_{\infty h}$
$(1/2)^- \otimes (1/2)^+$	0^{-+}	$\{\Sigma_u^-\}$
	1^{--}	$\{\Sigma_g^+, \Pi_g\}$

s-wave+s-wave
Ex. **$D\bar{D}$** threshold

Σ_u^- component in hybrids couple with Σ_u^- component in s-wave+s-wave !!!!

Recent lattice computation for $c\bar{c}$ hybrid 1^{-+} **decay** to

$$D_1\bar{D} : 258(133) \text{ MeV}$$

$$D^*\bar{D} : 88(18) \text{ MeV}$$

$$D^*\bar{D}^* : 150(118) \text{ MeV}$$

Shi et al. Phys. Rev. D 109, 094513 (2024)

Hybrid: Mixing with heavy-light

Coupled-channel equations:

$$\left[-\frac{1}{m_Q r^2}\partial_r r^2 \partial_r + \frac{1}{m_Q r^2}\begin{pmatrix} l(l+1)+2 & -2\sqrt{l(l+1)} & 0 \\ -2\sqrt{l(l+1)} & l(l+1) & 0 \\ 0 & 0 & l(l+1) \end{pmatrix} + \begin{pmatrix} E_{\Sigma_u^-}(r) & 0 & g(r) \\ 0 & E_{\Pi_u}(r) & 0 \\ g(r) & 0 & E_{\Sigma_u^-{}'}(r) \end{pmatrix}\right]\begin{pmatrix}\psi_\Sigma \\ \psi_\Pi \\ \psi_{\Sigma'}\end{pmatrix} = \mathcal{E}\begin{pmatrix}\psi_\Sigma \\ \psi_\Pi \\ \psi_{\Sigma'}\end{pmatrix}$$

Berwein, Brambilla, AM, Vairo, arXiv 2408.04719

No lattice results available on $g(r)$!!!

Branching ratio for H_1 hybrid:

Braaten, Bruschini Phys. Rev. D 109, 094051 (2024)

	1^{--}	0^{-+}	1^{-+}	2^{-+}
$B\bar{B}$	1	0	0	0
$B\bar{B}^* + B^*\bar{B}$	0	2	2	2
$B^*\bar{B}^*$	3	2	2	2

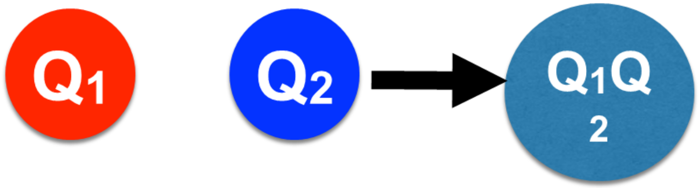
Tetraquarks
&
Pentaquarks

BOEFT: $QQ\bar{q}\bar{q}$ multiplets

Berwein, Brambilla, AM, Vairo,
arXiv 2408.04719

doubly heavy core

spin: $1/2 \otimes 1/2 = 0 \oplus 1$



color: $3 \otimes 3 = 6 \oplus 3^*$

flavor: $\{Q_1Q_2\}$ $[Q_1Q_2]$

J^P: 1^+ 0^+

light antiquarks

q q'

$\{qq'\}, 1^+$ $[qq'], 0^+$

Defines the Born-Oppenheimer static potentials $\Sigma_g^+, \{\Sigma_g^-, \Pi_g\}$

doubly heavy tetraquarks

QQ color state	Light spin K^{PC}	Static energies	Isospin I	l	J^P $S_Q = 0$	J^P $S_Q = 1$
anti-triplet $\bar{3}$	0^+	$\{\Sigma_g^+\}$	0	0	—	1^+
				1	1^-	—
	1^+	$\{\Sigma_g^-, \Pi_g\}$	1	0	0^-	—
				1	1^-	$(0,1,2)^+$

J^P for T_{cc}^+

Limited lattice inputs available on Born-Oppenheimer static potentials $\Sigma_g^+, \{\Sigma_g^-, \Pi_g\}$

Bicudo, Cichy, Peters, & Wagner
PRD 93, 034501 (2016)

BOEFT: $Q\bar{Q}q\bar{q}$ multiplets

Berwein, Brambilla, AM, Vairo,
arXiv 2408.04719

$Q\bar{Q}$ color state	Light spin K^{PC}	Static energies	l	J^{PC} $\{S_Q=0, S_Q=1\}$	Multiplets
Octet	0^{-+}	$\{\Sigma_u^-\}$	0	$\{0^{++}, 1^{+-}\}$	T_1^0
			1	$\{1^{--}, (0,1,2)^{-+}\}$	T_2^0
			2	$\{2^{++}, (1,2,3)^{+-}\}$	T_3^0
	1^{--}	$\{\Sigma_g^{+\prime}, \Pi_g\}$	1	$\{1^{+-}, (0,1,2)^{++}\}$	T_1^1
		$\{\Sigma_g^{+\prime}\}$	0	$\{0^{-+}, 1^{--}\}$	T_2^1
		$\{\Pi_g\}$	1	$\{1^{-+}, (0,1,2)^{--}\}$	T_3^1
		$\{\Sigma_g^{+\prime}, \Pi_g\}$	2	$\{2^{-+}, (1,2,3)^{--}\}$	T_4^1

J^{PC} for neutral partner of Z_c, Z_b states. Probably mixing between both channels required ?

J^{PC} for $X(3872)$

Lattice inputs only available on Born-Oppenheimer static potentials $\boldsymbol{\Sigma_u^-}$ for r/a >1

Prelovsek, Bahtiyar, & Petkovic
Phys. Lett. B 805 (2020)

BOEFT: Pentaquark multiplets

Berwein, Brambilla, AM, Vairo,

arXiv 2408.04719

$Q\bar{Q}qqq$

$Q\bar{Q}$ color state	Light spin $\boldsymbol{K}^P$	Static energies	l	J^P $\{S_Q=0, S_Q=1\}$
Octet	$(1/2)^+$	$(1/2)_g$	$1/2$	$\{1/2^-, (1/2, 3/2)^-\}$
	$(3/2)^+$	$(3/2)_g$	$3/2$	$\{3/2^-, (1/2, 3/2, 5/2)^-\}$

No lattice inputs available on Born-Oppenheimer static potentials for pentaquarks

$QQqq\bar{q}$

QQ color state	Light spin $\boldsymbol{K}^P$	heavy spin $S_Q=0$	heavy spin $S_Q=1$
sextet	$(1/2)^-$	$\{(1/2)^-\}$	$\{(1/2, 3/2)^+, (1/2, 3/2, 5/2)^+\}$
	$(3/2)^-$	$\{(3/2)^-\}$	$\{(1/2, 3/2)^+\}, \{(1/2, 3/2, 5/2)^+\}, \{(3/2, 5/2, 7/2)^+\}$
antitriplet	$(1/2)^-$	$\{(1/2)^+, (3/2)^+\}$	$\{(1/2, 3/2)^-\}$
	$(3/2)^-$	$\{(1/2)^+\}, \{(3/2)^+\}, \{(5/2)^+\}$	$\{(1/2, 3/2, 5/2)^-\}$

Summary/Outlook

- Born-Oppenheimer EFT: Tool based on QCD and Born-Oppenheimer approximation to study Exotic states.
- BOEFT: model-independent & systematic framework with inputs from lattice QCD.
- Tetraquark & Pentaquark states can be addressed in BOEFT with inputs from lattice QCD.
- Behavior of tetraquark / pentaquark static energy:
  - $Q\bar{Q}$ **systems:** Adjoint meson/ baryon behavior at small r $(\boldsymbol{r \to 0})$
  - QQ **systems:** Triplet or sextet meson / baryon behavior at small r $(\boldsymbol{r \to 0})$**.**
  - Heavy meson pair or heavy meson baryon threshold at large r $(\boldsymbol{r \to \infty})$
  - $Q\bar{Q}$ systems: Avoided crossing between tetraquark and quarkonium static energy (Isopsin=0)
- Inputs needed from lattice QCD: adjoint meson or baryon spectrum, triplet & sextet meson or baryon spectrum, computation of tetraquark & pentaquark static energies.
- Phenomenological studies of X(3872), Z_c & Z_b and T_cc are underway. Stay Tuned !!

Brambilla, Lai, AM, Vairo arXiv:2212.09187

Hybrid: Summary

- Hybrids ($Q\bar{Q}g$): Color singlet state of color octet $Q\bar{Q}$ + gluon. (Q = c, b)

  ✓ **Isoscalar neutral mesons (Isospin=0)**

  ✓ Candidates for hybrids based on **mass, quantum numbers,** and **decays** to **quarkonium:**

Charm sector:

- $\boldsymbol{X(4160)}$: could be **charm hybrid** $\boldsymbol{H_1[2^{-+}](4155)}$.
- $\boldsymbol{X(4630)}$: could be **charm hybrid** $\boldsymbol{H_1[(1/2^{-+})](4507)}$.
- $\boldsymbol{\psi(4390)}$: could be **charm hybrid** $\boldsymbol{H_1[1^{--}](4507)}$.
- $\boldsymbol{\psi(4710)}$: could be **charm hybrid** $\boldsymbol{H_1[(1^{--})](4812)}$.
- $\boldsymbol{\chi_{c1}(4685)}$: could be **charm hybrid** $\boldsymbol{H_2[(1^{++})](4667)}$.

Bottom sector:

- $\boldsymbol{\Upsilon(10753)}$: could be **bottom hybrid** $\boldsymbol{H_1[(1^{--})](10786)}$.

DISCLAIMER!!!

All the above interpretation can differ accounting for decays to **heavy- light pair threshold states** and **hybrid-quarkonium mixing**.

What is an XYZ Meson ???

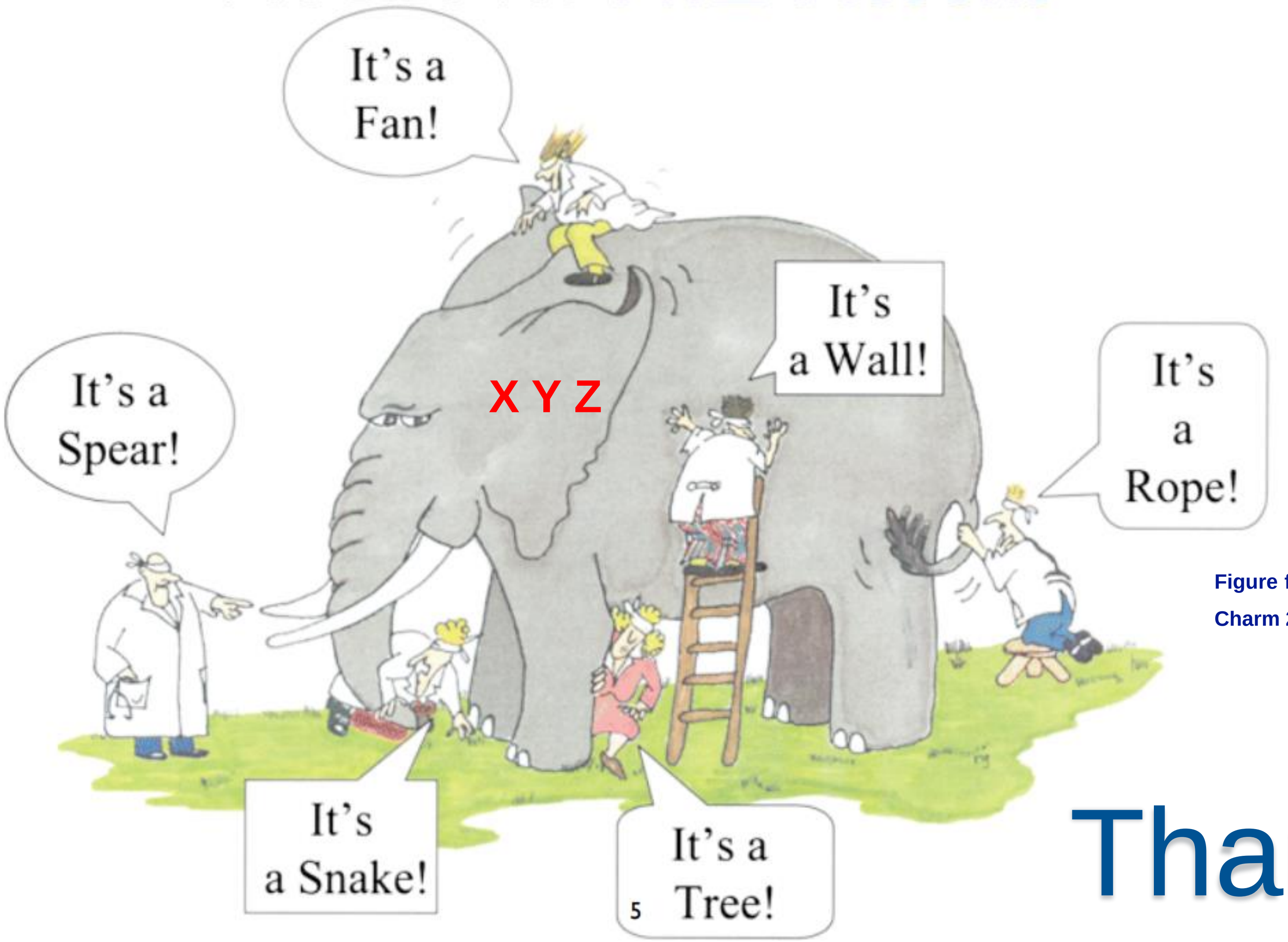


Figure from Eric Braaten talk:
Charm 2020 conference

Thank you!!

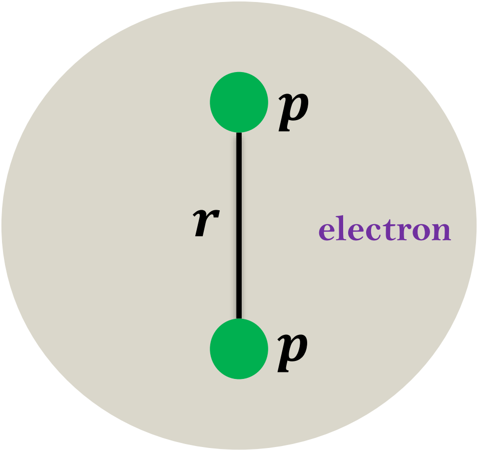
Perhaps BO-EFT can address whole picture together !!! .

Backup Slides

Born-Oppenheimer Philosophy

- Sharp difference between time or energy scales of heavy & light degrees of freedom.

Ex. H_2^+ molecule: 2 protons & 1 electron. $m_p \sim 1\,\mathrm{GeV} \gg m_e \sim 0.5\,\mathrm{MeV}$



Protons (nuclei) move very slowly compared to electrons and can be considered **static** (fixed) when considering the motion of the electrons

Electrons instantaneously adjust as **r** changes

1. Solve electron Schrödinger eq. for fixed **r**

$$H_{\mathrm{el}}(\mathbf{r})\,|\psi^i_{\mathrm{el}};\mathbf{r}\rangle = E^i_{\mathrm{el}}(r)\,|\psi^i_{\mathrm{el}};\mathbf{r}\rangle$$

$\boldsymbol{E^i_{\mathrm{el}}(r)}$: Electronic static energy

2. Solve nuclei (proton) Schrödinger eq. with $\boldsymbol{E^i_{\mathrm{el}}(r)}$ as **potential**.

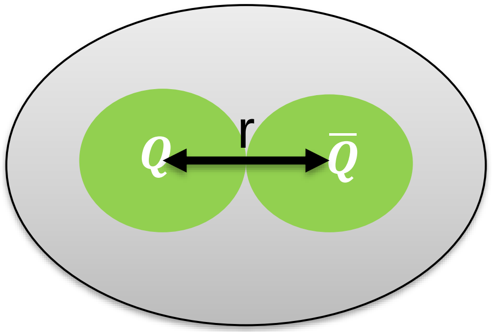
QCD states with 2-heavy quarks (XYZ mesons): analogous of molecules in atomic systems !!!

Heavy quarks ↔ nuclei **Gluons & light quarks ↔ electrons**

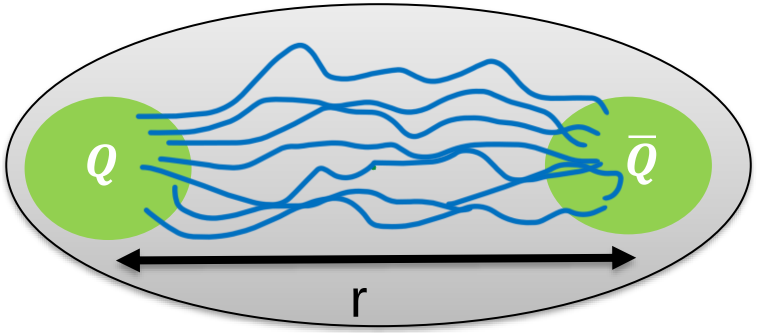
Lattice input required for gluon and light-quark energies

BOEFT: Potentials

Brambilla, AM, Vairo
arXiv 2406.xxxxx



Short-distance ($r \to 0$)



Large-distance ($r \to \infty$)

LDF-quantum #: $\kappa = \{K^{PC}, f\}$

BO-quantum #: Λ_η^σ

$$V_s(r) = -\frac{4\alpha_s}{3r}, \; V_o(r) = \frac{\alpha_s}{6r}$$

$$V_T(r) = -\frac{2\alpha_s}{3r}, \; V_\Sigma(r) = \frac{\alpha_s}{3r}$$

Short-distance behavior of BO-Potentials:

$Q\bar{Q}$: $$E^{(0)}_{\Sigma_g^+}(r) = V_s(r) + b_{\Sigma_g^+} r^2 + \cdots$$

$Q\bar{Q}X$: $$E^{(0)}_{\Lambda_\eta^\sigma}(r) = V_o(r) + \Lambda_{H_\kappa} + b_{\Lambda_\eta^\sigma} r^2 + \cdots$$

QQX: $$E^{(0)}_{\Lambda_\eta^\sigma}(r) = V_l(r) + \Lambda_{H_\kappa,l} + b_{\kappa\lambda,l}\, r^2 + \cdots$$

$$(l = T, \Sigma)$$

Long-distance behavior of BO-Potentials:

- String behavior (only pure SU(3) gauge theory)

$$E_N(r) = \sqrt{\sigma^2 r^2 + 2\pi\sigma\,(N - 1/12)}$$

K. Juge, J. Kuti, C. Morningstar, Phys. Rev. Lett. 90 (2003)

- Mixing with pair of heavy-light states based on BO-quantum number Λ_η^σ representations

BOEFT: Potentials

$$\Lambda_{H_\kappa} = \lim_{T\to\infty} \frac{i}{T} \langle \mathrm{vac}| H^a_\kappa(T/2,\, \boldsymbol{R})\, \phi^{ab}\,(T/2, -T/2)\, H^{a\dagger}_\kappa(-T/2,\, \boldsymbol{R})|\mathrm{vac}\rangle$$

Foster, Michael (UKQCD) Phys. Rev. D 59 (1999)

Campbell, Jorysz, Michael Phys. Lett. B 167 (1986)

- Gluelump / adjoint meson or baryon mass for $Q\bar{Q}X$ states
- Triplet meson or baryon / Sextet meson or baryon mass for QQX states

Gluelump masses:

J^{PC}	H	$\Lambda_H^{\mathrm{RS}} r_0$	$\Lambda_H^{\mathrm{RS}}/\mathrm{GeV}$
1^{+-}	B_i	2.25(39)	0.87(15)
1^{--}	E_i	3.18(41)	1.25(16)
2^{--}	$D_{\{i}B_{j\}}$	3.69(42)	1.45(17)
2^{+-}	$D_{\{i}E_{j\}}$	4.72(48)	1.86(19)
3^{+-}	$D_{\{i}D_jB_{k\}}$	4.72(45)	1.86(18)
0^{++}	$\mathbf{B}^2$	5.02(46)	1.98(18)
4^{--}	$D_{\{i}D_jD_kB_{l\}}$	5.41(46)	2.13(18)
1^{-+}	$(\mathbf{B}\wedge\mathbf{E})_i$	5.45(51)	2.15(20)

Bali, Pineda Phys. Rev. D. 69 (2004)

Most recent results on gluelump spectrum:

Herr, Schlosser, Wagner Phys. Rev. D 109 (2024)

Gluelump spectrum with 2+1 dynamical light quarks

Marsh, Lewis Phys. Rev. D 89 (2014):

$m\left(1^{--}\right) - m\left(1^{+-}\right) \approx 300\ \mathrm{MeV}$ $\quad m\left(2^{--}\right) - m\left(1^{+-}\right) \approx 700\ \mathrm{MeV}$

Adjoint meson spectrum (1^{--} & 0^{-+}):

Foster, Michael (UKQCD) Phys. Rev. D 59 (1999)

$m_A\left(1^{--}\right) - m_G\left(1^{+-}\right) = -10(103)\ \mathrm{MeV}$ $\quad m_A\left(0^{-+}\right) - m_G\left(1^{+-}\right) = 34(161)\ \mathrm{MeV}$

No results available on adjoint baryon, triplet meson or baryon / sextet meson or baryon masses