Compton Amplitude of the Pion using Feynman-Hellmann

Jordan Mckee, Kadir Utku Can, Ross Young, James Zanotti

The University of Adelaide
QCDSF Collaboration

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Introduction

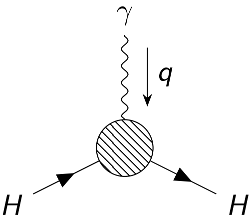
Motivation

- Interested in determining the internal electromagnetic structure of the pion
- Experimentally determined in scattering experiments (elastic or deep inelastic scattering)
- Theoretically determined using Quantum Chromodynamics (QCD)
- Due to its non-perturbative nature at low energies, take a numerical approach using Lattice QCD

Electromagnetic Structure

Determine internal structure of hadrons via scattering experiments

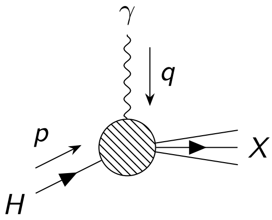
Elastic



Form Factors $F_\pi(Q^2)$

Momentum Transfer
$Q^2 = -q^2 = -(p - p')^2$

Inelastic



Structure Functions $F_{1,2}(x, Q^2)$
Compton Amplitude $\mathcal{F}_{1,2}(\omega, Q^2)$

Bjorken Scaling $x = \frac{Q^2}{2p\cdot q} = \frac{1}{\omega}$

Lattice QCD

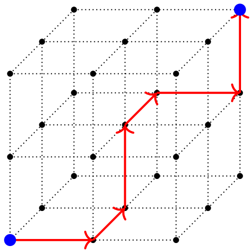
Determine physical structures theoretically using QCD

→ non-perturbative at low energies due to α_s increasing
→ Path Integral approach
→ Numerical calculations

Euclidean Path Integral Equation

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\overline{\psi}\,\mathcal{D}\psi\mathcal{D}A\ O[\overline{\psi}, \psi, A] e^{-S[\overline{\psi},\psi,A]}$$

Choice of O provides 2,3,... point correlation functions
→ Energy states of system



Lattice Spacing a
Lattice Size $N_{x,y,z,t}$

$L^3 \times T = 32^3 \times 64$

Feynman-Hellmann Theorem

Perturb system by modifying the action

$$S \rightarrow S + \lambda\mathcal{O}$$

$\rightarrow$ Energy states of the system shift

Energy Shift $$\Delta E_\lambda(\mathbf{p}) = \lambda \frac{\partial E_\lambda(\mathbf{p})}{\partial\lambda}\bigg|_{\lambda=0} + \frac{\lambda^2}{2}\frac{\partial^2 E_\lambda(\mathbf{p})}{\partial\lambda^2}\bigg|_{\lambda=0} + \mathcal{O}(\lambda^3)$$

Relate energy shift to structure

Feynman-Hellmann theorem $$\frac{\partial E_\lambda(\mathbf{p})}{\partial\lambda}\bigg|_{\lambda=0} = \frac{1}{2E(\mathbf{p})}\langle H(\mathbf{p})|\mathcal{O}|H(\mathbf{p})\rangle$$

Determine physical structure $\rightarrow$ compute at different λ shifts

Deep Inelastic Scattering

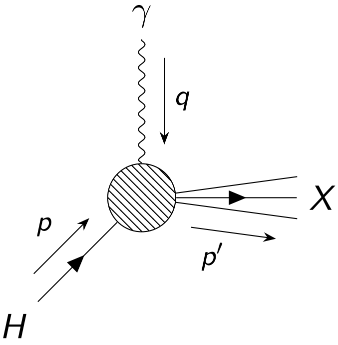
Scattering Amplitude

$$\mathcal{M} \propto \langle X | J^\mu | H(p) \rangle$$

with photon current J^μ inserted

Inclusive Cross Section

$$\sigma \propto \sum_X |\mathcal{M}|^2$$
$$\propto \sum_X \langle H(p) | J^\mu | X \rangle \langle X | j^\nu | H(p) \rangle$$
$$= \langle H(p) | J^\mu J^\nu | H(p) \rangle$$



Equivalent to Compton scattering

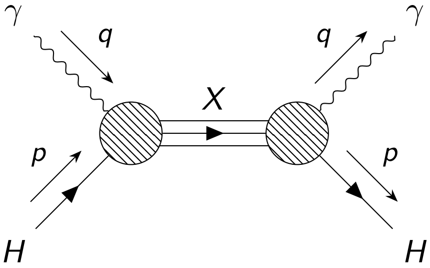
Compton Scattering

General description

Hadronic Tensor $W_{\mu\nu}$

$$\sigma \propto W_{\mu\nu}$$

Inaccessible on Euclidean lattice



Related quantity (spin-averaged, forward)

Compton Tensor $$T_{\mu\nu}(p,q) = i\int d^4 z e^{iq\cdot z}\langle p,s|\mathcal{T}\{\mathcal{J}_\mu(z),\mathcal{J}_\mu(0)\}|p,s\rangle$$

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Compton Structure Functions

To determine the general decomposition, apply constraints:

- Hermiticity
- Parity, Time-Reversal
- Ward Identity $q^\mu T_{\mu\nu} = 0$

$$T_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) \mathcal{F}_1(\omega, Q^2) + \left(p_\mu - \frac{p \cdot q}{q^2} q_\mu \right) \left(p_\nu - \frac{p \cdot q}{q^2} q_\nu \right) \frac{\mathcal{F}_2(\omega, Q^2)}{p \cdot q}$$

Isolate $\mathcal{F}_1$ by setting $\mu, \nu = 3,\ p_z = q_z = 0$

Relate Compton $\mathcal{F}_{1,2}$ to Ordinary $F_{1,2}$ via the optical theorem:

$$\mathrm{Im}\mathcal{F}_{1,2}(\omega, Q^2) = 2\pi F_{1,2}(x, Q^2)$$

Feynman-Hellmann Application

Modify QCD action according to:

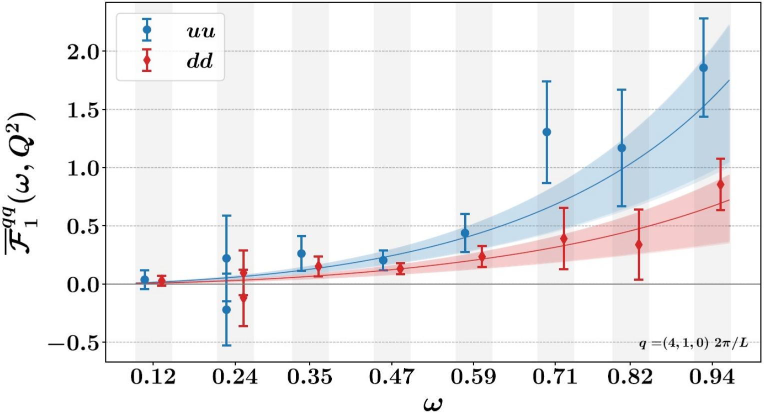
$$S(\lambda) = S + \lambda \int d^4 z (e^{iq\cdot z} + e^{-iq\cdot z}) \mathcal{J}_3(z)$$

Renormalised vector current $\mathcal{J}_3(z) = Z_V \bar{q}(z) i\gamma_3 q(z)$

Second order energy shift related to $\mathcal{F}_1$

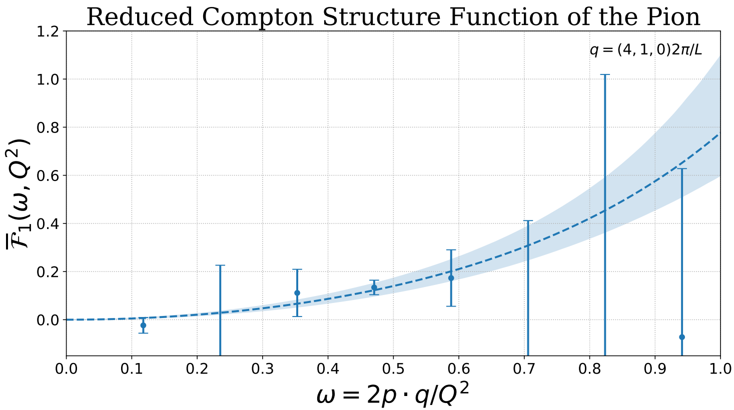
$$\left.\frac{\partial^2 E(\mathbf{p})}{\partial^2 \lambda}\right|_{\lambda=0} = -\frac{\mathcal{F}_1(\omega, Q^2)}{E(\mathbf{p})}$$

Nucleon Compton Structure Function (QCDSF Collaboration)



[K. U. Can et al. Phys. Rev. 2020]

Pion Compton Structure Function

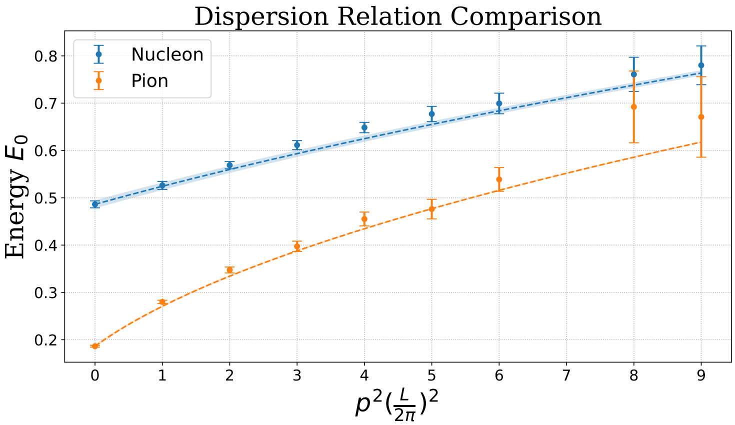


Higher uncertainty corresponds to higher momenta

$\rightarrow 2_{nd}$ point

$p = [0, 2, 0](\frac{2\pi}{L})$

Dispersion Relations



Boosted systems are more susceptible to noise

Pion becomes dominated by kinetic energy
$\rightarrow$ utilise noise reduction techniques

Noise Reduction

All Mode Averaging (AMA)

Compute solves with a lower precision (sloppy $\epsilon = 10^{-3}$) and correct using a strict ($\epsilon = 10^{-12}$) solve

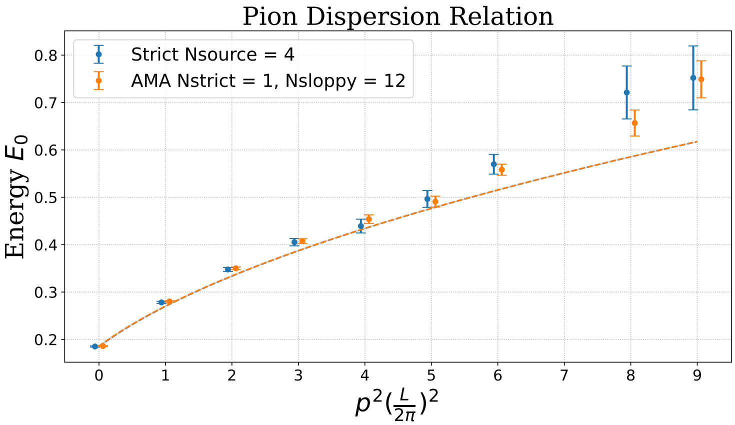
Correction Factor
$$C_{\text{corr}} = \frac{1}{N_{\text{strict}}} \sum_{i=1}^{N_{\text{strict}}} \left(C^{i}_{\text{strict}} - C^{i}_{\text{sloppy}} \right)$$

Improved Sloppy
$$C^{j}_{\text{imp}} = C^{j}_{\text{sloppy}} + C_{\text{corr}}$$

AMA
$$C_{\text{AMA}} = \frac{1}{N_{\text{sloppy}}} \left(\sum_{i=1}^{N_{\text{strict}}} C^{i}_{\text{strict}} + \sum_{j=N_{\text{strict}}+1}^{N_{\text{sloppy}}} C^{j}_{\text{imp}} \right)$$

[E. Shintani et al. Phys. Rev. 2015]

AMA Dispersion Relation



Equal cost comparison

1 Strict equivalent to 4 Sloppy

Elastic Scattering

Characterise with matrix element

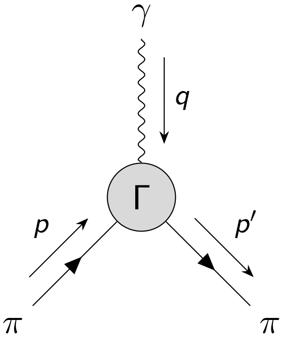
$$\langle H(p')|\mathcal{V}_\mu(0)|H(p)\rangle = \Gamma_\mu$$

Structure in Vertex function Γ_μ

general form decomposed by previous constraints

$$\Gamma_\mu = -i(p' + p)_\mu F_\pi(Q^2)$$

Pion Form Factor $F_\pi(Q^2)$
Related to transverse charge distribution



Pion Form Factors

Modify the QCD action according to:

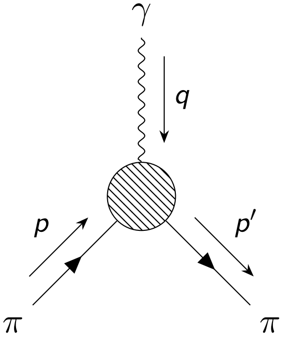
$$S(\lambda) = S + \lambda \int d^4 z (e^{iq\cdot z} + e^{-iq\cdot z}) \mathcal{V}_\mu(z)$$

Vector current $\mathcal{V}_\mu(z) = \overline{\psi}(z)\gamma_\mu\psi(z)$
choose $\mu = 4$

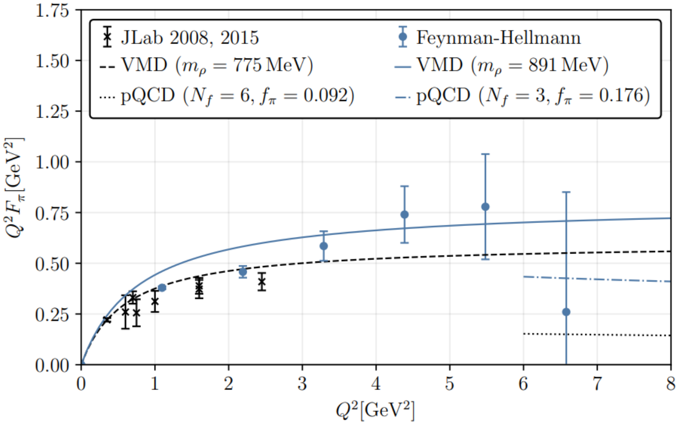
Feynman-Hellmann application

Provides first order energy shift in Breit frame $(\mathbf{p}' = -\mathbf{p})$

$$\left.\frac{\partial E_\pi(\mathbf{p})}{\partial \lambda}\right|_{\lambda=0} = F_\pi(Q^2)$$

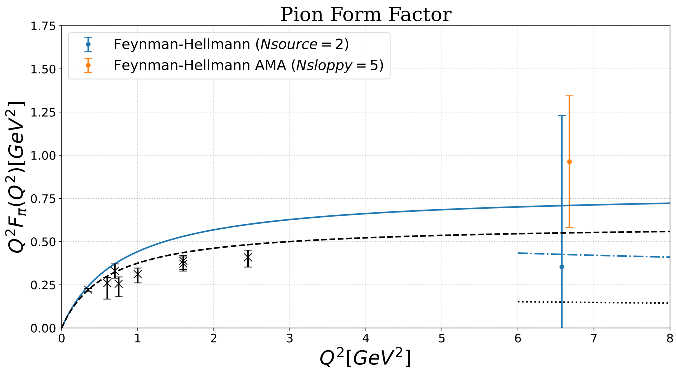


Pion Form Factor F_π (QCDSF Collaboration)



$N_{\text{conf}} = 1700$ [A. J. Chambers et al. Phys. Rev. 2017]

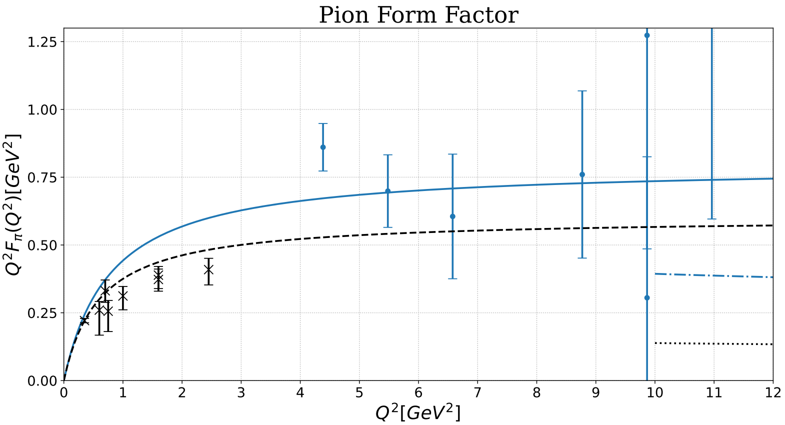
F_π Comparison



Improved analysis & half total cost using $N_{\text{conf}} = 199$

$$Q^2 F_\pi = 0.330 \pm 0.757 \quad [GeV^2] \qquad Q^2 F_{\pi,AMA} = 0.963 \pm 0.381 \quad [GeV^2]$$

AMA F_π Preliminary Results



Equal cost comparison to Alex's results using $N_{\text{conf}} = 195$, $N_{\text{sloppy}} = 18$

Where am I going from here?

- Can push this further → increase N_{sloppy}, N_{conf}
- Interested to apply this to smaller quark masses, as well as vary the lattice characteristics
- Apply AMA to the nucleon for the electric G_E and magnetic form factors G_M of the proton and determine G_E/G_M

Future Work

- Applying AMA to determine the Compton amplitude
- Combining AMA with another noise reduction technique, momentum smearing (Ian van Schalkwyk, check out his poster!)

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Thanks for Listening!

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Lattice Specifications

Gauge ensemble details

$2+1$ flavours

Lattice Size : $L^3 \times T = 32^3 \times 64$

Lattice Spacing : $a = 0.074(2) fm$

Parameters : $\beta = 5.50 \qquad \kappa_l = 0.120900 \qquad \kappa_s = 0.120900$

Masses [GeV] : $m_\pi = 0.467(12), \quad m_N = 1.250(39)$

Local Vector Renormalisation : $Z_V = 0.8611(84)$

Dispersion Relations

Dispersion relation from analiticity and Crossing symmetry

$$\mathcal{F}_1(\omega, Q^2) - \mathcal{F}_1(0, Q^2) = \frac{2\omega^2}{\pi} \int_1^\infty d\omega' \frac{Im\mathcal{F}_1(\omega', Q^2)}{\omega'(\omega'^2 - \omega^2 - i\epsilon)}$$

$$\mathcal{F}_2(\omega, Q^2) = \frac{2\omega}{\pi} \int_1^\infty d\omega' \frac{Im\mathcal{F}_2(\omega', Q^2)}{\omega'^2 - \omega^2 - i\epsilon}$$

Optical theorem relates $\mathcal{F}_1$ to F_1

$$Im\mathcal{F}_1(\omega, Q^2) = 2\pi F_1(x, Q^2)$$

Together provides the relation

$$\overline{\mathcal{F}}_1(\omega, Q^2) = 4\omega^2 \int_0^1 dx \frac{xF_1(x, Q^2)}{1 - x^2\omega^2 - i\epsilon}$$

$$\mathcal{F}_2(\omega, Q^2) = 4\omega \int_0^1 dx \frac{F_2(x, Q^2)}{1 - x^2\omega^2 - i\epsilon}$$

Dispersion Relations

Dispersion relations not particularly easy to reverse. To avoid poles the condition $|\omega| < 1$, physically this keeps the intermediate state X from becoming on-shell.

Reduced/Subtracted Compton amplitude is defined:

$$\overline{\mathcal{F}}_1(\omega, Q^2) = \mathcal{F}_1(\omega, Q^2) - \mathcal{F}_1(0, Q^2)$$

Taylor expanding at fixed Q^2 defines the moments expansion:

$$\overline{\mathcal{F}}_1(\omega, Q^2) = \sum_{n=1}^{\infty} 2\omega^{2n} M_{2n}^{(1)}(Q^2)$$

$$\mathcal{F}_2(\omega, Q^2) = \sum_{n=1}^{\infty} 4\omega^{2n-1} M_{2n}^{(2)}(Q^2)$$

Compton Amplitude Fit Function

with the moments defined by:

$$M_{2n}^{(1)}(Q^2) = 2\int_0^1 dx x^{2n-1} F_1(x, Q^2)$$

$$M_{2n}^{(2)}(Q^2) = \int_0^1 dx x^{2n-2} F_2(x, Q^2)$$

Require condition on moments

$$M_2^{(1)}(Q^2) \geq M_4^{(1)}(Q^2) \geq \cdots \geq M_{2n}^{(1)}(Q^2) \geq \cdots \geq 0$$

Utilise Bayesian Analysis to determine the Moments (distribution) I model up to order ω^8 (Moment M_8)

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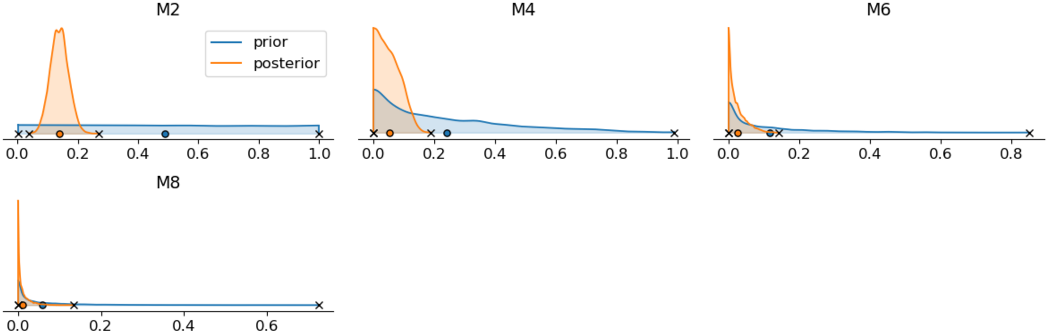
Model Coefficients

Taking a look at the distributions of the coefficients



Only the first coefficient is sufficiently isolated

$$M_2 = 0.14404 \pm^{0.02604}_{0.02558}$$

Physical Interpretations of Form Factors

Transverse Charge and Magnetisisation Densities

Form factors can be interpreted as the Fourier transforms of the transverse charge and magnetisation densities of the system

$$\rho_{E,p}(\mathbf{b}) = \int_0^\infty \frac{d|Q|}{2\pi} |Q| J_0(|Q||\mathbf{b}|) F_{1,p}(Q^2)$$

$$\rho_{M,p}(\mathbf{b}) = |\mathbf{b}| \sin^2\phi \int_0^\infty \frac{d|Q|}{2\pi} Q^2 J_1(|Q||\mathbf{b}|) F_{2,p}(Q^2)$$

J_1 cylindrical Bessel function of the first kind. ϕ angle between impact parameter b and the proton polarisation

Physical Interpretations of Form Factors

Electric and Magnetic Radii

$$\langle r^2 \rangle_{E,p} = -\frac{6}{G_{E,p}(0)} \frac{dG_{E,p}(Q^2)}{dQ^2}\bigg|_{Q^2=0}$$

$$\langle r^2 \rangle_{M,p} = -\frac{6}{G_{M,p}(0)} \frac{dG_{M,p}(Q^2)}{dQ^2}\bigg|_{Q^2=0}$$

In the forward limits

$$G_{E,p}(0) = F_{1,p}(0) = 1 \qquad \text{Charge Conservation}$$

$$G_{M,p}(0) = 1 + F_{2,p}(0) = \mu_p \qquad \text{Magnetic Moment}$$

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Pion Form Factor Models

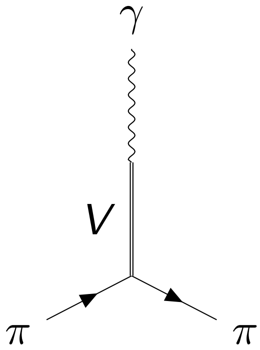
Low Q^2 Region
Vector Meson Dominance (VMD)

$$F_\pi(Q^2) \approx \frac{1}{1 + Q^2/m_\rho^2}$$

High Q^2 Region
Perturbative QCD (pQCD)

$$F_\pi(Q^2) \to \frac{16}{Q^2}\alpha_s(Q^2) f_\pi$$

Pion decay constant f_π
Strong coupling constant α_s



Form Factor Experimental Results

Jefferson Lab (JLab) CEBAF (2008, 2015)
Primary method uses pion electroproduction off the nucleon
F_π up to $2.45 GeV^2$.

Kelly (2004)
Parameterised model using experimental results

$$G_{En}(Q^2) = \frac{A\tau}{1 + B\tau} G_D(Q^2)$$

where $\tau = Q^2/4m_p^2$ and dipole form factor
$G_D(Q^2) = (1 + Q^2/\Lambda^2)^{-2}$