Formal Developments in EFTs

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EFT WG

November 16, 2023

Standard Model Effective Field Theory

Two objectives for formal developments:

- Improve computational ability
- Expose structure

$$\mathcal{L}_{\text{SMEFT}} = \mathcal{L}_{\text{SM}} + \sum_i \frac{C_i^{(6)}}{\Lambda^2}\mathcal{O}_i^{(6)} + \sum_i \frac{C_i^{(8)}}{\Lambda^4}\mathcal{O}_i^{(8)} + \dots$$

Theory developments for theorists

Theory developments for experimentalists

Theory developments for theorists

Operator basis

- Hilbert series
  [Lehman, Martin '15]
  [Henning et.al. '15-'17,'22]
  [. . .]
- Automated tools for matching
  [CoDEx, Matchmakereft, STrEAM, Matchete, SuperTracer, MatchingTools]
- Results to dimension 12
  [Li et.al. '20-'22]
  [Murphy '20]
  [Harlander et.al. '23]

On-shell methods

- On-shell basis
  [Ma et.al. '19,'21,'22]
  [Aoude et.al. '19,'20]
  [Shadmi et.al. '20]
  [. . .]
- On-shell matching
  [De Angelis, Durieux '23]
- Recursion relations
- On-shell RGE
  [Cheung, Shen '15]
  [. . .]

Theory developments for experimentalists

Entanglement

[Aoude et.al. '22,'23]

[Severi, Vryonidou '22]

[. . .]

Positivity

[Adams et.al. '06]

[Bellazzini, Riva '18]

[Riembau '22]

[Remmen, Rodd '19-'22]

[. . .]

Geometry

[Alonso et.al. '15-'17]

[Craig et.al. '20-'23]

[AH et.al. '19-'22]

[Assi et.al. '23]

[Jenkins et.al. '23]

[. . .]

Scalar sector of the SMEFT

Higgs field $H = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_2 + i\phi_1 \\ \phi_4 - i\phi_3\end{pmatrix}$

$$\mathcal{L} = \frac{1}{2} g_{IJ}(\phi)(\partial_\mu \phi^I)(\partial^\mu \phi^J) - V(\phi) + \ldots$$

- $g_{IJ}(\phi)$: metric in field space
- $V(\phi)$: potential
- $\ldots$: higher-derivative operators, e.g., $\lambda_{IJKL}(\phi)(\partial_\mu\phi)^I(\partial^\mu\phi)^J(\partial_\nu\phi)^K(\partial^\nu\phi)^L$

Field redefinitions

We can redefine the fields

$$\phi^I \to \varphi^I(\phi)$$

The derivative of the scalar transforms as a vector

$$\partial_\mu \phi^I \to \left(\frac{\delta \varphi^I(\phi)}{\delta \phi^J} \right) \partial_\mu \phi^J$$

Therefore, the metric transforms as a tensor

$$g_{IJ}(\phi) \to \left(\frac{\delta \phi^K}{\delta \varphi^I(\phi)} \right) \left(\frac{\delta \phi^L}{\delta \varphi^J(\phi)} \right) g_{KL}(\varphi)$$

Field-space geometry

Christoffel symbol

$$\Gamma^I_{JK} = \frac{1}{2} g^{IL} \left(g_{JL,K} + g_{LK,J} - g_{JK,L} \right)$$

Riemann curvature

$$R^I{}_{JKL} = \partial_K \Gamma^I_{LJ} + \Gamma^I_{KN} \Gamma^N_{LJ} - (K \leftrightarrow L)$$

Covariant derivative ∇_I

Amplitudes and geometry

Two-derivative theory

$$A_4^{i_1 i_2 i_3 i_4} = R^{i_1 i_3 i_2 i_4} s_{34} + R^{i_1 i_2 i_3 i_4} s_{24}$$

Examples: $A(W_L W_L \to W_L W_L)$ and $A(W_L W_L \to hh)$ [Alonso et.al. '15]

Amplitudes and geometry

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Examples: $A(W_L W_L \to W_L W_L)$ and $A(W_L W_L \to hh)$ [Alonso et.al. '15]

$$\begin{aligned} A_5^{i_1 i_2 i_3 i_4 i_5} =& \nabla^{i_3} R^{i_1 i_4 i_2 i_5} s_{45} + \nabla^{i_4} R^{i_1 i_3 i_2 i_5} s_{35} + \nabla^{i_4} R^{i_1 i_2 i_3 i_5} s_{25} \\ &+ \nabla^{i_5} R^{i_1 i_3 i_2 i_4} s_{34} + \nabla^{i_5} R^{i_1 i_2 i_3 i_4} (s_{24} + s_{45}) \end{aligned}$$

Amplitudes depend on geometric invariants! [Volkov '73]

Fermion Geometry

Add fermions [Assi et.al. '23]

$$\mathcal{L} = \mathcal{L}_{\text{scalar}} + \frac{1}{2} i k_{\bar{p}r}(\phi)(\bar{\psi}^{\bar{p}}\gamma^{\mu}\overleftrightarrow{D}_{\mu}\psi^{r}) + i\omega_{\bar{p}rI}(\phi)(D^{\mu}\phi)^{I}(\bar{\psi}^{\bar{p}}\gamma_{\mu}\psi^{r}) - \bar{\psi}^{\bar{p}}\mathcal{M}_{\bar{p}r}(\phi)\psi^{r}$$

Define fermion metric, where $\omega^{\pm}_{\bar{p}rI} = \omega_{\bar{p}rI} \pm \frac{1}{2}\partial_I k_{\bar{p}r}$

$$\bar{g}_{ab} = \begin{pmatrix} g_{IJ} & (\bar{\psi}\omega^{-})_{rI} & (\omega^{+}\psi)_{\bar{r}I} \\ -(\bar{\psi}\omega^{-})_{pJ} & 0 & k_{\bar{r}p} \\ -(\omega^{+}\psi)_{\bar{p}J} & -k_{\bar{p}r} & 0 \end{pmatrix}$$

Amplitudes and geometry

Fermion-scalar scattering

$$A_{pI\bar{r}J} = (\bar{u}_{\bar{r}} \not{\phi}_J u_p) \bar{R}_{\bar{r}pIJ}$$

Examples: $A(\bar{\psi}\psi \to W_L W_L)$ and $A(\bar{\psi}\psi \to hh)$ [Assi et.al. '23]

Amplitudes and geometry

Fermion-scalar scattering

$$A_{pI\bar{r}J} = (\bar{u}_{\bar{r}} \not{\phi}_J u_p) \bar{R}_{\bar{r}pIJ}$$

Examples: $A(\bar{\psi}\psi \to W_L W_L)$ and $A(\bar{\psi}\psi \to hh)$ [Assi et.al. '23]

$$A_{pI\bar{r}JK} = (\bar{u}_{\bar{r}} \not{\phi}_J u_p) \bar{\nabla}_K \bar{R}_{\bar{r}pIJ} + (\bar{u}_{\bar{r}} \not{\phi}_K u_p) \bar{\nabla}_J \bar{R}_{\bar{r}pIK}$$

LHC is measuring the Higgs geometry!

Efficiency in computations

Numerical results for Higgs decay to two photons [Hays et.al. '20]

$$
\begin{aligned}
\frac{\Gamma^{\hat{m}_W}_{\rm SMEFT}(h\to\gamma\gamma)}{\Gamma^{\hat{m}_W}_{\rm SM}(h\to\gamma\gamma)} \simeq & \quad 1 - 788 f_1^{\hat{m}_W} + 394^2\,(f_1^{\hat{m}_W})^2 - 351\,(\tilde{C}^{(6)}_{HW} - \tilde{C}^{(6)}_{HB})\, f_3^{\hat{m}_W} \\
+ & \quad 979\,\tilde{C}^{(6)}_{HD}(\tilde{C}^{(6)}_{HB} + 0.80\ \tilde{C}^{(6)}_{HW} - 1.02\,\tilde{C}^{(6)}_{HWB}) + 2228\,\delta G_F^{(6)}\ f_1^{\hat{m}_W} \\
+ & \quad 2283\,\tilde{C}^{(6)}_{HWB}(\tilde{C}^{(6)}_{HB} + 0.66\ \tilde{C}^{(6)}_{HW} - 0.88\,\tilde{C}^{(6)}_{HWB}) \\
- & \quad 788\left[\left(\tilde{C}^{(6)}_{H\Box} - \frac{\tilde{C}^{(6)}_{HD}}{4}\right) f_1^{\hat{m}_W} + f_2^{\hat{m}_W}\right] - 1224\,(f_1^{\hat{m}_W})^2
\end{aligned}
$$

where

$$
\begin{aligned}
f_1^{\hat{m}_W} \simeq & \quad f_1^{\hat{\alpha}_{ew}} = \left[\tilde{C}^{(6)}_{HB} + 0.29\ \tilde{C}^{(6)}_{HW} - 0.54\,\tilde{C}^{(6)}_{HWB}\right] \\
f_2^{\hat{m}_W} \simeq & \quad f_2^{\hat{\alpha}_{ew}} = \left[\tilde{C}^{(8)}_{HB} + 0.29\ (\tilde{C}^{(8)}_{HW} + \tilde{C}^{(8)}_{HW,2}) - 0.54\,\tilde{C}^{(8)}_{HWB}\right] \\
f_3^{\hat{m}_W} \simeq & \quad f_3^{\hat{\alpha}_{ew}} = \left[\tilde{C}^{(6)}_{HW} - \tilde{C}^{(6)}_{HB} - 0.66\,\tilde{C}^{(6)}_{HWB}\right]
\end{aligned}
$$

Efficiency in computations

All that is coming from a single component of this covariant amplitude

$$A_{ABI} = \frac{1}{2}\nabla_I g_{AB}[ab]^2$$

Renormalization Group Equations

We can calculate the RGE for the geometric quantities [AH, Jenkins, Manohar '22]

$$(16\pi^2)\mu\frac{d}{d\mu}R_{IKJL} = \frac{1}{2}\Big\{\frac{1}{4}\gamma R_{IKJL} + R_{I\ J}^{\ M\ N}\left[V_{;(MNKL)} + (t_{N;K}\cdot t_{M;L}) + (t_{N;L}\cdot t_{M;K})\right]$$
$$+ R_{IJ}^{\ \ MN}\left[-\frac{1}{6}(t_{N;M}\cdot t_{K;L}) + 4(t_{N;L}\cdot t_{M;K})\right]$$
$$+ R_{IK}^{\ \ MN}\left[-\frac{1}{6}(t_{N;M}\cdot t_{J;L}) + 4(t_{N;L}\cdot t_{M;J})\right]$$
$$+(I\leftrightarrow K, J\leftrightarrow L) - (J\leftrightarrow L) - (I\leftrightarrow K)\Big\} + (I\leftrightarrow J, K\leftrightarrow L)$$

where

$$(t_{I;J}\cdot t_{K;L}) = (\nabla_J t_{AI})g^{AB}(\nabla_L t_{BK})$$

1-loop divergence

The 1-loop divergence can be calculated from the second variation of the action

$$\delta^2 S = \frac{1}{2}\int d^4x \left[g_{IJ}(\mathcal{D}_\mu\eta)^I(\mathcal{D}^\mu\eta)^J + X_{IJ}\eta^I\eta^J\right]$$

The infinite part of the 1-loop functional integral is ['t Hooft, '73]

$$\Delta S = \frac{1}{32\pi^3\epsilon}\int d^4x \left[\tfrac{1}{12}\mathrm{Tr}[Y_{\mu\nu}Y^{\mu\nu}] + \tfrac{1}{2}\mathrm{Tr}[X^2]\right]$$

where

$$[Y_{\mu\nu}]^I{}_J = [\mathcal{D}_\mu, \mathcal{D}_\mu]^I{}_J$$

1-loop divergence for scalars

The 1-loop divergence can be calculated from the second variation of the action

$$\delta^2 S = \frac{1}{2}\int d^4x \left[g_{IJ}(\mathcal{D}_\mu \eta)^I (\mathcal{D}^\mu \eta)^J - R_{IkJL}(D_\mu \phi)^K (D^\mu \phi)^L \eta^I \eta^J - (\nabla_I \nabla_J V)\eta^I \eta^J \right]$$

Covariant geometric quantities

$$[Y_{\mu\nu}]^I{}_J = R^I{}_{JKL}(D_\mu \phi)^K (D_\nu \phi)^L + \nabla_J t^I_C F^C_{\mu\nu}$$

1-loop effective action for scalars and gauge fields

The 1-loop divergence can be calculated from the second variation of the action

$$\delta^2 S = \frac{1}{2}\int d^4x \left[g_{ij}(\mathcal{D}_\mu \eta)^i(\mathcal{D}^\mu \eta)^j + X_{ij}\eta^i\eta^j\right]$$

Covariant geometric quantities

$$[Y_{\mu\nu}]^i{}_j = \bar{R}^i{}_{jkl}(D_\mu Z)^k(D_\nu Z)^l + \bar{\nabla}_j \bar{t}^i_C F^C_{\mu\nu}$$

$$(D_\mu Z)^i = \begin{bmatrix}(D_\mu \phi)^I \\ F^A_{\mu\nu}\end{bmatrix}$$

SMEFT RGE at dimension eight

$$
\begin{aligned}
{}^{8}\dot{C}^{(1)}_{H^6D^2} =& -96\ {}^{6}C_{H^6}{}^{6}C_{H^4\Box} - 12\ {}^{6}C_{H^6}{}^{6}C_{H^4D^2} + \left(352\lambda + 20g_1^2 + \frac{20}{3}g_2^2\right)\left({}^{6}C_{H^4\Box}\right)^2 \\
&+ \left(-23\lambda + \frac{1}{8}g_1^2 + \frac{161}{24}g_2^2\right)\left({}^{6}C_{H^4D^2}\right)^2 + \left(-64\lambda - 2g_1^2 + 12g_2^2\right){}^{6}C_{H^4\Box}{}^{6}C_{H^4D^2} \\
&- 22g_2^2\ {}^{6}C_{H^4\Box}{}^{6}C_{W^2H^2} + 6g_1^2\ {}^{6}C_{H^4\Box}{}^{6}C_{B^2H^2} - \frac{32}{3}g_1g_2\ {}^{6}C_{H^4\Box}{}^{6}C_{WBH^2} \\
&+ 8g_2^2\ {}^{6}C_{H^4D^2}{}^{6}C_{W^2H^2} + 6g_1^2\ {}^{6}C_{H^4D^2}{}^{6}C_{B^2H^2} + \frac{43}{3}g_1g_2\ {}^{6}C_{H^4D^2}{}^{6}C_{WBH^2} \\
&+ 512\lambda\left({}^{6}C_{G^2H^2}\right)^2 + \left(192\lambda + 4g_2^2\right)\left({}^{6}C_{W^2H^2}\right)^2 + \left(64\lambda + 12g_1^2\right)\left({}^{6}C_{B^2H^2}\right)^2 \\
&+ \left(-3g_1^2 - 3g_2^2\right)\left({}^{6}C_{WBH^2}\right)^2 + \frac{80}{3}g_1g_2\ {}^{6}C_{W^2H^2}{}^{6}C_{WBH^2} + \frac{8}{3}g_1g_2\ {}^{6}C_{B^2H^2}{}^{6}C_{WBH^2} \\
&+ \left(68\lambda + \frac{1}{2}g_1^2 - \frac{31}{6}g_2^2\right){}^{8}C^{(1)}_{H^6D^2} + \left(-8\lambda + 7g_1^2 + \frac{17}{3}g_2^2\right){}^{8}C^{(2)}_{H^6D^2},
\end{aligned}
$$

Outlook

Summary:

- Scattering amplitudes depend on geometry
- Geometry exposes structure and simplifies computations
- LHC is measuring the Higgs geometry!

Thank you!

Amplitudes and geometry

Four-derivative interactions

$$A_{4,\lambda}^{i_1 i_2 i_3 i_4} = \frac{1}{2}\lambda^{i_1 i_2 i_3 i_4} s_{12} s_{34} + \frac{1}{2}\lambda^{i_1 i_3 i_2 i_4} s_{13} s_{24} + \frac{1}{2}\lambda^{i_2 i_3 i_1 i_4} s_{23} s_{14}$$

Amplitudes and geometry

Four-derivative interactions

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$$A_{5,\lambda}^{i_1 i_2 i_3 i_4 i_5} = \frac{1}{2}\nabla^{i_5}\lambda^{i_1 i_2 i_3 i_4} s_{12} s_{34} + \frac{1}{2}\nabla^{i_5}\lambda^{i_1 i_3 i_2 i_4} s_{13} s_{24} + \frac{1}{2}\nabla^{i_5}\lambda^{i_2 i_3 i_1 i_4} s_{23} s_{14} + \text{cyclic}$$

Amplitudes depend on geometric invariants!

Geometric Soft Theorem

A new soft theorem for any theory of scalars (no potential)

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$$\lim_{q\to 0} A_{n+1}^{i_1\cdots i_n i} = \nabla^i A_n^{i_1\cdots i_n}$$

Geometric Soft Theorem

A new soft theorem for any theory of scalars (no potential)

$$\lim_{q\to 0} A_{n+1}^{i_1\cdots i_n i} = \nabla^i A_n^{i_1\cdots i_n}$$

With a potential

$$\lim_{q\to 0} A_{n+1}^{i_1\cdots i_n i} = \nabla^i A_n^{i_1\cdots i_n} + \sum_{a=1}^{n} \frac{\nabla^i V^{i_a}{}_{j_a}}{(p_a+q)^2 - m_{j_a}^2} \left(1 + q^\mu \frac{\partial}{\partial p_a^\mu}\right) A^{i_1\cdots j_a\cdots i_n}$$

Field basis invariance

Even though the soft theorem contains a covariant derivative (which acts on masses) and $q^\mu \frac{\partial}{\partial p^\mu}$ (which acts on momenta), the full soft theorem preserves the on-shell condition

$$\left(\nabla^i + \sum \frac{\nabla^i V}{(p+q)^2 - m^2}\left(1 + q\frac{\partial}{\partial p}\right)\right)(p_a^2 \delta^{i_a}_{j_a} - V^{i_a}_{j_a})\mathcal{O}^{i_1 \cdots j_a \cdots i_n} \stackrel{!}{=} 0$$

Examples: two-derivative theory

Soft limit of four-particle amplitude

$$\lim_{p_4\to 0} A_4^{i_1 i_2 i_3 i_4} = \lim_{p_4\to 0}\left(R^{i_1 i_3 i_2 i_4} s_{34} + R^{i_1 i_2 i_3 i_4} s_{24}\right) = 0 = \nabla^{i_4} A_3^{i_1 i_2 i_3}$$

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Soft limit of five-particle amplitude

$$\begin{aligned}\lim_{p_5\to 0} A_5^{i_1 i_2 i_3 i_4 i_5} = \lim_{p_5\to 0}\Big(&\nabla^{i_3} R^{i_1 i_4 i_2 i_5} s_{45} + \nabla^{i_4} R^{i_1 i_3 i_2 i_5} s_{35} + \nabla^{i_4} R^{i_1 i_2 i_3 i_5} s_{25} \\ &+ \nabla^{i_5} R^{i_1 i_3 i_2 i_4} s_{34} + \nabla^{i_5} R^{i_1 i_2 i_3 i_4}(s_{24} + s_{45})\Big) \\ =&\nabla^{i_5}\left(R^{i_1 i_3 i_2 i_4} s_{34} + R^{i_1 i_2 i_3 i_4} s_{24}\right) = \nabla^{i_5} A_4^{i_1 i_2 i_3 i_4}\end{aligned}$$

Examples: potential

Soft limit of four-particle amplitude

$$\lim_{p_4\to 0} A_4^{i_1 i_2 i_3 i_4} = \lim_{p_4\to 0}\left(-V^{i_1 i_2 i_3 i_4} - \sum_j \left(\frac{V^{i_1 i_2 j} V_j^{i_3 i_4}}{s_{12}-m_j^2} + \frac{V^{i_1 i_3 j} V_j^{i_2 i_4}}{s_{13}-m_j^2} + \frac{V^{i_1 i_4 j} V_j^{i_2 i_3}}{s_{14}-m_j^2}\right)\right)$$

$$= \nabla^{i_4} A_3^{i_1 i_2 i_3} + \sum_{a=1}^{3} \frac{\nabla^{i_4} V_{j_a}^{i_a}}{(p_a+p_4)^2 - m_{j_a}^2} A^{i_1\cdots j_a \cdots i_3}$$

Examples: higher-derivative interactions

Soft limit of four-particle amplitude

$$\lim_{p_4\to 0} A_{4,\lambda}^{i_1 i_2 i_3 i_4} = \lim_{p_4\to 0}\left(\frac{1}{2}\lambda^{i_1 i_2 i_3 i_4} s_{12}s_{34} + \frac{1}{2}\lambda^{i_1 i_3 i_2 i_4} s_{13}s_{24} + \frac{1}{2}\lambda^{i_2 i_3 i_1 i_4} s_{23}s_{14}\right) = 0 = \nabla^{i_4} A_{3,\lambda}^{i_1 i_2 i_3}$$

Examples: higher-derivative interactions

Soft limit of four-particle amplitude

$$\lim_{p_4\to 0} A^{i_1 i_2 i_3 i_4}_{4,\lambda} = \lim_{p_4\to 0}\left(\frac{1}{2}\lambda^{i_1 i_2 i_3 i_4} s_{12}s_{34} + \frac{1}{2}\lambda^{i_1 i_3 i_2 i_4} s_{13}s_{24} + \frac{1}{2}\lambda^{i_2 i_3 i_1 i_4} s_{23}s_{14}\right) = 0 = \nabla^{i_4} A^{i_1 i_2 i_3}_{3,\lambda}$$

Soft limit of five-particle amplitude

$$\lim_{p_5\to 0} A^{i_1 i_2 i_3 i_4 i_5}_{5} = \lim_{p_5\to 0}\left(\frac{1}{2}\nabla^{i_5}\lambda^{i_1 i_2 i_3 i_4} s_{12}s_{34} + \frac{1}{2}\nabla^{i_5}\lambda^{i_1 i_3 i_2 i_4} s_{13}s_{24} + \frac{1}{2}\nabla^{i_5}\lambda^{i_2 i_3 i_1 i_4} s_{23}s_{14} + \text{cyclic}\right)$$

$$= \nabla^{i_5}\left(\frac{1}{2}\lambda^{i_1 i_2 i_3 i_4} s_{12}s_{34} + \frac{1}{2}\lambda^{i_1 i_3 i_2 i_4} s_{13}s_{24} + \frac{1}{2}\lambda^{i_2 i_3 i_1 i_4} s_{23}s_{14}\right) = \nabla^{i_5} A^{i_1 i_2 i_3 i_4}_{4}$$

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Example: pions

Two-derivative theory with vanishing soft limit [Adler '65]
Soft limit of $(n+1)$-particle amplitude

$$\lim_{q\to 0} A_{n+1}^{i_1\cdots i_n i} = \nabla^i A_n^{i_1\cdots i_n} \stackrel{!}{=} 0$$

Equivalent to $\nabla^{i_5} R^{i_1 i_2 i_3 i_4} = 0$. Adler zero holds for symmetric cosets, where the curvature is covariantly constant.

Example: Nambu-Goldstone bosons

Take a theory with some nonlinearly realized symmetry and a vanishing potential. The Lagrangian is invariant under the symmetry transformation

$$\phi^I \to \phi^I + \mathcal{K}^I(\phi), \qquad \mathcal{K}^I : \text{Killing vector}$$

Soft theorem can be recast

$$\lim_{q\to 0} \mathcal{K}_i A_{n+1}^{i_1\cdots i_n i} = \mathcal{K}_i \nabla^i A_n^{i_1\cdots i_n} \stackrel{!}{=} \sum_{a=1}^{n} \nabla_{j_a} \mathcal{K}^{i_a} A_n^{i_1\cdots j_a \cdots i_n}$$

(e.g. [Kampf, Novotny, Shifman, Trnka, '20])

Double soft theorem

We also have a new double soft theorem (vanishing potential)

$$\lim_{q_a,q_b\to 0} A_{n+2}^{i_1\cdots i_n i_a i_b} = \frac{1}{2}\sum_{c\neq a,b}\frac{s_{ac}-s_{bc}}{s_{ac}+s_{bc}} R^{i_a i_b i_c}{}_{j_c} A_n^{i_1\cdots j_c\cdots i_n} + \nabla^{(i_a}\nabla^{i_b)} A_n^{i_1\cdots i_n}$$

This generalizes the double-soft theorem for pions [Arkani-Hamed, Cachazo, Kaplan, '08]

Triple soft theorem

We also have a new triple soft theorem (vanishing potential)

$$\lim_{q_a,q_b,q_c\to 0} A_{n+3}^{i_1\cdots i_n i_a i_b i_c}$$
$$= \sum_{d\neq a,b,c}\left(\frac{1}{2}\frac{s_{ad}-s_{bd}}{s_{ad}+s_{bd}}R^{i_a i_b i_d}{}_{j_d}\nabla^{i_c}+\frac{1}{3}\frac{s_{ad}-s_{bd}}{s_{ad}+s_{bd}+s_{cd}}\nabla^{i_c}R^{i_a i_b i_d}{}_{j_d}\right)A_n^{i_1\cdots j_d\cdots i_n}$$
$$+\frac{1}{3}\frac{s_{ac}-s_{bc}}{s_{ab}+s_{ac}+s_{bc}}R^{i_a i_b i_c}{}_{j_d}\nabla^{j_d}A_n^{i_1\cdots i_n}+(a\leftrightarrow c)+(b\leftrightarrow c)$$
$$+\nabla^{(i_a}\nabla^{i_b}\nabla^{i_c)}A_n^{i_1\cdots i_n}$$

This has no analog for pions.

Gauge fields

We can add gauge fields to the Lagrangian

$$\mathcal{L} = \frac{1}{2} g_{IJ}(\phi)(D_\mu \phi)^I (D^\mu \phi)^J - \frac{1}{4} g_{AB}(\phi) F^A_{\mu\nu} F^{B\mu\nu}$$

with

$$(D_\mu \phi)^I = \partial_\mu \phi^I + A^B_\mu t^I_B$$

$t^I_B(\phi)$: Killing vector

Combined field-space metric

We now combine the two metrics g_{IJ} and g_{AB}

$$\bar{g}_{ij} = \begin{bmatrix} g_{IJ} & 0 \\ 0 & -g_{AB}\eta_{\mu\nu} \end{bmatrix}$$

From this metric, we define a covariant derivative $\bar{\nabla}$ and a curvature $\bar{R}_{ijkl}$

Scattering Amplitudes

From this we calculate the scattering of two gauge bosons + scalars from g_{IJ}, g_{AB}

$$A_{ABI}^{++} = \frac{1}{2}\nabla_I g_{AB}[12]^2$$

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$$A^{++}_{ABI} = \frac{1}{2}\nabla_I g_{AB}[12]^2$$

$$A^{++}_{ABIJ} = \frac{1}{2}\bar{\nabla}_I \nabla_J g_{AB}[12]^2$$

Scattering Amplitudes

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$$A^{++}_{ABI} = \tfrac{1}{2}\nabla_I g_{AB}[12]^2$$

$$A^{++}_{ABIJ} = \tfrac{1}{2}\bar{\nabla}_I \nabla_J g_{AB}[12]^2$$

$$A^{++}_{ABIJK} = \tfrac{1}{2}\bar{\nabla}_{(I}\bar{\nabla}_J \nabla_{K)} g_{AB}[12]^2 + \tfrac{1}{6}\nabla^L g_{AB}[12]^2 \frac{1}{s_{345}}\left[s_{34}(R_{IJKL} - 2R_{IKJL}) + s_{35}(R_{IKJL} - 2R_{IJKL}) + s_{45}(R_{IJKL} + R_{IKJL})\right]$$

Geometric Soft Theorem

A new soft theorem for any theory of scalars + gauge bosons

$$\lim_{q\to 0} A_{n+1}^{i_1\cdots i_n i} = \bar{\nabla}^i A_n^{i_1\cdots i_n}$$

Examples

We take the soft limit for four particles

$$\lim_{p_4 \to 0} A^{++}_{ABIJ} = \frac{1}{2}\bar{\nabla}_I \nabla_J g_{AB}[12]^2 = \bar{\nabla}_J A^{++}_{ABI}$$

Examples

We take the soft limit for four particles

$$\lim_{p_4 \to 0} A^{++}_{ABIJ} = \frac{1}{2}\bar{\nabla}_I \nabla_J g_{AB}[12]^2 = \bar{\nabla}_J A^{++}_{ABI}$$

We take the soft limit for five particles

$$\lim_{p_5 \to 0} A^{++}_{ABIJK} = \lim_{p_5 \to 0} \frac{1}{2}\bar{\nabla}_{(I}\bar{\nabla}_J \nabla_{K)} g_{AB}[12]^2$$
$$+ \frac{1}{6}\nabla^L g_{AB}[12]^2 \frac{1}{s_{345}} \left[s_{34}(R_{IJKL} - 2R_{IKJL}) + s_{35}(R_{IKJL} - 2R_{IJKL}) + s_{45}(R_{IJKL} + R_{IKJL})\right]$$
$$= \bar{\nabla}_K A^{++}_{ABIJ}$$

Fields vs particles

First, we expand around the vev

$$\phi^I = v^I + \bar{\phi}^I$$

At the vev, we can flatten the metric and diagonalize the mass matrix using tetrads

$$g_{IJ}(v)e_i^I(v)e_j^J(v) = \delta_{ij},$$
$$V_{IJ}(v)e_i^I(v)e_j^J(v) = m_i^2\delta_{ij}$$

The tetrad is the overlap of the mass eigenstate with the scalar field

$$\langle p^i|\bar{\phi}^J(x)|0\rangle = e^{iJ}(v)e^{ip\cdot x}$$

LSZ reduction and tetrads

The scattering amplitude is

$$A_n^{i_1\cdots i_n}(p_1,\cdots,p_n)\delta^D(p_1+\cdots+p_n)=\langle p_1^{i_1}\cdots p_n^{i_n}|0\rangle$$

which is related to the correlator using LSZ reduction

$$\langle p_1^{i_1}\cdots p_n^{i_n}|0\rangle=(-i)^{n+1}\left[\prod_{a=1}^{n}\lim_{p_a^2\to m_{i_a}^2}(p_a^2-m_{i_a}^2)e_{I_a}^{i_a}\right]\langle T\bar{\phi}^{I_1}(p_1)\cdots\bar{\phi}^{I_n}(p_n)\rangle$$

Wavefunction normalization factors are tetrads!

Proof of theorem

Start with the Euler-Lagrange equations

$$\partial_\mu \mathcal{J}_I^\mu = \frac{\delta L}{\delta \phi^I} = \partial_I L \qquad \text{where} \qquad \mathcal{J}_I^\mu = \frac{\delta L}{\delta \partial_\mu \phi^I}$$

Inserting this into correlators gives

$$\langle \partial_\mu \mathcal{J}_I^\mu \rangle = \lim_{q \to 0} A_{I_1 \cdots I_n I} - \tfrac{1}{2} \langle \partial_K g_{IJ} \partial_\mu (\phi^J \overleftrightarrow{\partial}^\mu \phi^K) \rangle_{\text{ext}}$$

$$\langle \partial_I L \rangle = \partial_I A_{I_1 \cdots I_n} + \tfrac{1}{2} \langle \partial_I g_{JK} \partial_\mu \phi^J \partial^\mu \phi^K - \partial_I \partial_J \partial_K V \phi^J \phi^K \rangle_{\text{ext}}$$

On-shell recursion

We can use the soft theorem in recursion relations

$$A_n(0) = \frac{1}{2\pi i} \oint \frac{dz}{z} \frac{A_n(z)}{F(z)} = -\sum_{\alpha} \mathrm{Res}_{z=z_\alpha^\pm} \left(\frac{A_n(z)}{zF(z)} \right)$$