


# Accelerating the search for mass bumps using the Data-Directed Paradigm

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## Introduction

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What? → Train a neural network to **identify mass bumps in real data** without the need of simulation or analytical fit to estimate the background

- Rapidly scan many different regions of the observable-space
- Complementary to the standard analysis approach

Why? → Exploit the **discovery potential of the data**

- Impossible to cover all possible searches with the traditional analysis
- Many possible resonances in unexplored final states

Existing searches for two-body resonances<sup>[1]</sup>

	$e$	$\mu$	$\tau$	$q/g$	$b$	$t$	$\gamma$	$Z/W$	$H$	BSM → SM <sub>1</sub> × SM <sub>1</sub>				BSM → SM <sub>1</sub> × SM <sub>2</sub>			BSM → complex		
										$q/g$	$\gamma/\pi^0s$	$b$	...	$tZ/H$	$bH$	...	$\tau q\bar{q}'$	$e q\bar{q}'$	$\mu q\bar{q}'$
$e$	[37,38]	[39,40]	[39]	∅	∅	∅	[41]	[42]	∅	∅	∅	∅	∅	∅	∅	∅	∅	[43,44]	∅
$\mu$		[37,38]	[39]	∅	∅	∅	[41]	[42]	∅	∅	∅	∅	∅	∅	∅	∅	∅	∅	[43,44]
$\tau$			[45,46]	∅	[47]	∅	∅	∅	∅	∅	∅	∅	∅	∅	∅	∅	∅	[48,49]	∅
$q/g$				[29,30,50,51]	[52]	∅	[53,54]	[55]	∅	∅	∅	∅	∅	∅	∅	∅	∅	∅	∅
$b$					[29,52,56]	[57]	[54]	[58]	[59]	∅	∅	∅	∅	∅	[60]	∅	∅	∅	∅
$t$						[61]	∅	[62]	[63]	∅	∅	∅	∅	∅	[64]	[60]	∅	∅	∅
$\gamma$							[65,66]	[67-69]	[68,70]	∅	∅	∅	∅	∅	∅	∅	∅	∅	∅
$Z/W$								[71]	[71]	∅	∅	∅	∅	∅	∅	∅	∅	∅	∅
$H$								[72,73]	[74]	∅	∅	∅	∅	∅	∅	∅	∅	∅	∅
$q/g$										∅	∅	∅	∅	∅	∅	∅	∅	∅	∅
BSM → SM <sub>1</sub> × SM <sub>1</sub>										∅	∅	∅	∅	∅	∅	∅	∅	∅	∅
$\gamma/\pi^0s$										[75]	∅	∅	∅	∅	∅	∅	∅	∅	∅
$b$											[76,77]	∅	∅	∅	∅	∅	∅	∅	∅
...																			
...																			

[1] J. H. Kim et al., J. High Energy. Phys. 2020, 30 (2020), arXiv:1907.06659 [hep-ph]

## Data-Directed Paradigm

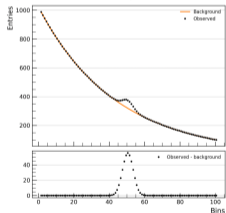
- The **Data Directed Paradigm (DDP)** is a search strategy to efficiently identify regions of interest in the data. It requires two ingredients:

Property of the SM  
on which deviations can be searched for

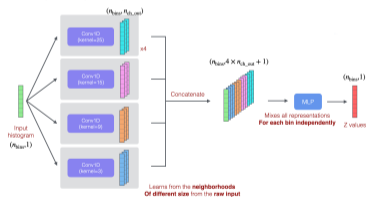
+

Tool to scan the observable-space  
in search for deviations

e.g. smoothly falling invariant mass



e.g. NN getting statistical significance for bumps



- Proof of concept performed for **symmetries**<sup>[2]</sup> and **bump searches**<sup>[3]</sup>

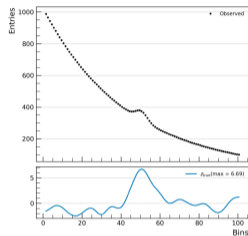
[2] S. Volkovich et al., Eur. Phys. J. C **82**, 265 (2022), arXiv:2107.11573 [hep-ex]

[3] M. Birman et al., Eur. Phys. J. C **82**, 508 (2022), arXiv:2203.07529 [hep-ph]

## Bump search with DDP

- Bump search performed with a **network mapping invariant mass distribution to statistical significance**

- **Input:** vector of bin entries from invariant mass histogram
- **Target:** vector of statistical significance  $Z$  from likelihood-ratio test
- Representative of an ideal analysis without modelling uncertainties



- Exploit its full potential with as **many final states** as possible
  - Multiple combinations of **objects** at the LHC
  - **Selections** on variables such as  $H_T$ ,  $E_{Tmiss}$ , leading object  $p_T$ , etc

Electron	Leptonic Z
Muon	Boosted hadronic W/Z
Photon	Boosted top
Jet	High mass jet ( $m > 200$ GeV)

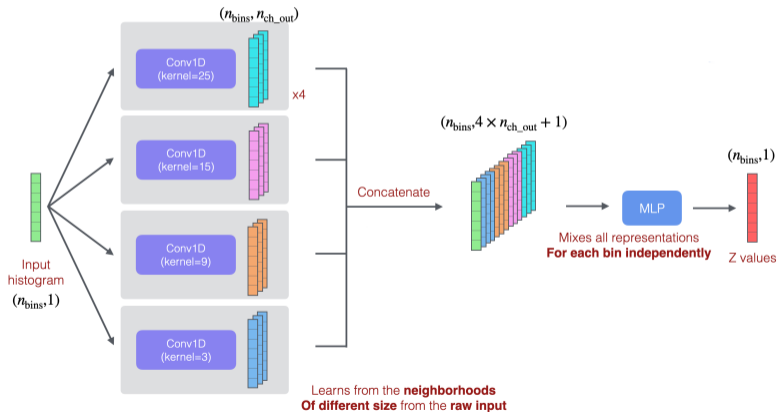
## Neural network and synthetic data generation

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# Implementation

## Architecture:

- Use of 1D convolution layers followed by a dense layer
- Intuitive and agnostic to the number of bins in the histogram



# Synthetic data generation

## ➤ Data generation workflow:

### 1. Obtain background shapes

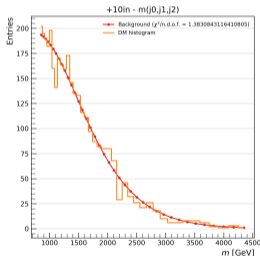
#### ➤ Analytical functions

$$be^{-ax}, \quad ax + b, \quad \frac{1}{ax} + b, \quad \frac{1}{ax^2} + b, \quad \frac{1}{ax^3} + b,$$

$$\frac{1}{ax^4} + b, \quad a(x - x_2)^2 + y_2, \quad -a \cdot \ln(x) + b,$$

$$(y_1 - y_2) \cos(a(x - b)) + y_2, \quad \cosh(a(x - x_2)) + b.$$

#### ➤ Fits to simulation data

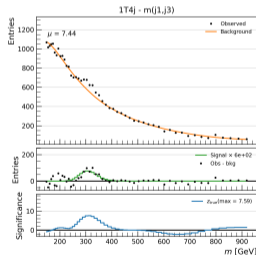


## ➤ Training data:

- Histograms with 30 to 100 bins
- Broad dynamic range, from 10 to 100k entries per bin
- Signals injected from broad significance range, from 1 to 20σ

### 2. Inject signal

- Select **background**
- Generate **Gaussian signal**
- Combine both into **observed histogram**
- Poisson fluctuate the histogram
- Calculate **true significance** with likelihood-ratio test





## Dark Machines sample description

- Using the **Dark Machines** dataset<sup>[4]</sup>
  - Designed to test anomaly detection techniques
  - Contains all of the **highest cross-section processes** at the LHC
  - Generation with Madgraph and Pythia, including fast detector simulation using Delphes
  - Events divided into signal regions/channels e.g. channel 3, which is more inclusive with cuts on  $E_{Tmiss} > 100$  GeV and  $H_T \geq 600$  GeV
  - Dataset equivalent to  $10 \text{ fb}^{-1}$

SM processes			
Physics process	Process ID	$\sigma$ (pb)	$N_{tot} (N_{10\text{fb}^{-1}})$
$pp \rightarrow jj(+2j)$	njets	19718 $_{H_T > 600\text{GeV}}$	415331302 (197179140)
$pp \rightarrow l^2\nu(+2j)$	w_jets	10537 $_{H_T > 100\text{GeV}}$	135692164 (105366237)
$pp \rightarrow \gamma j(+2j)$	gam_jets	7927 $_{H_T > 100\text{GeV}}$	123709226 (79268824)
$pp \rightarrow l^+l^-(+2j)$	z_jets	3753 $_{H_T > 100\text{GeV}}$	60076409 (37529592)
$pp \rightarrow t\bar{t}(+2j)$	tbar	541	13590811 (5412187)
$pp \rightarrow t + \text{jets}(+2j)$	single_top	130	7223883 (1297142)
$pp \rightarrow \bar{t} + \text{jets}(+2j)$	single_topbar	112	7179922 (1116396)
$pp \rightarrow W^+W^-(+2j)$	ww	82.1	17740278 (821354)
$pp \rightarrow W^\pm t(+2j)$	wtop	57.8	5252172 (577541)
$pp \rightarrow W^\pm \bar{t}(+2j)$	wtopbar	57.8	4723206 (577541)
$pp \rightarrow \gamma\gamma(+2j)$	2gam	47.1	17464818 (470656)
$pp \rightarrow W^\pm\gamma(+2j)$	Wgam	45.1	18633683 (450672)
$pp \rightarrow ZW^\pm(+2j)$	zw	31.6	13847321 (315781)
$pp \rightarrow Z\gamma(+2j)$	Zgam	29.9	15909980 (299439)
$pp \rightarrow ZZ(+2j)$	zz	9.91	7118820 (99092)
$pp \rightarrow h(+2j)$	single_higgs	1.94	2596158 (19383)
$pp \rightarrow t\bar{t}\gamma(+2j)$	tbarGam	1.55	95217 (15471)
$pp \rightarrow t\bar{t}Z$	tbarZ	0.59	300000 (5874)
$pp \rightarrow t\bar{t}h(+1j)$	tbarHiggs	0.46	200476 (4568)
$pp \rightarrow \gamma t(+2j)$	atop	0.39	2776166 (3947)
$pp \rightarrow t\bar{t}W^\pm$	tbarW	0.35	279365 (3495)
$pp \rightarrow \gamma \bar{t}(+2j)$	atopbar	0.27	4770857 (2707)
$pp \rightarrow Zt(+2j)$	ztop	0.26	3213475 (2554)
$pp \rightarrow Z\bar{t}(+2j)$	ztopbar	0.15	2741276 (1524)
$pp \rightarrow t\bar{t}\bar{t}$	4top	0.0097	399999 (96)
$pp \rightarrow t\bar{t}W^+W^-$	tbarWW	0.0085	150000 (85)

[4] T. Aarrestad et al., SciPost Phys. 12, 043 (2022), arXiv:2105.14027 [hep-ph]

## Histogram production

- Consider all possible **combinations of objects and selections**

- With 0 to 4 objects per type
- Additional kinematic cuts:  
 $E_{Tmiss} > 200, 500 \text{ GeV}$ ; leading object  $p_T > 100, 200, 400 \text{ GeV}, \dots$

$$\begin{aligned} & 1\mu + 3j + E_{Tmiss} > 200 \text{ GeV} \\ & + 0e, 0\gamma, 0T, 0Z, 0Wh, 0HM \end{aligned}$$

- Split the sub-dataset according to **jet multiplicity**

- 0 jet, 1 jet, 2 jets, ... ,  $\geq 6$  jets (depends on the available stat)
- Should allow to improve S/B
- Help reducing look-elsewhere effect (since bump should appear at the same place in neighboring jet multiplicities)

$$\begin{aligned} & 1W + 1Z + 3j + p_T(Z) > 100 \text{ GeV} \\ & + 0e, 0\gamma, 0T, 0HM \end{aligned}$$

- Build variables** from available objects

- Mass distributions of the objects and their combinations
- Transverse mass distributions including  $E_{Tmiss}$
- For jets, use only the 4 leading jets using b-tagging information

...

- Find the **maximum of the histogram** and start from there

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- $$1\mu + 3j + E_{Tmiss} > 200 \text{ GeV}$$

$$+ 0e, 0\gamma, 0T, 0Z, 0Wh, 0HM$$

- $$\text{|||} \quad m(\mu j_1), m(\mu j_1 E_{Tmiss}), \dots$$

$$\text{|||} \quad m(\mu j_1 j_2), m(\mu j_1 j_2 E_{Tmiss}), \dots$$

$$\text{|||} \quad \dots$$

- $$1W + 1Z + 3j + p_T(Z) > 100 \text{ GeV}$$

$$+ 0e, 0\gamma, 0T, 0HM$$

- $$\text{|||} \quad m(Wh, j_1), m(Wh, j_1, E_{Tmiss}) \dots$$

$$\text{|||} \quad m(j_1 j_2), m(j_1 j_2, E_{Tmiss}) \dots$$

$$\text{|||} \quad \dots$$

- $$\text{|||} \quad \dots$$

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$$\text{|||} \quad m(\mu j_1 j_2), m(\mu j_1 j_2 E_{Tmiss}), \dots$$

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$$+ 0e, 0\gamma, 0T, 0HM$$

- $$\text{|||} \quad m(Wh, j_1), m(Wh, j_1, E_{Tmiss}) \dots$$

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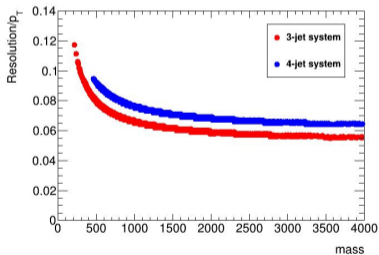
$$\text{|||} \quad \dots$$

- $$\text{|||} \quad \dots$$

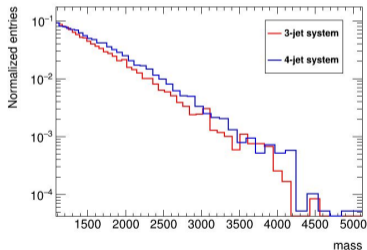
Total of **30 000 mass histograms**

# Processing and calibration

- In real data, signal width determined by **detector resolution**
  - Different for different final states and/or mass
  - Narrow signal should produce a bump in few bins
  
- **Rebin histogram** to reflect the detector resolution we would see in actual experimental data
  - $e, \mu, \gamma$ : Delphes Card formulas, depends on  $p_T$  and  $\eta$
  - jets: ATLAS report on jet resolution, depends on  $p_T$
  - $HM/T/Wh$  accounts for 3/3/2 jets
  - Using  $p_T \approx m/2$  approximation



Resolution is higher for  $m(4j)$  than  $m(3j)$ , also larger for small masses



Binning reflects this with larger bin width when resolution is smaller

## Dark Machines datasets

- **Background-only datasets:**

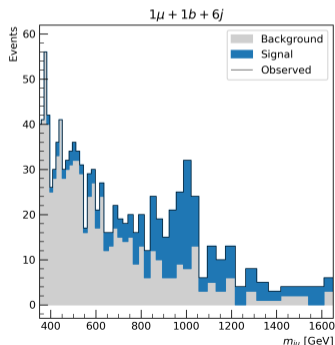
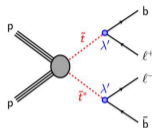
- Obtain shapes to be included in the training sample
- Assess false positive rate

- **BSM signal datasets:**

- Simulated signal data added on top of the backgrounds
- Test the network in a more realistic scenario
- Different levels of difficulty (e.g. cross-section, mass values, etc)

Some of the new physics models we have available include:

- Q RPV stop  $\rightarrow b\ell$
- Q  $W' \rightarrow WZ \rightarrow \ell\nu qq, qq\nu\nu$
- Q  $LQ \rightarrow beb\mu, bebe, \nu\nu\nu$
- Q  $Z' \rightarrow 3l$



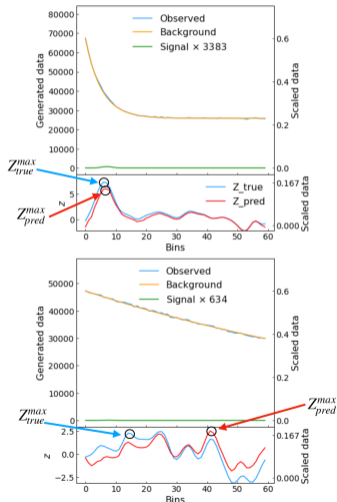
## Performance and finding BSM signals

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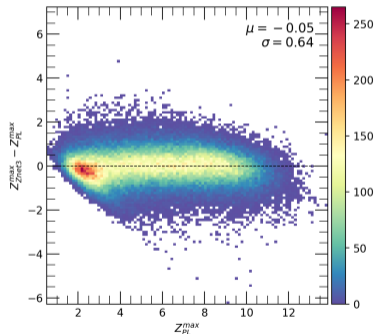
## Performance over synthetic data

- ▶ Performance quantified in terms of the **difference between predicted and true maximum significance**
  - ▶  $Z_{true}^{max}$ : maximal significance calculated via the likelihood ratio test
  - ▶  $Z_{pred}^{max}$ : maximal predicted significance
- ▶ Majority of entries should have  $Z_{pred}^{max} - Z_{true}^{max}$  close to 0 with the smallest variance possible
- ▶ **Non-zero significance** for background-only histograms
  - ▶ Due to look-elsewhere-effect
  - ▶ Could artificially bias the performance at low significance

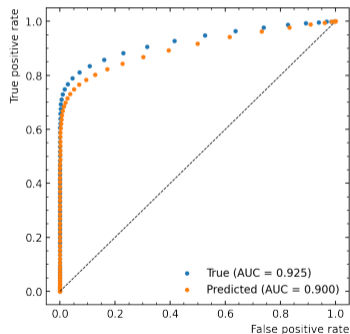


## Performance on the testing sample

- Trained model **accurately predicts maximum significance** with no bias and a variance of  $\pm 0.64$



- Excellent discriminating performance** of signal and background with an AUC of 0.900



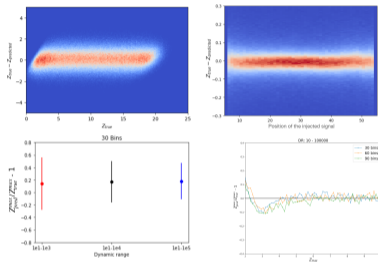
## Performance stability

- Good agreement between  $Z_{pred}^{max}$  and  $Z_{true}^{max}$  is **stable**

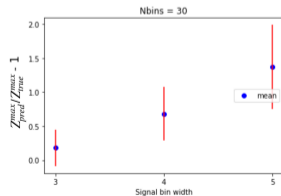
- For all  $Z_{true}^{max}$
- Over all mass range
- For all dynamic ranges
- For linear combinations of the functions

- Training with fixed signal width has **increased bias as signal width increases**

- Same behaviour regardless of number of bins and dynamic range
- Accurate calibration takes time (and the DDP should work fast)
- Room for improvement, yet well defined behaviour

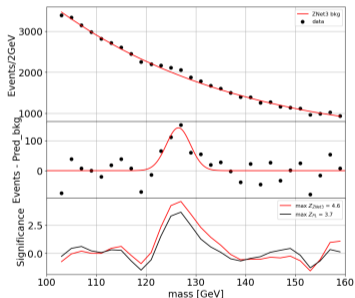
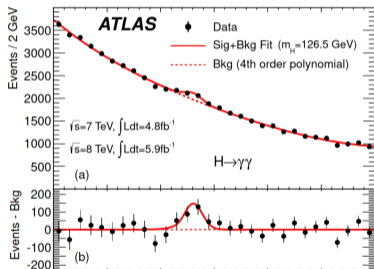


Mean &amp; sigma as a function of signal width



## Finding BSM signals

- ▶ Promising results when **finding the Higgs bump**
  - ▶ Data sampled from the ATLAS plot<sup>[5]</sup> using a digitizer



- ✔ Predicted resonance at the correct mass
- ❗ Predicted significance of  $4.6\sigma$  whereas the ATLAS significance is  $3.7\sigma$

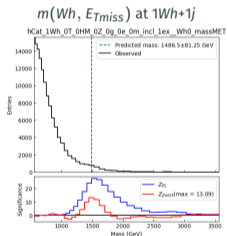
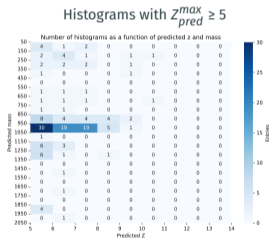
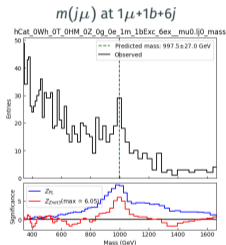
[5] ATLAS Collaboration, Physics Letters B 716, 1–29 (2012), arXiv:1207.7214 [hep-ex]

# Finding BSM signals

- Tested over **simulated BSM signals** added to the Dark Machines background

Q RPV stop  $\rightarrow b\ell$

Q  $W' \rightarrow WZ$



- Successfully finds an excess at the expected mass of the stop at 1TeV
- Successfully finds bump for  $W'$  (with a boosted Z in the final state)
- Other signals tested and successfully found include  $LQ \rightarrow beb\mu, bebe$  and  $Z' \rightarrow 3\ell$
- False-positive rate of 0.1% when tested over background-only sample

## Conclusion

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- **Data directed paradigm bump search** to scan unexplored final states in search for resonances
  - Target smoothly falling invariant mass spectra across a variety of final states
  - Exploit full potential of data without the need of simulation or analytical fit
- **Network implementation and performance**
  - Using Dark Machines datasets with highest cross-section processes at the LHC
  - Produce mass histograms with binning that reflects detector resolution
  - Successfully finds Higgs bump and BSM signals on top of the Dark Machines background, such as RPV stop and  $W'$
- **Future developments**
  - Application to real experimental data, focusing on Run 2
  - Use full MC simulation data with basic selections
  - First iteration using single-lepton trigger and same objects
  - Eventually adding more objects, such as large-R jets

## Appendix

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# Calculation of target significance

For each bin  $i$ , we perform a **hypothesis test** and obtain a significance value  $z_i$  (see [arXiv:1007.1727](https://arxiv.org/abs/1007.1727) ↗)

- **Step 1:** Define the shape of the signal  $S$  for which we are looking, e.g. a Gaussian centered on bin  $i$

- We are currently using the mass values on the x-axis

$$s_i = \int_{\text{bin } i} f(x; x_0, \sigma) dx \approx f(x; x_0, \sigma) \Delta x$$

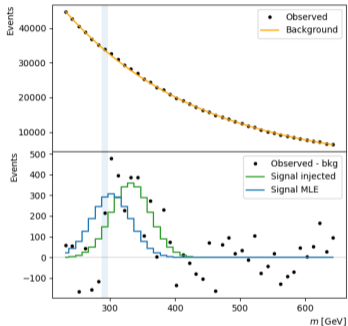
$$\begin{cases} x_0 = \text{center of bin } i \\ \sigma = \text{width of bin } i \times \text{given number of bins} \end{cases}$$

- **Step 2:** Obtain the **maximum likelihood estimator**  $\hat{\mu}$  for this signal  $S$

- Expected value is  $N_{\text{exp}} = b_i + \mu s_i$


$$\hat{\mu} = \arg \min - \ln[L(\mu)]$$

$$- \ln[L(\mu)] = - \sum_i N_{\text{obs}} \ln(N_{\text{exp}}) - N_{\text{exp}} + \ln\left(\frac{1}{N_{\text{obs}}!}\right)$$



- **Observed data** ( $N_{\text{obs}}$ ) is the synthetic data
- **Unfluctuated background** ( $b$ ) is our null hypothesis
- **Injected signal** is the one added into the observed data
- **Test signal** is the signal that maximizes the likelihood for each bin

# Calculation of target significance

For each bin  $i$ , we perform a **hypothesis test** and obtain a significance value  $z_i$  (see [arXiv:1007.1727](https://arxiv.org/abs/1007.1727) )

- ▶ **Step 3:** Obtain the **likelihood** for both the background-only hypothesis and the maximum-likelihood estimator (MLE) signal hypothesis

- ▶  $-\ln[L(\mu = 0)] \rightarrow$  Background-only ( $N_{\text{exp}} = b_i$ )

- ▶  $-\ln[L(\mu = \hat{\mu})] \rightarrow$  Signal MLE ( $N_{\text{exp}} = b_i + \hat{\mu}s_i$ )

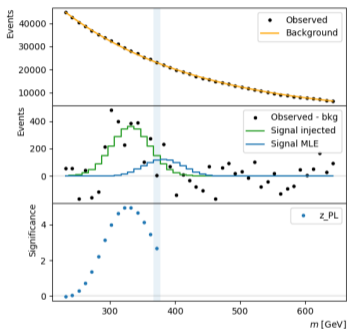
- ▶ **Step 4:** Obtain the significance using the **profile likelihood ratio** test

If  $\hat{\mu} \geq 0$ :

$$z_i = \sqrt{q_0} \quad \text{with} \quad q_0 = -2 \ln \left[ \frac{L(0)}{L(\hat{\mu})} \right]$$

If  $\hat{\mu} < 0$ :

$$z_i = -\sqrt{-q_0} \quad \text{with} \quad q_0 = \begin{cases} -2 \ln \left[ \frac{L(\hat{\mu})}{L(0)} \right], & \text{if two-sided} \\ 0, & \text{otherwise} \end{cases}$$



- ▶ **Observed data** ( $N_{\text{obs}}$ ) is the synthetic data
- ▶ **Unfluctuated background** ( $b$ ) is our null hypothesis
- ▶ **Injected signal** is the one added into the observed data
- ▶ **Test signal** is the signal that maximizes the likelihood for each bin

# Background functions in the framework

$$ax + b$$

```
def linear(x, p0, p1):  
    return p0 * x + p1
```

$$ae^{-bx}$$

```
def exponential(x, p0, p1):  
    return p0 * np.exp(-p1 * x)
```

$$\frac{1}{ax} + b$$

```
def one_over_x(x, p0, p1):  
    return 1/(p0*x) + p1
```

$$a(x - x_{max})^2 + y_{min}$$

```
def parabola_half(x, p0, p1, x2, y2):  
    return p0*(x - x2)**2 + y2
```

$$\frac{1}{ax^2} + b$$

```
def one_over_x_squared(x, p0, p1):  
    return 1/(p0*x**2) + p1
```

$$\Delta y \cos[a(x - b)] + y_{max}$$

```
def cos_quarter(x, p0, p1, y1, y2):  
    return (y1 - y2)*np.cos(p0*(x - p1)) + y1
```

$$\frac{1}{ax^3} + b$$

```
def one_over_x_cubed(x, p0, p1):  
    return 1/(p0*x**3) + p1
```

$$-a \ln(x) + b$$

```
def ln_negative(x, p0, p1):  
    return -p0*np.log(x) + p1
```

$$\frac{1}{ax^4} + b$$

```
def one_over_x_to_4th(x, p0, p1):  
    return 1/(p0*x**4) + p1
```

$$\cosh[a(x - x_{max})] + b$$

```
def cosh_half(x, p0, p1, x2):  
    return np.cosh(p0*(x - x2)) + p1
```

$$\frac{1}{ax^n} + b$$

```
def one_over_x_to_nth(x, p0, p1, n):  
    return 1/(p0*x**n) + p1
```

→ where  $n$  is taken randomly from the range [0.01, 10]