# Towards a theory of hadron resonances

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- Short summary & outlook


Master reference: Mai, UGM, Urbach, Phys. Rept. **1001** (2023) 1

# Introduction

# QCD Lagrangian

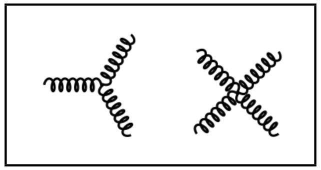


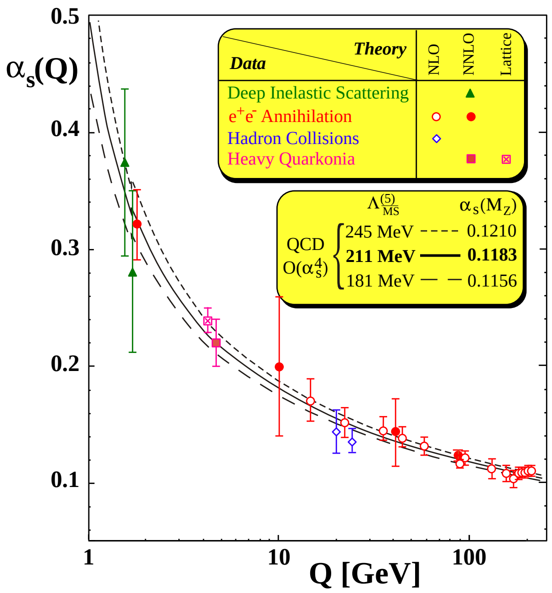
- $$\mathcal{L}_{\rm QCD} = -\frac{1}{4}\, G^a_{\mu\nu} G^{\mu\nu,a} + \sum_f \bar{q}_f (i\not{D} - \mathcal{M}) q_f + \ldots$$

$$D_\mu = \partial_\mu - i g A^a_\mu \lambda^a/2$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g[A^b_\mu, A^c_\nu]$$

$$f = (u, d, s, c, b, t)$$






- running of $\alpha_s = \dfrac{g^2}{4\pi} \Rightarrow \Lambda_{\rm QCD} = 210 \pm 14\,$MeV $\quad (N_f = 5, \overline{MS}, \mu = 2\,$GeV)

- light (u,d,s) and heavy (c,b,t) quark flavors:

| $m_{\rm light} \ll \Lambda_{\rm QCD}$ | $m_{\rm heavy} \gg \Lambda_{\rm QCD}$ |
|---|---|
| $m_u = 2.2^{+0.6}_{-0.4}$ MeV | $m_c = 1.28 \pm 0.03$ GeV |
| $m_d = 4.7^{+0.5}_{-0.4}$ MeV | $m_b = 4.18^{+0.04}_{-0.03}$ GeV |
| $m_s = 96^{+8}_{-4}$ MeV | $m_t = 173.1 \pm 0.6$ GeV |

# Limits of QCD

- **light quarks:** $\mathcal{L}_{\rm QCD} = \bar{q}_L\, i\not{D} q_L + \bar{q}_R\, i\not{D} q_R + \mathcal{O}(m_f/\Lambda_{\rm QCD})$
  - L and R quarks decouple $\Rightarrow$ chiral symmetry
  - spontaneous chiral symmetry breaking $\Rightarrow$ pseudo-Goldstone bosons
  - pertinent EFT $\Rightarrow$ chiral perturbation theory (CHPT)
- **heavy quarks:** $\mathcal{L}_{\rm QCD} = \bar{Q}_f\, iv\cdot D\, Q_f + \mathcal{O}(\Lambda_{\rm QCD}/m_f)$
  - independent of quark spin and flavor
    $\Rightarrow$ SU(2) spin and SU(2) flavor symmetries (HQSS and HQFS)
  - pertinent EFT $\Rightarrow$ heavy quark effective field theory (HQEFT)
- **heavy-light systems:**
  - heavy quarks act as matter fields coupled to light pions
  - combine CHPT and HQEFT

# Why excited states?



- The spectrum of QCD is its **least** understood feature

  $\rightarrow$ why only $qqq$ and $\bar{q}q$ states? XYZ states? “exotics”? glueballs?

  $\rightarrow$ important players: **hadronic molecules** $\leftrightarrow$ nuclear physics

  $\rightarrow$ the quark model is much too simple . . .

  $\rightarrow$ need insight from EFTs $\leftrightarrow$ symmetries!

- Many recent high-precision data (utilizing e.g. double polarization exp’s)

  $\rightarrow$ ELSA at Bonn, CEBAF at Jefferson Lab, LHCb at CERN,
  BESIII at BEPCII, GlueX at JLab12, . . ., PANDA at FAIR, . . .

- Lattice QCD can get ground-states at almost physical pion masses

  $\rightarrow$ most distinctive feature of excited states: *decays*

  $\rightarrow$ only captured for very few states in lattice QCD

  $\rightarrow$ must explore this (almost complete) *terra incognita*

# Lesson 1
# What is a resonance?

# What is a resonance?

- *"Not every bump is a resonance and not every resonance is a bump"*

Moorhouse 1960ties

- Resonances have **complex** properties (mass & width, photo-couplings, . . .)

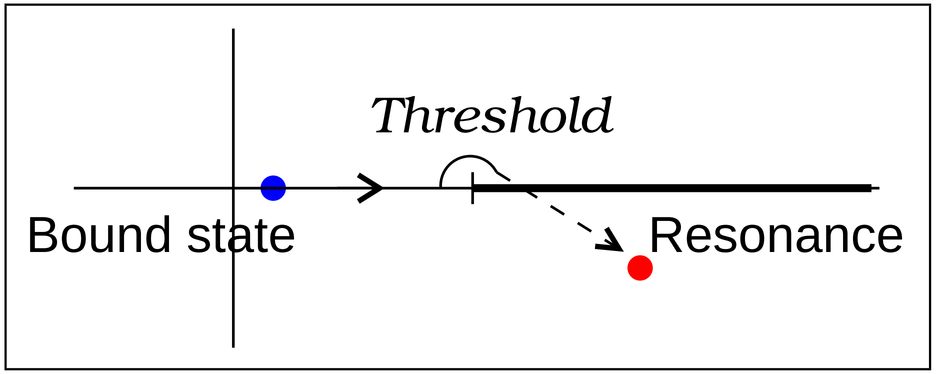
$\hookrightarrow$ these intrinsic properties do not depend on the experiment or theory (model)

- Resonances correspond to S-matrix poles on unphysical Riemann sheets

$\hookrightarrow$ only model-independent definition !

$\hookrightarrow$ matrix-elements from analytic cont.

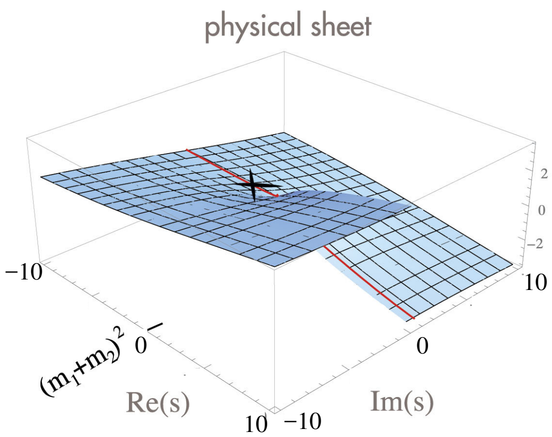
to the resonance pole $p_R$ $\hookrightarrow$ pics next slide

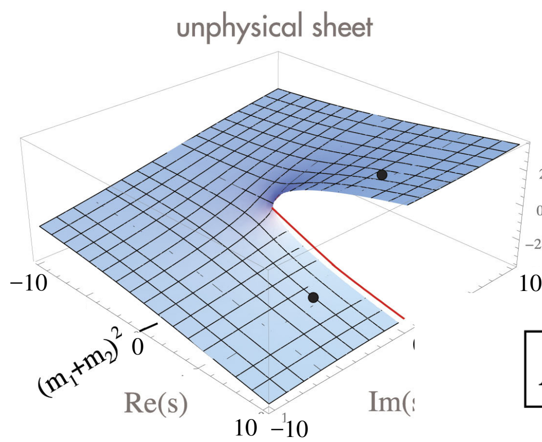



- That's all nice in the continuum, but . . .

# Pictures of resonances

- Resonances as complex poles on unphsyical sheets:



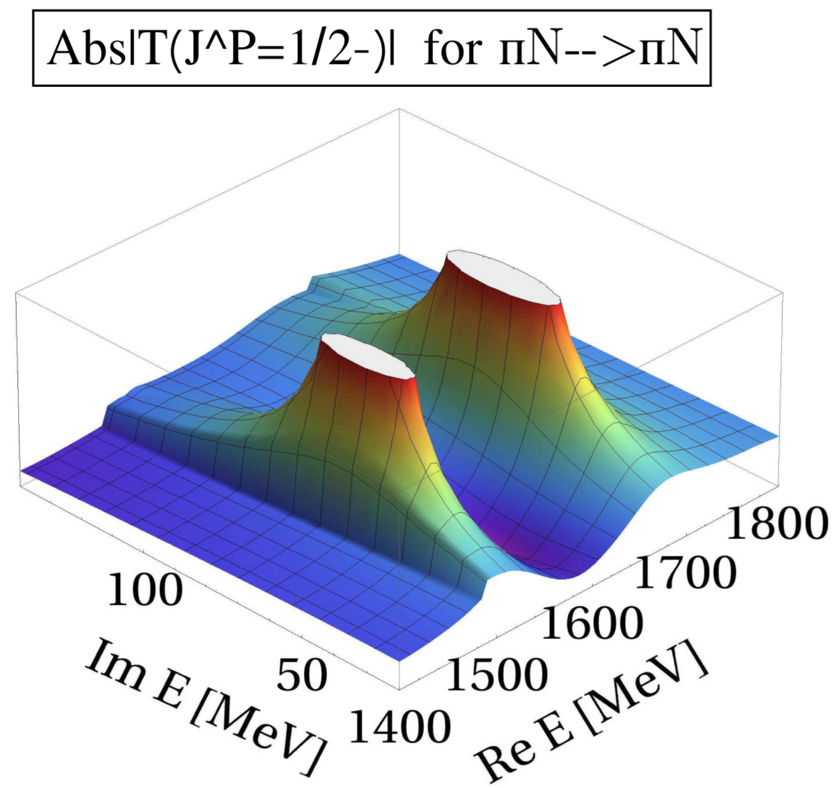

- A view of the two close-by baryon resonances:

  the two lowest nucleon excitations in the

  $S_{11}$ partial wave of $\pi N \to \pi N$

  JüBo approach, D. Rönchen et al.

http://collaborations.fz-juelich.de/ikp/meson-baryon/main



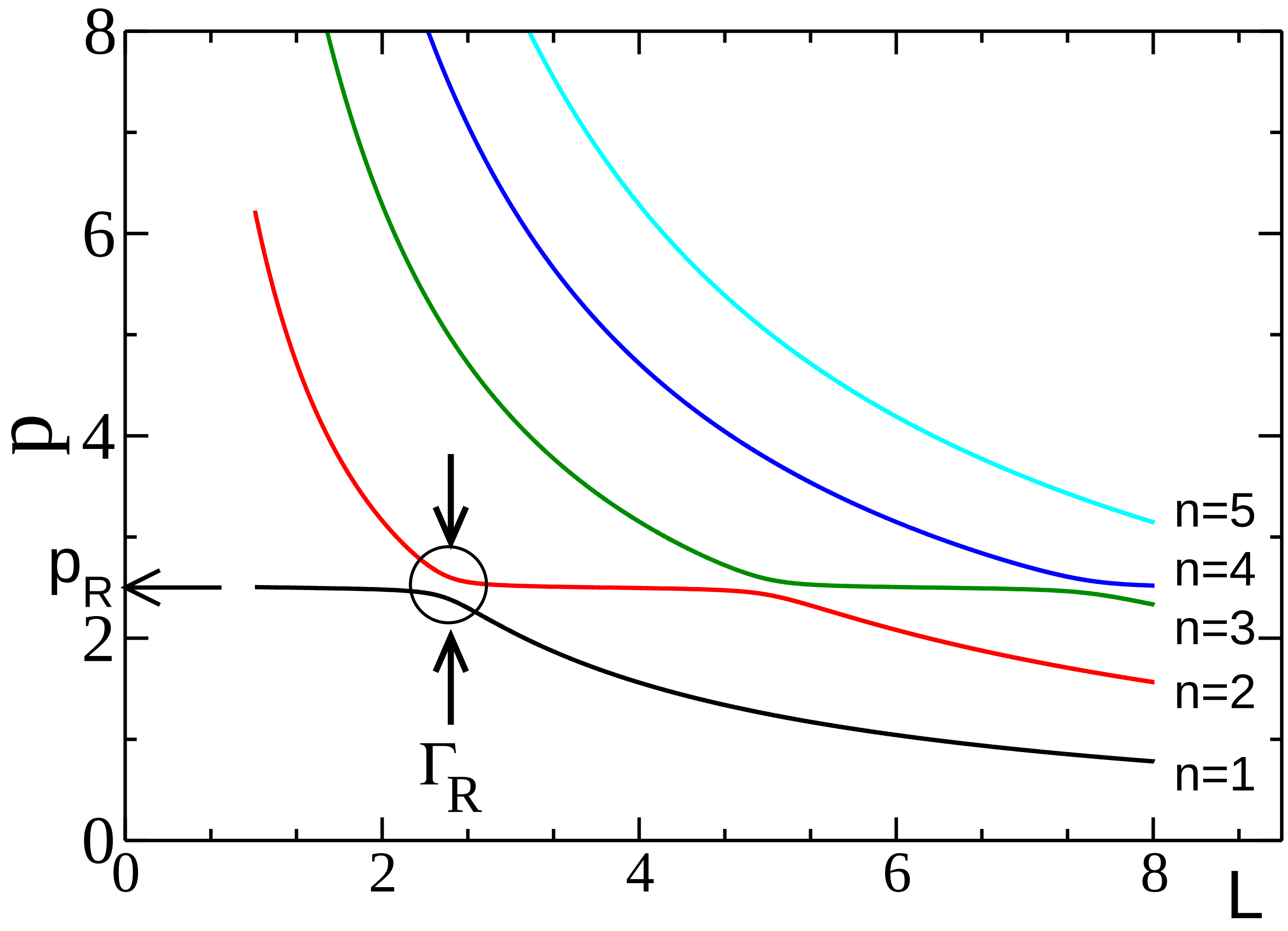
# Resonances in a box



- Resonances in a box: not eigenstates of the Hamiltonian
  $\Rightarrow$ volume dependence of the energy spectrum

Lüscher, Wiese, ...

- consider a narrow resonance $\rightarrow$ *avoided level crossing*



# Lesson 2
# Well separated resonances

# Isolated resonances in a box

- Two identical particles of mass $m$ in a box, no interaction:

$$E = 2\sqrt{m^2 + |\vec{p}\,|^2}\,, \quad p_i = \frac{2\pi}{L} n_i\,, \quad n_i \in \mathbb{Z}$$

- turn on interaction $\rightarrow$ scattering phase $\rightarrow$ Lüscher formula: Lüscher 1985

$$\delta(p) = -\phi(q) \bmod \pi\,, \quad q = \frac{pL}{2\pi}$$

$$\phi(q) = -\frac{\pi^{3/2} q}{\mathcal{Z}_{00}(1;q^2)}\,, \quad \mathcal{Z}_{00}(1;q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\vec{n}\in\mathbb{Z}^3} \frac{1}{\vec{n}^{\,2} - q^2}$$

- assume resonance with mass $m_R > 2m$ $\rightarrow$ effective range expansion (Breit-Wigner shape):

$$\tan\left(\delta - \frac{\pi}{2}\right) = \frac{E^2 - m_R^2}{m_R \Gamma_R}$$

[not general!]

$\Rightarrow$ measure the phase shift in the resonance region and fit $m_R, \Gamma_R$
& extension to moving frames    Rummukainen, Gottlieb (1995) + ...

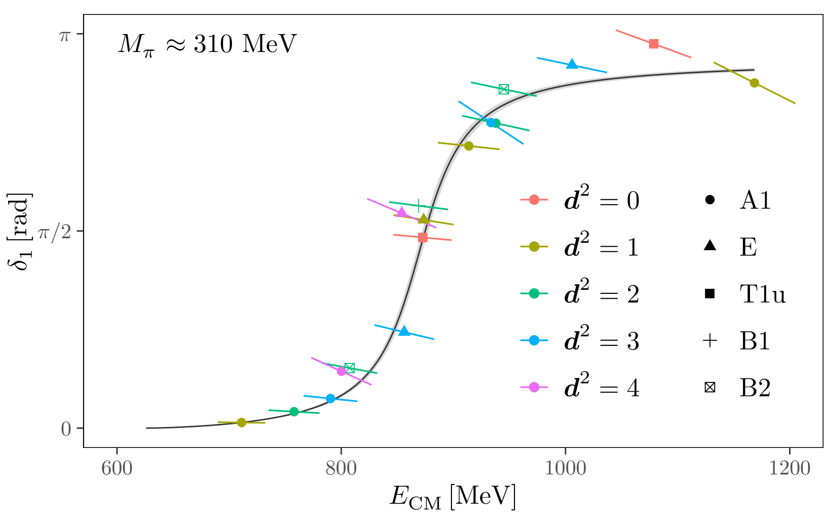
# Results for the $\rho(770)$-meson I



- The $\rho(770)$ is a well separated meson resonance in the $\pi\pi$ system
- P-wave $\pi\pi$ scattering, $M_\pi = 280 - 500$ MeV, three different $a$, three different $L$, boosts $\vec{d} = 0, 1, 2, 3, 4$, all irreps

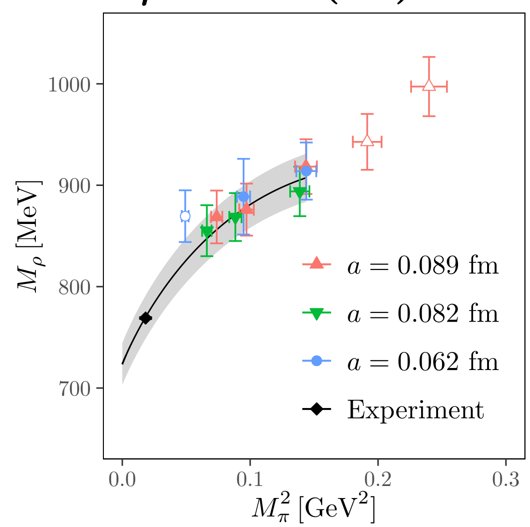
Werner, Urbach et al. [ETMC] (2019)

- Phase shift



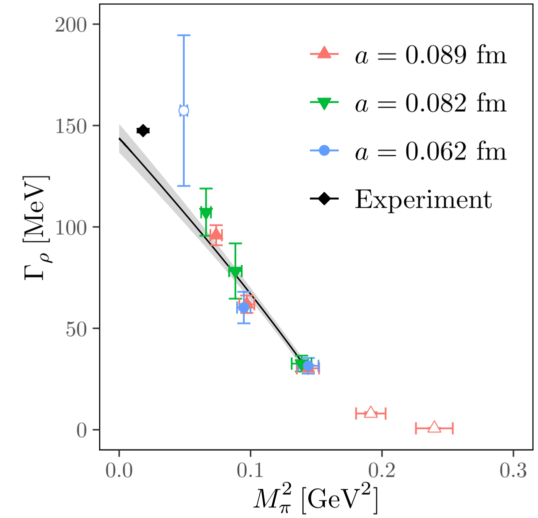

- Mass

$M_\rho = 770(19)$ MeV




- Width

$\Gamma_\rho = 130(7)$ MeV



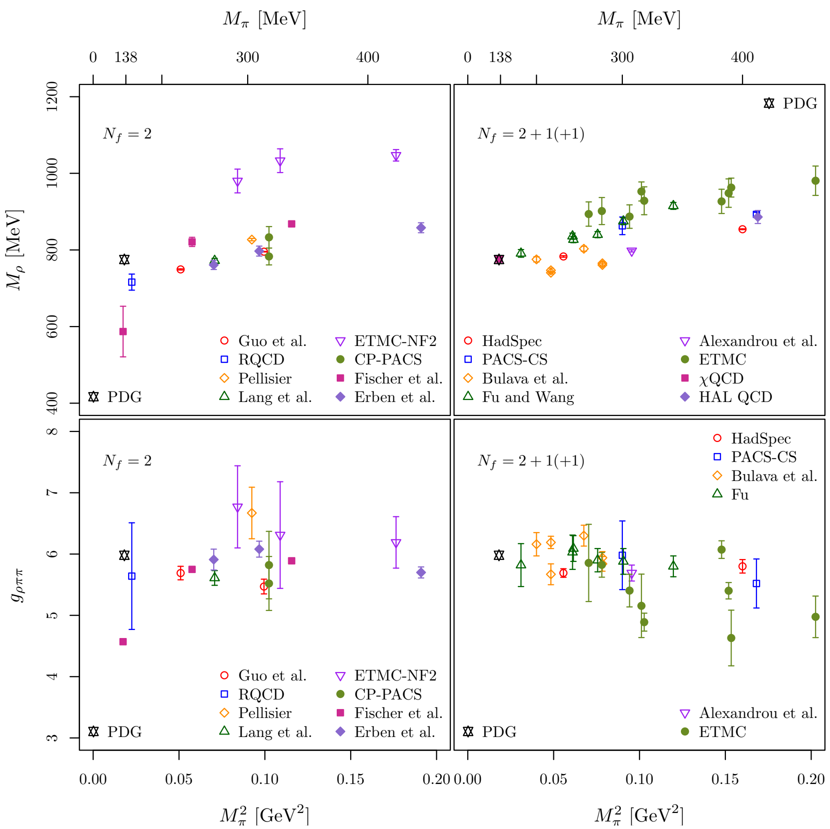

- consistent with other collaborations world-wide
- pioneered in: Feng, Jansen, Renner (2011)

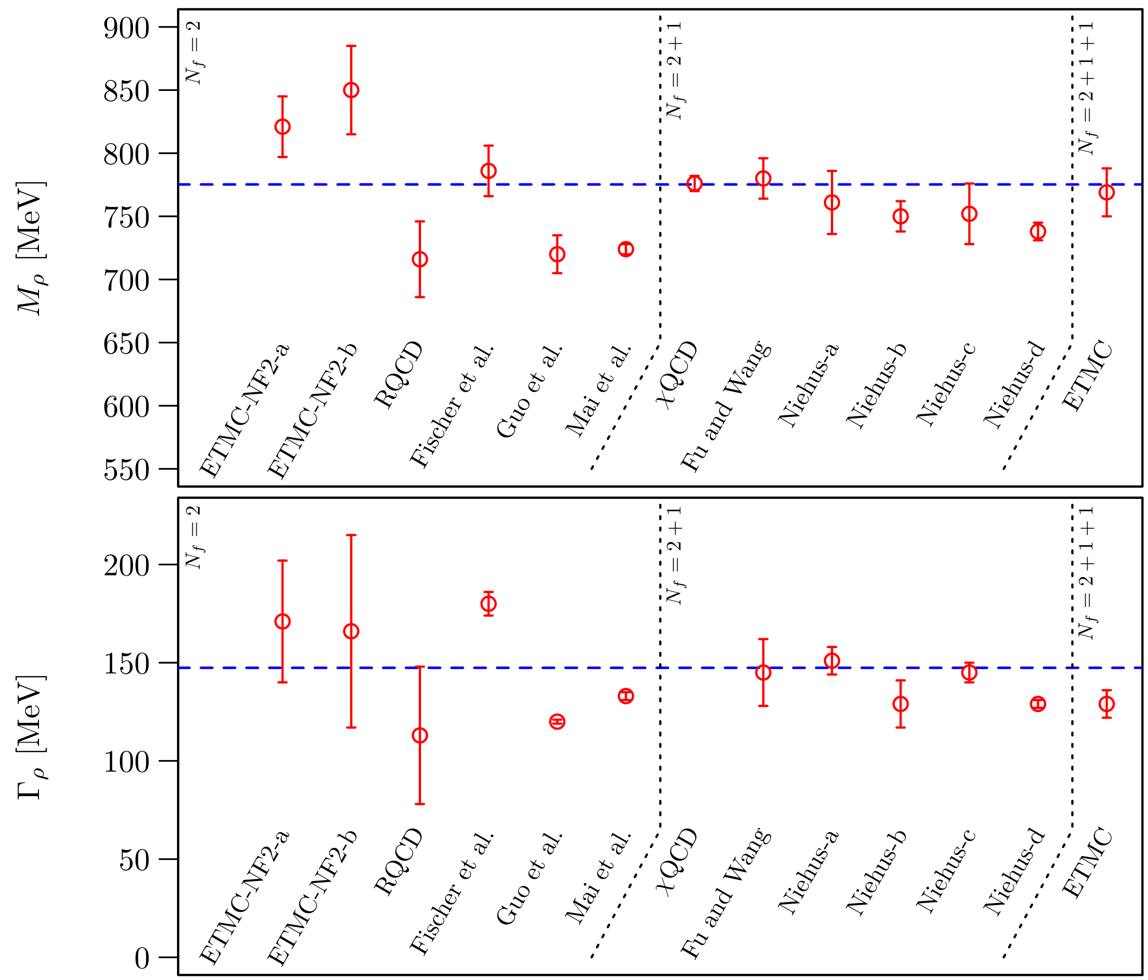
# Results for the $\rho(770)$-meson II



Mai, UGM, Urbach, Phys. Rept. **1001** (2023) 1

- Collect all available results based on dynamical simulations:



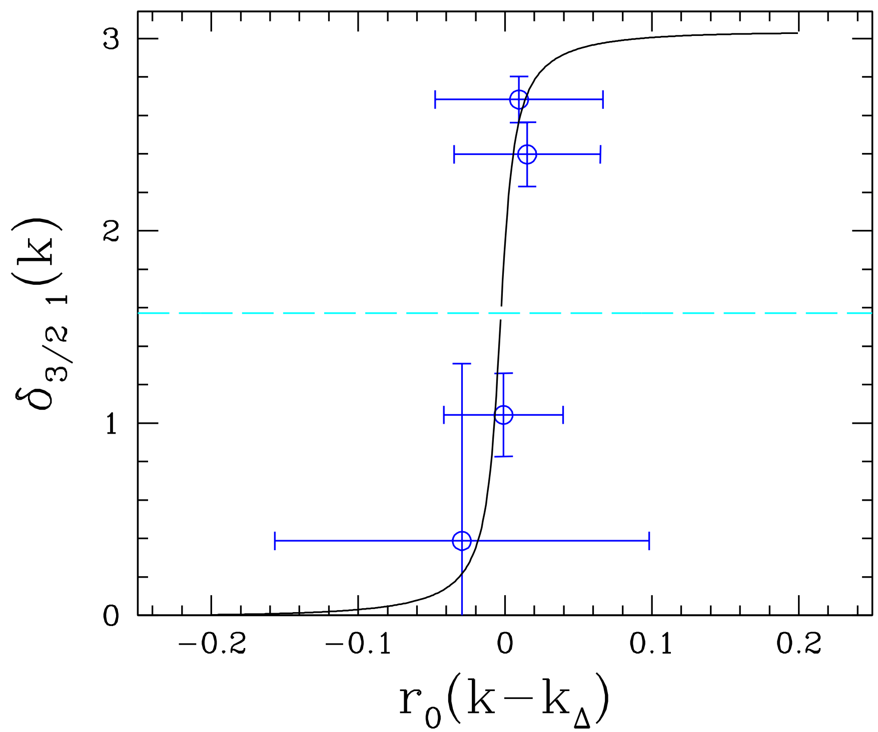

- Mass comes out mostly correct
- Width (coupling constant) can still be improved

# Results for the $\Delta(1232)$

Mai, UGM, Urbach, Phys. Rept. **1001** (2023) 1

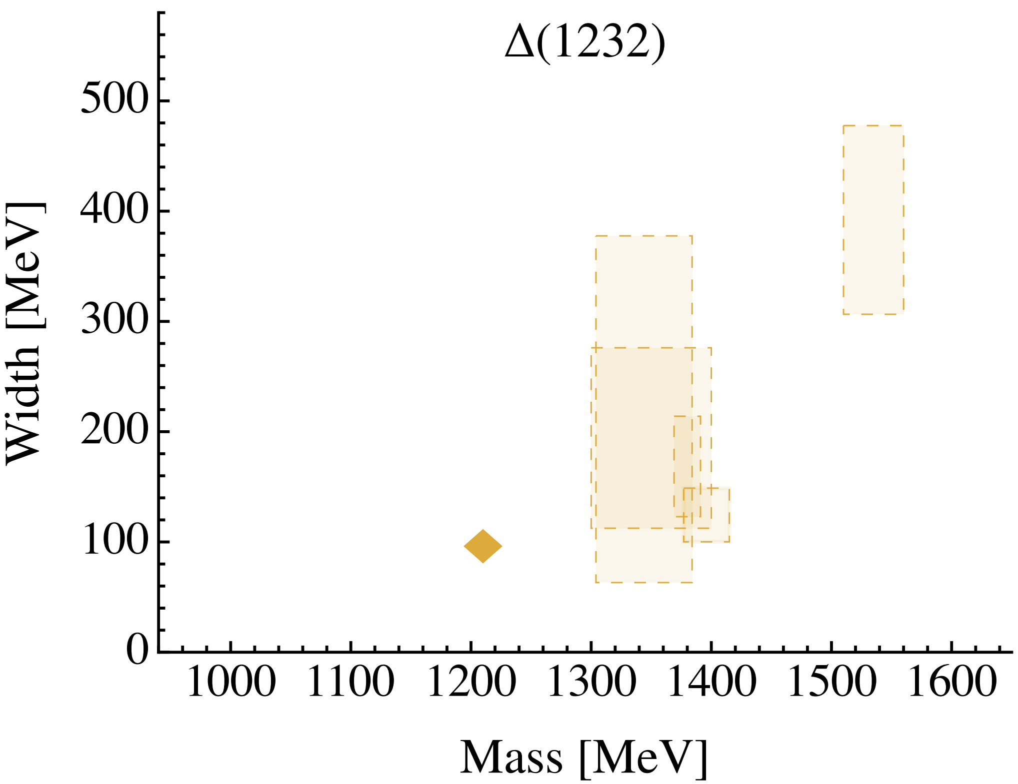
- The $\Delta(1232)$ is a well separated baryon resonance in the $\pi N$ system

- $l = 1, I = 3/2\ \pi N$ phase shift




QCDSF-Bonn-Jülich coll., see UGM, J. Phys. Conf. Ser. **295** (2011) 012001

- State-of-the-art (2022):




- More work required!

# Lesson 3: Coupled channels / thresholds

# Extension to coupled channels

- Isolated (well-separated) resonances are the exception
- Coupled channel effects, close-by thresholds: $f_0(980), a_0(980), \Lambda(1405), \ldots$
- various extensions of Lüscher's approach:

⋆ purely quantum mechanical treatment

Feng, He, Liu, Li, . . .

⋆ non-relativistic EFT (NREFT)

Beane, Savage, Bernard, Lage, UGM, Rusetsky, Briceno, Davoudi, Luu, . . .

⋆ finite-volume unitarized CHPT

Döring, UGM, Rusetsky, Oset, . . .

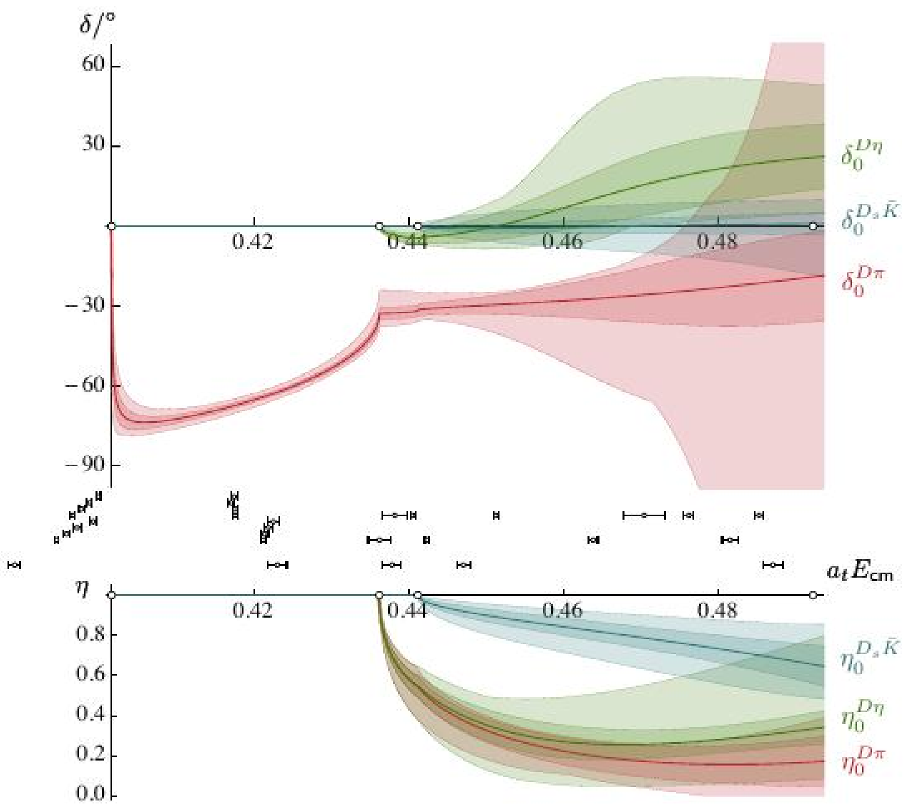
- Mostly done in the meson sector, not much for baryons
- Be aware of methods that can mislead you (must respect symmetries!)

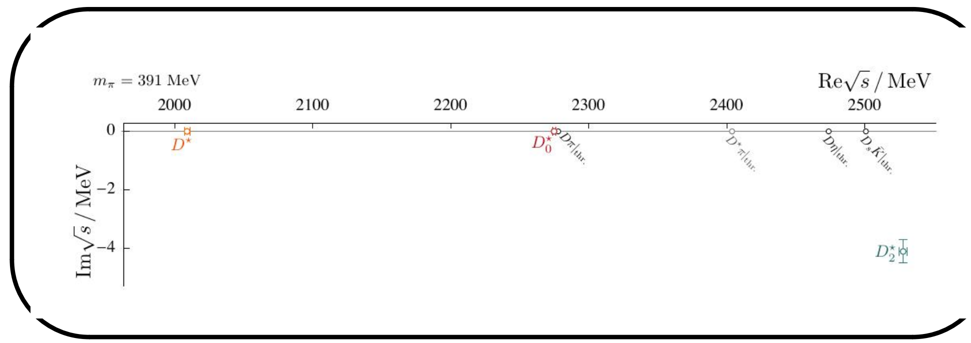
# Coupled channel scattering on the lattice



Moir, Peardon, Ryan, Thomas, Wilson, JHEP **1610** (2016) 011

- $D\pi$, $D\eta$, $D_s\bar{K}$ scattering with $I = 1/2$:
- 3 volumes, one $a_s$, one $a_t$, $M_\pi \simeq 390$ MeV, various K-matrix type extrapolations



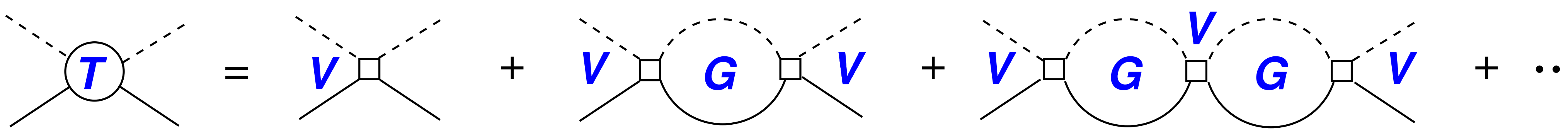

- S-wave pole at $(2275.9 \pm 0.9)$ MeV
- close to the $D\pi$ threshold
- consistent w/ $D_0^\star(2300)$ of PDG
- BUT: chiral symmetry ignored... :-(

# Coupled channel dynamics



Kaiser, Weise, Siegel (1995), Oset, Ramos (1998), Oller, UGM (2001), Kolomeitsev, Lutz (2002),Jido et al. (2003), Guo et al. (2006), . . .

- $D\phi$ bound states: Poles of the T-matrix (potential from CHPT and unitarization)




- Unitarized CHPT as a non-perturbative tool:

$$T^{-1}(s) = \mathcal{V}^{-1}(s) - G(s)$$

- $\mathcal{V}(s)$: derived from the SU(3) chiral Lagrangian, 6 LECs up to NLO $\longrightarrow$ next slide

- $G(s)$: 2-point scalar loop function, regularized w/ a subtraction constant $a(\mu)$

- $T, \mathcal{V}, G$: all these are matrices, channel indices suppressed

# Coupled channel dynamics cont'd



Barnes et al. (2003), van Beveren, Rupp (2003), Kolomeitsev, Lutz (2004), Guo et al. (2006), . . .

- NLO effective chiral Lagrangian for coupled channel dynamics

Guo, Hanhart, Krewald, UGM, Phys. Lett. B **666** (2008) 251

$$\mathcal{L}_{\rm eff} = \mathcal{L}^{(1)} + \mathcal{L}^{(2)}$$

$$\mathcal{L}^{(1)} = \mathcal{D}_\mu D \mathcal{D}^\mu D^\dagger - M_D^2 D D^\dagger\ , \quad D = (D^0, D^+, D_s^+)$$

$$\mathcal{L}^{(2)} = D\left[-h_0\langle\chi_+\rangle - h_1\chi_+ + h_2\langle u_\mu u^\mu\rangle - h_3 u_\mu u^\mu\right] D$$

$$+\mathcal{D}_\mu D\left[h_4\langle u^\mu u^\nu\rangle - h_5\{u^\mu, u^\nu\}\right]\mathcal{D}_\nu D$$

with $u_\mu \sim \partial_\mu\phi\ ,\ \ \chi_+ \sim \mathcal{M}_{\rm quark}\ ,\ \ldots$

- LECs:

$\hookrightarrow$ $h_0$ absorbed in masses

$\hookrightarrow$ $h_1 = 0.42$ from the $D_s$-$D$ splitting

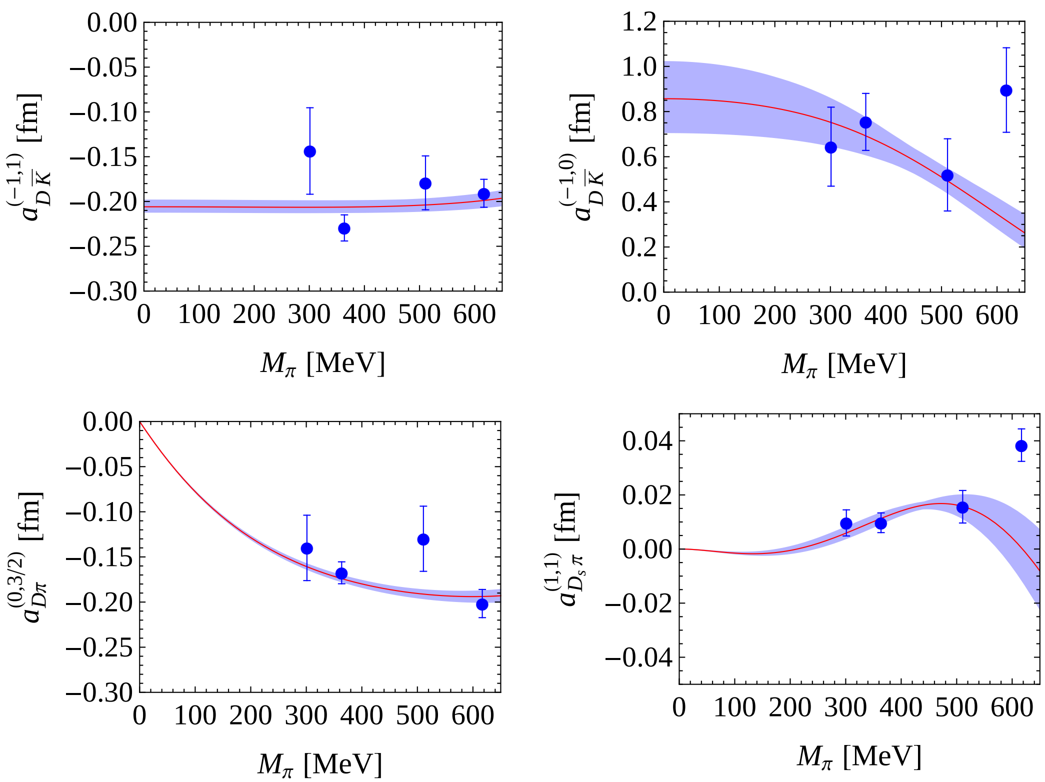
$\hookrightarrow$ $h_{2,3,4,5}$ from a fit to lattice data ($D\pi \to D\pi, D\bar{K} \to D\bar{K}, \ldots$)

Liu, Orginos, Guo, Hanhart, UGM, Phys. Rev. D **87** (2013) 014508

# Fit to lattice data

Liu, Orginos, Guo, Hanhart, UGM, PRD **87** (2013) 014508

- Fit to lattice data in 5 “simple” channels: no disconnected diagrams




[$M_\pi = 600$ MeV not fitted]

- Prediction: Pole in the $(S, I) = (1, 0)$ channel: $2315^{+18}_{-28}$ MeV

  Experiment: $M_{D^\star_{s0}(2317)} = (2317.8 \pm 0.5)$ MeV PDG2022

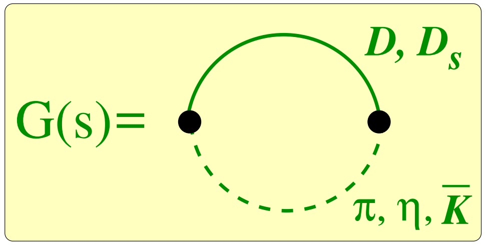
# Finite volume formalism



- Goal: postdict the finite volume (FV) energy levels for $I = 1/2$ and compare with the recent LQCD results from Moir et al. using the already fixed LECs
  $\to$ parameter-free insights into the $D_0^\star(2300)$

- In a FV, momenta are quantized: $\vec{q} = \frac{2\pi}{L}\vec{n}\,, \quad \vec{n} \in \mathbb{Z}^3$

$\Rightarrow$ Loop function $G(s)$ gets modified: $\int d^3\vec{q} \to \frac{1}{L^3}\sum_{\vec{q}}$




$$\tilde{G}(s,L) = G(s) = \lim_{\Lambda\to\infty}\left[\frac{1}{L^3}\sum_{\vec{n}}^{|\vec{q}|<\Lambda} I(\vec{q}) - \int_0^\Lambda \frac{q^2 dq}{2\pi^2} I(\vec{q})\right]$$

Döring, UGM, Rusetsky, Oset, Eur. Phys. J. A**47** (2011) 139

- FV energy levels from the poles of $\tilde{T}(s,L)$:

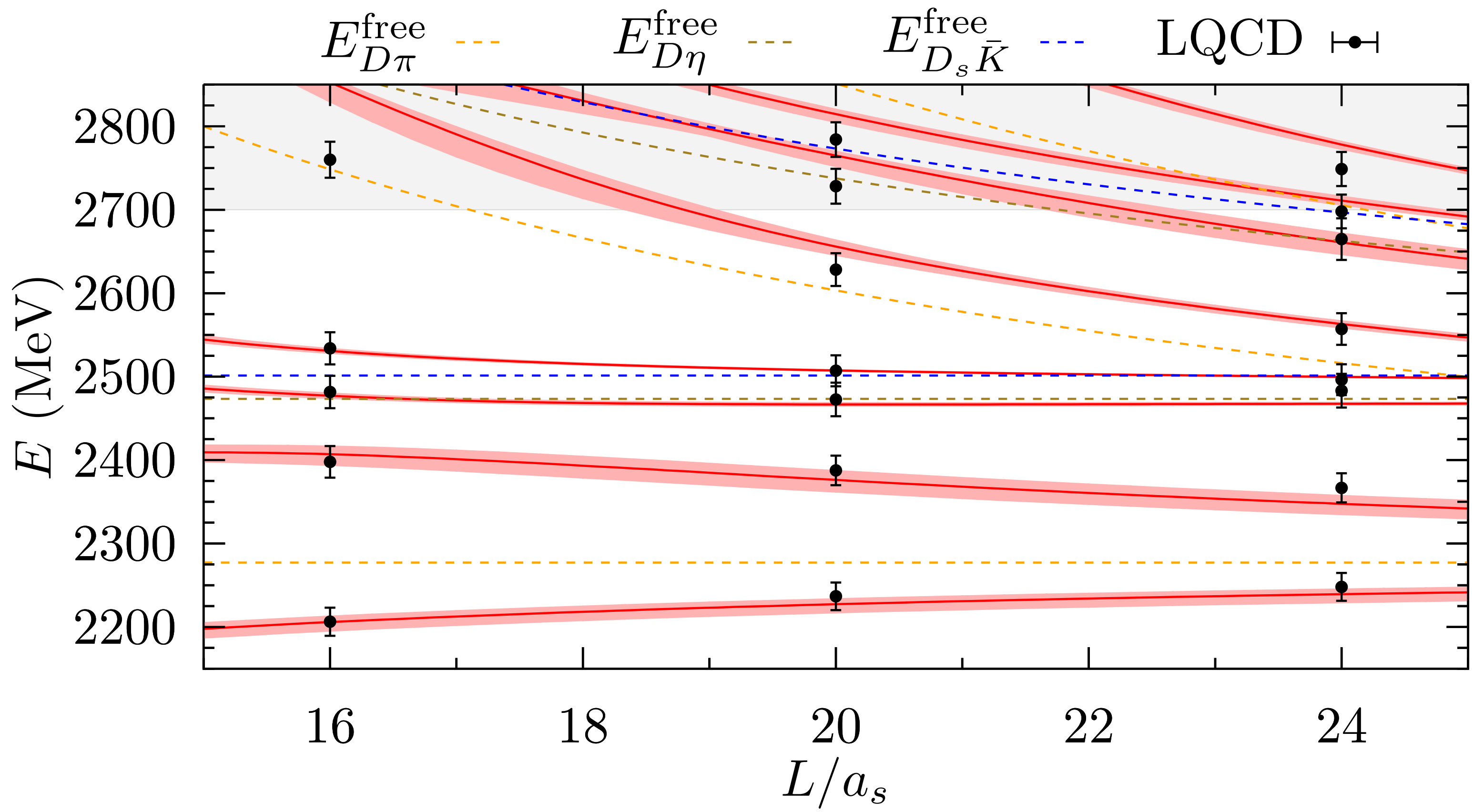
$$\tilde{T}^{-1}(s,L) = V^{-1}(s) - \tilde{G}(s,L)$$

# What about the $D_0^\star(2300)$?



- Results for $I = 1/2$ $D\phi$ scattering

Albaladejo, Fernandez-Soler, Guo, Nieves, Phys. Lett. B **767** (2017) 465




- this is NOT a fit!
- all LECs taken from the earlier study of Liu et al. (discussed before)

# What about the $D_0^\star(2300)$? – cont'd

Albaladejo, Fernandez-Soler, Guo, Nieves (2017)

- reveals a two-pole scenario! [cf. $\Lambda(1405)$]

- understood from group theory

$$\bar{3} \otimes 8 = \underbrace{\bar{3} \oplus 6}_{\text{attractive}} \oplus \overline{15}$$

- this was seen earlier in various calc's

  Kolomeitsev, Lutz (2004), F. Guo, Shen, Chiang, Ping, Zou (2006), F. Guo, Hanhart, UGM (2009), Z. Guo, UGM, Yao (2009)

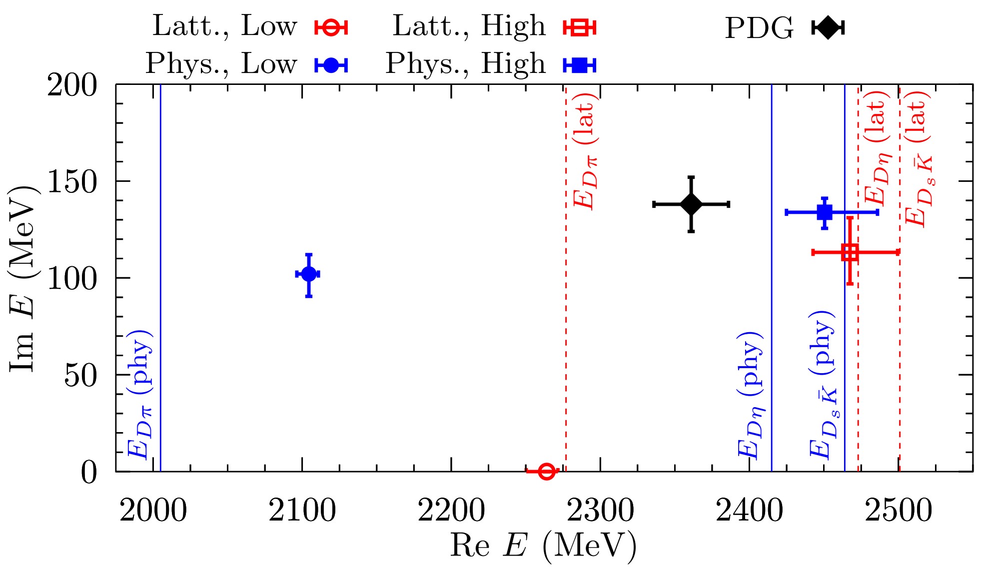
- Again: important role of **chiral symmetry**

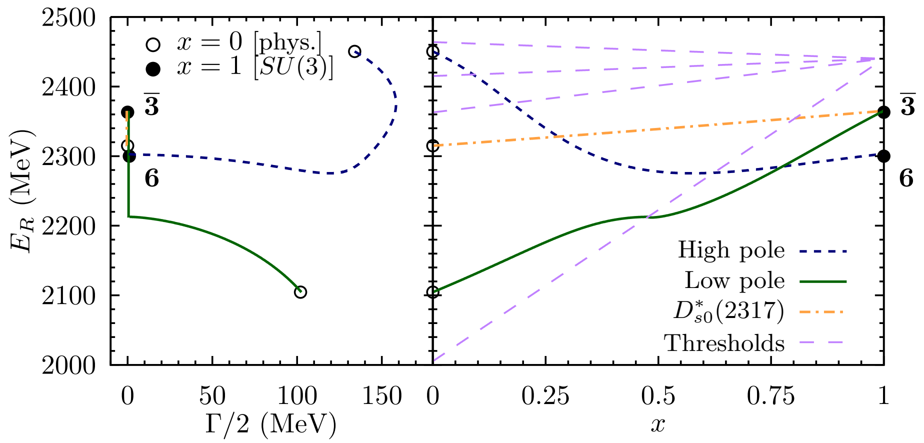
- Easy lattice QCD test:

  sextet pole becomes a bound state

  for $M_\phi > 575$ MeV in the SU(3) limit

  Du et al., Phys.Rev. D **98** (2018) 094018



# Two-pole scenario in the heavy-light sector



- Two states in various $I = 1/2$ states in the heavy meson sector $(M, \Gamma/2)$

| | Lower [MeV] | Higher [MeV] | PDG2021 [MeV] |
|---|---|---|---|
| $D_0^\star$ | $\left(2105^{+6}_{-8}, 102^{+10}_{-11}\right)$ | $\left(2451^{+36}_{-26}, 134^{+7}_{-8}\right)$ | $(2343 \pm 10, 115 \pm 8)$ |
| $D_1$ | $\left(2247^{+5}_{-6}, 107^{+11}_{-10}\right)$ | $\left(2555^{+47}_{-30}, 203^{+8}_{-9}\right)$ | $(2412 \pm 9, 157 \pm 15)$ |
| $B_0^\star$ | $\left(5535^{+9}_{-11}, 113^{+15}_{-17}\right)$ | $\left(5852^{+16}_{-19}, 36 \pm 5\right)$ | — |
| $B_1$ | $\left(5584^{+9}_{-11}, 119^{+14}_{-17}\right)$ | $\left(5912^{+15}_{-18}, 42^{+5}_{-4}\right)$ | — |

Lattice QCD: $M_{B_{s0}^\star} = 5711(13)(19)$ MeV , $M_{B_{s1}} = 5750(17)(19)$ MeV

Lang et al., Phys.Lett. B **750** (2015) 17

$\longrightarrow$ but is there further experimental support for this?

# Amplitude Analysis of $B \to D\pi\pi$

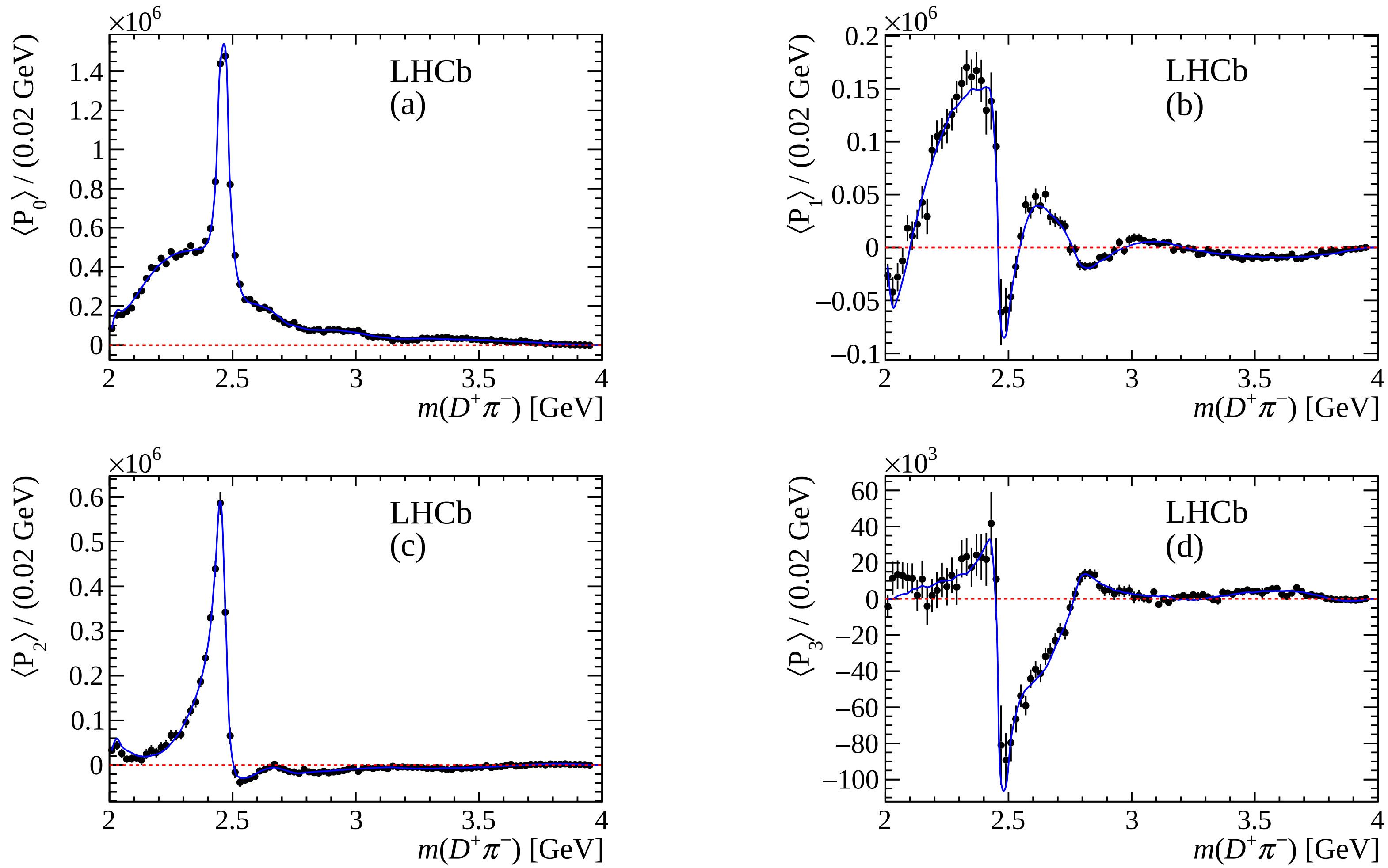
# Data for $B \to D\pi\pi$



- Recent high precision results for $B \to D\pi\pi$ from LHCb

Aaji et al. [LHCb], Phys. Rev. D **94** (2016) 072001, . . .

- Spectroscopic information in the angular moments ($D\pi$ FSI):



# Chiral Lagrangian for $B \to D$ transitions



Savage, Wise, Phys. Rev. D39 (1989) 3346

- Consider $\bar{B} \to D$ transition with the emission of two light pseudoscalars (pions)

  $\hookrightarrow$ chiral symmetry puts constraints on one of the two pions

  $\hookrightarrow$ the other pion moves fast and does not participate in the final-state interactions

- Chiral effective Lagrangian:

$$\begin{aligned}\mathcal{L}_{\rm eff} = \bar{B}\big[&c_1\,(u_\mu t M + M t u_\mu) + c_2\,(u_\mu M + M u_\mu)\,t \\ &+ c_3\,t\,(u_\mu M + M u_\mu) + c_4\,(u_\mu\langle M t\rangle + M\langle u_\mu t\rangle) \\ &+ c_5\,t\langle M u_\mu\rangle + c_6\langle (M u_\mu + u_\mu M)\,t\rangle\big]\partial^\mu D^\dagger\end{aligned}$$

with

$\bar{B} = (B^-, \bar{B}^0, \bar{B}_s^0)\,, \quad D = (D^0, D^+, D_s^+)$

$M$ is the matter field for the fast-moving pion

$t = uHu$ is a spurion field for Cabbibo-allowed decays

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\to$ only some combinations of the LECs $c_i$ appear

# Theory of $B \to D\pi\pi$



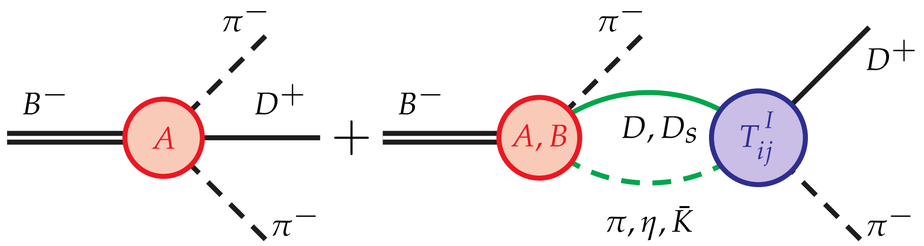
Du, Albadajedo, Fernandez-Soler, Guo, Hanhart, UGM, Nieves, Phys. Rev. **D98** (2018) 094018

- $B^- \to D^+\pi^-\pi^-$ contains coupled-channel $D\pi$ FSI

- consider $S, P, D$ waves: $\mathcal{A}(B^- \to D^+\pi^-\pi^-) = \mathcal{A}_0(s) + \mathcal{A}_1(s) + \mathcal{A}_2(s)$

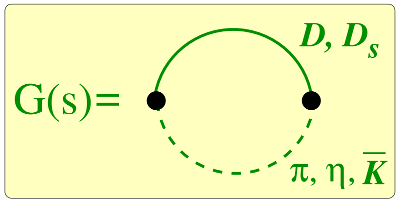
$\to$ P-wave: $D^\star$, $D^\star(2680)$; D-wave: $D_2(2460)$ as by LHCb

$\to$ S-wave: use coupled channel ($D\pi, D\eta, D_s\bar{K}$) amplitudes with all parameters fixed before




$\to$ only two parameters in the S-wave (one combination of the LECs $c_i$ and one subtraction constant in the $G_{ij}$)

$$\mathcal{A}_0(s) \propto E_\pi \left[2 + G_{D\pi}(s)\left(\frac{5}{3}T^{1/2}_{11}(s) + \frac{1}{3}T^{3/2}_{11}(s)\right)\right]$$
$$+\frac{1}{3}E_\eta G_{D\eta}(s)T^{1/2}_{21}(s) + \sqrt{\frac{2}{3}}E_{\bar{K}}G_{D_s\bar{K}}(s)T^{1/2}_{31}(s)$$
$$+C\, E_\eta G_{D\eta}(s)T^{1/2}_{21}(s)$$



# Theory of $B \to D\pi\pi$ continued



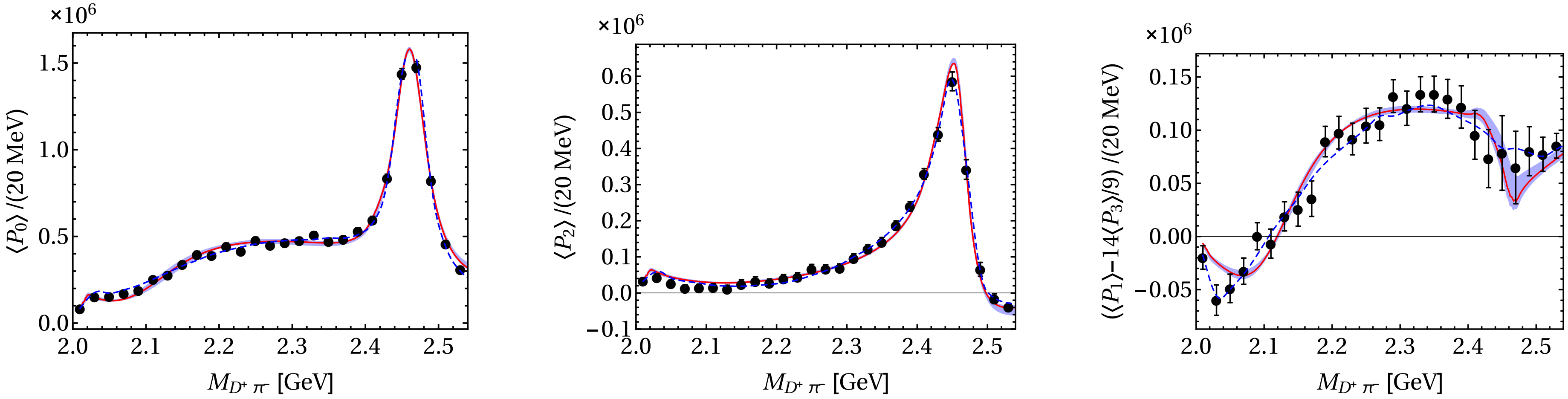
Du, Albadajedo, Fernandez-Soler, Guo, Hanhart, UGM, Nieves, Yao, Phys. Rev. **D98** (2018) 094018

- More appropriate combinations of the angular moments:

$$\langle P_0 \rangle \propto |\mathcal{A}_0|^2 + |\mathcal{A}_1|^2 + |\mathcal{A}_2|^2$$

$$\langle P_2 \rangle \propto \tfrac{2}{5}|\mathcal{A}_1|^2 + \frac{2}{7}|\mathcal{A}_2|^2 + \frac{2}{\sqrt{5}}|\mathcal{A}_0||\mathcal{A}_2|\cos(\delta_2 - \delta_0)$$

$$\langle P_{13} \rangle = \langle P_1 \rangle - \frac{14}{9}\langle P_3 \rangle \propto \frac{2}{\sqrt{3}}|\mathcal{A}_0||\mathcal{A}_1|\cos(\delta_1 - \delta_0)$$



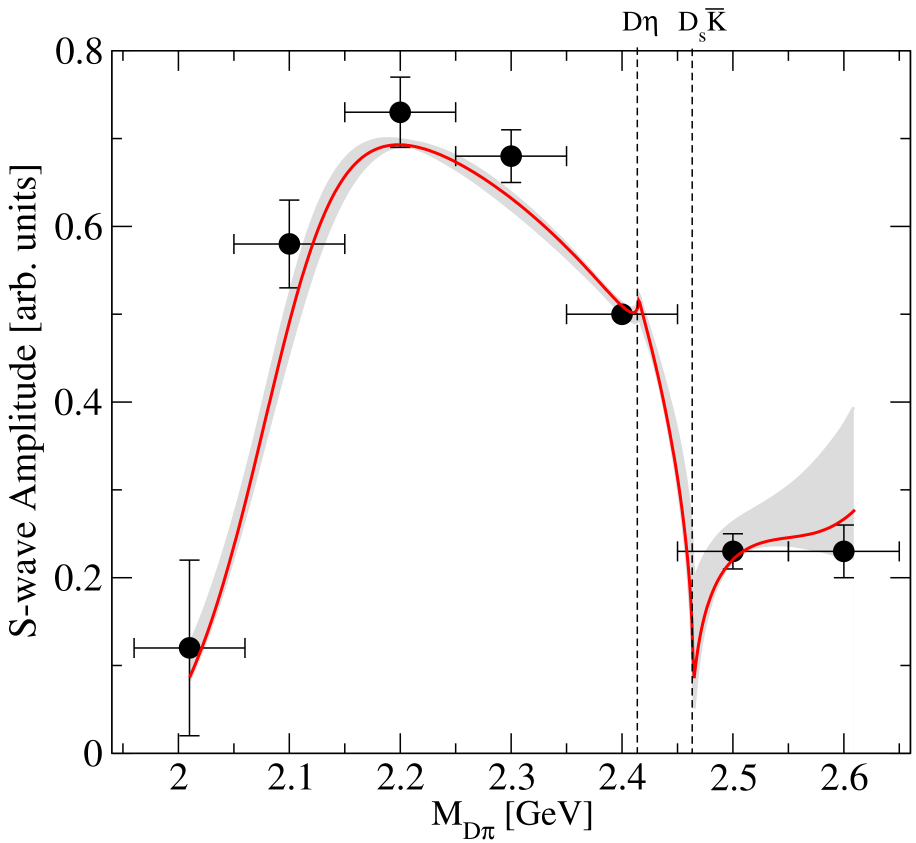

- The **S-wave** $D\pi$ can be very well described using pre-fixed amplitudes
- Fast variation in [2.4,2.5] GeV in $\langle P_{13} \rangle$: cusps at the $D\eta$ and $D_s\bar{K}$ thresholds

$\hookrightarrow$ should be tested experimentally

# A closer look at the S–wave



- LHCb provides anchor points, where the strength and the phase of the S-wave were extracted from the data and connected by cubic spline



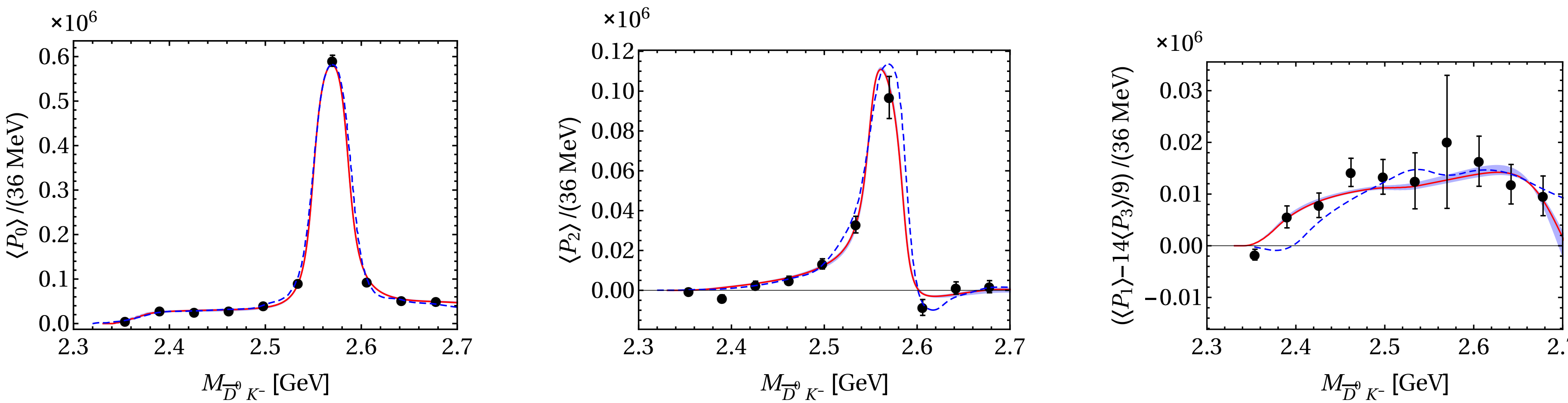

- Higher mass pole at 2.46 GeV clearly amplifies the cusps predicted in our amplitude

# Theory of $B_s^0 \to \bar{D}^0 K^- \pi^+$



Du, Albadajedo, Fernandez-Soler, Guo, Hanhart, UGM, Nieves, Yao, Phys. Rev. **D98** (2018) 094018

- LHCb has also data on $B_s^0 \to \bar{D}^0 K^- \pi^+$, but less precise
- Same formalism as before, one different combination of the LECs $c_i$
- same resonances in the P- and D-wave as LHCb $\hookrightarrow$ one parameter fit!




$\Rightarrow$ these data are also well described

$\Rightarrow$ better data for $\langle P_{13} \rangle$ would be welcome

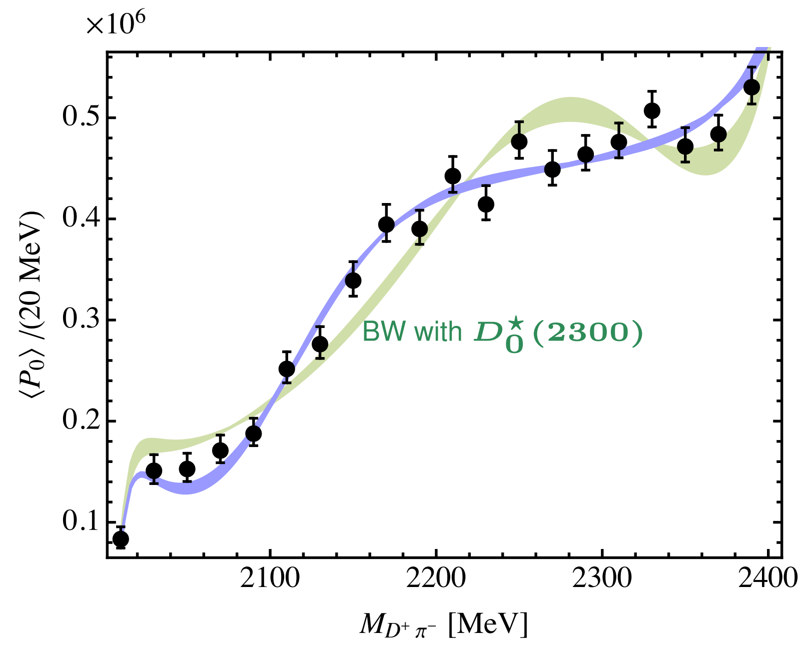
$\Rightarrow$ even more channels, see Du, Guo, UGM, Phys. Rev. D **99** (2019) 114002

# Where is the lowest charm-strange meson?



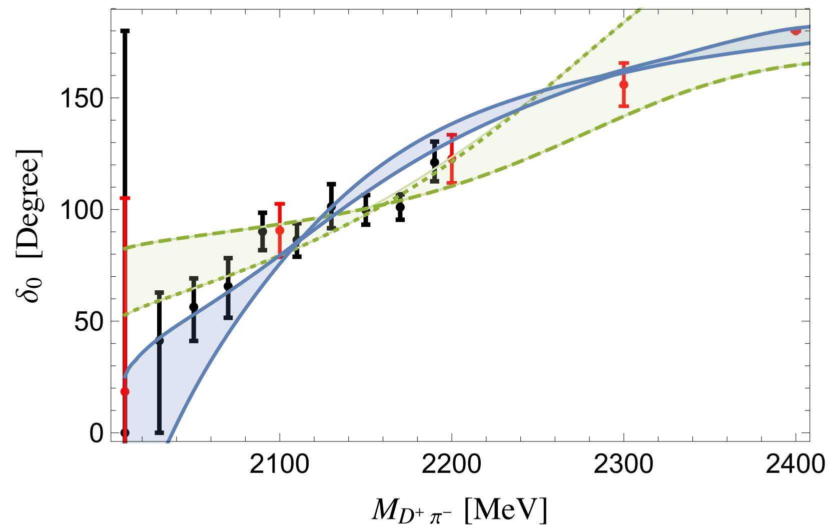
Du, Guo, Hanhart, Kubis, UGM, Phys.Rev.Lett. **126** (2021) 192001 [2012.04599]

- Precise analysis of the LHCb data on $B^- \to D^+\pi^-\pi^-$ using UChPT and Khuri-Treiman eq's (3-body unit.) → spares
  Aaji et al. [LHCb], Phys. Rev. D **94** (2016) 072001




- Breit-Wigner description **not** appropriate for the S-wave but UChPT and the dispersive analysis are! Gardner, UGM, Phys.Rev.D **65** (2002) 094004

- First determination of the $D\pi$ phase shift




- The lowest charm-strange meson is located at:

$$\left(2105^{+6}_{-8} - i\,102^{+10}_{-11}\right) \text{ MeV}$$

- Recently confirmed by Lattice QCD!
  Cheung et al. [HadSpec], JHEP **02** (2021) 100 [2008.06432]

# Lesson 4:
# Hadronic molecules

# What are hadronic molecules ?



- QCD offers yet another set of bound states, first seen in **nuclear physics**
  $\hookrightarrow$ **hadronic molecules** (made of 2 or 3 hadrons)

- Bound states of two hadrons in an S-wave very close a 2-particle threshold or between two close-by thresholds $\Rightarrow$ particular decay patterns

- weak binding entails a large spatial extension

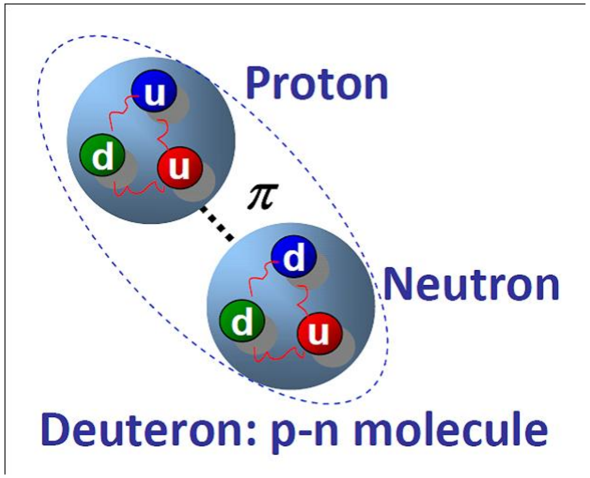
- the classical example:

  $\star$ the deuteron

  $m_p + m_n = 938.27 + 939.57$ MeV,

  $m_d = m_p + m_n - E_B \to E_B = 2.22$ MeV

  $r_d = 2.14$ fm  $[r_p = 0.85$ fm]




- other examples: $\Lambda(1405)$, $f_0(980)$, $X(3872)$, $\ldots$

$\Rightarrow$ how to distinguish these from compact multi-quark states ?

# Compositeness criterion



Weinberg (1965), Morgan (1991), Tornquist (1995), Baru et al. (2003), . . .

- Wave fct. of a bound state with a compact & a two-hadron component in S-wave:

$$|\Psi\rangle = \begin{pmatrix} \sqrt{Z}|\psi_0\rangle \\ \chi(\vec{k})|h_1 h_2\rangle \end{pmatrix}$$

compact comp. w/ probability $\sqrt{Z}$

two-hadron comp. w/ relative w.f. $\chi(\vec{k})$

- consider the scattering amplitude and compare with the ERE:

$$a = -2\frac{1-Z}{2-Z}\left(\frac{1}{\gamma}\right) + \mathcal{O}\left(\frac{1}{\beta}\right)\,,\quad r = -\frac{Z}{1-Z}\left(\frac{1}{\gamma}\right) + \mathcal{O}\left(\frac{1}{\beta}\right)$$

$$\gamma = \sqrt{2\mu E_B}$$

$a$ = scattering length, $\gamma/E_B$ = binding momentum/energy (**shallow** b.s.)

$\mu$ = reduced mass of the two-particle system, $\beta$ = range of forces

$\Rightarrow$ pure molecule ($Z = 0$): maximal scattering length $a = -1/\gamma$
natural effective range $r = \mathcal{O}(1/\beta)$

$\Rightarrow$ compact state ($Z = 1$): the scattering length is $a = -\mathcal{O}(1/\beta)$
effective range diverges, $r \to -\infty$

# The deuteron



Weinberg, Phys. Rev. **137** (1965) B672

- The deuteron: shallow neutron-proton bound state ($E_B \ll m_d$):

$$E_B = 2.22\,\text{MeV} \to \gamma = 45.7\,\text{MeV} \;= 0.23\,\text{fm}^{-1}$$

- range of forces set by the one-pion-exchange:

$$1/\beta \sim 1/M_\pi \simeq 1.4\,\text{fm}$$

- set $Z = 0$ in the Weinberg formula:

$$a_{\rm mol} = -(4.3 \pm 1.4)\,\text{fm}$$

- this is consistent with the data:

$$a = -5.419(7)\,\text{fm}\,,\;\; r = 1.764(8)\,\text{fm}$$

> One begins to suspect that Nature is doing her best to keep us from learning whether the "elementary" particles deserve that title. (Weinberg, 1965)

# Extension to resonances



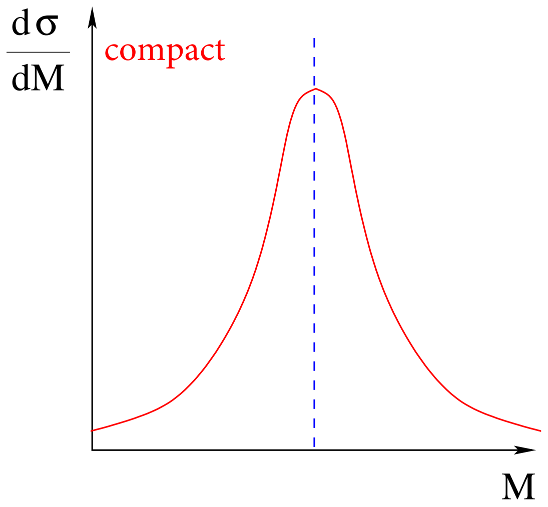
Baru et al. (2003), Braaten, Lu (2007), Aceti, Oset (2012), Guo, Oller (2016), . . .

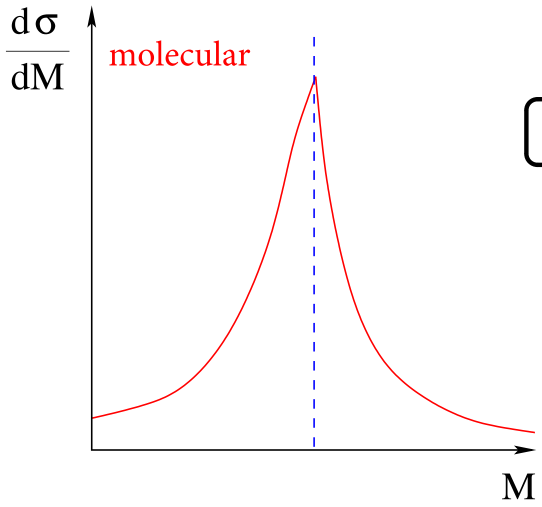
- Still assume closeness to a two-particle threshold:

$$T(E) = \frac{g^2/2}{E - E_r + (g^2/2)(ik + \gamma) + i\Gamma_0/2}$$

with $E = k^2/(2\mu)$ , $\Gamma_0$ accounts for the inelasticities of other channels

- leads to very different **line shapes** for compact and molecular states:




$M = m_1 + m_2 + E$

$k^2$ term dominates $\rightarrow$ symmetric

$g^2$ term dominates $\rightarrow$ asymmetric/cusp

- extension to instable particles/additional poles have also been worked out

# Some candidates



- Prominent examples in the light quark sector:

  $f_0(980)$, $a_0(980)$, the two $\Lambda(1405)$, . . .

  ↪ see next lesson

- Prominent examples in the $c\bar{c}$ spectrum:

  $X(3872)$, $Z_c(3900)$, $Y(4260)$, $Y(4660)$, . . .

- Prominent examples of heavy-light mesons:

  $D^\star_{s0}(2317)$, $D_{s1}(2460)$, $D^\star_{s1}(2860)$, . . .

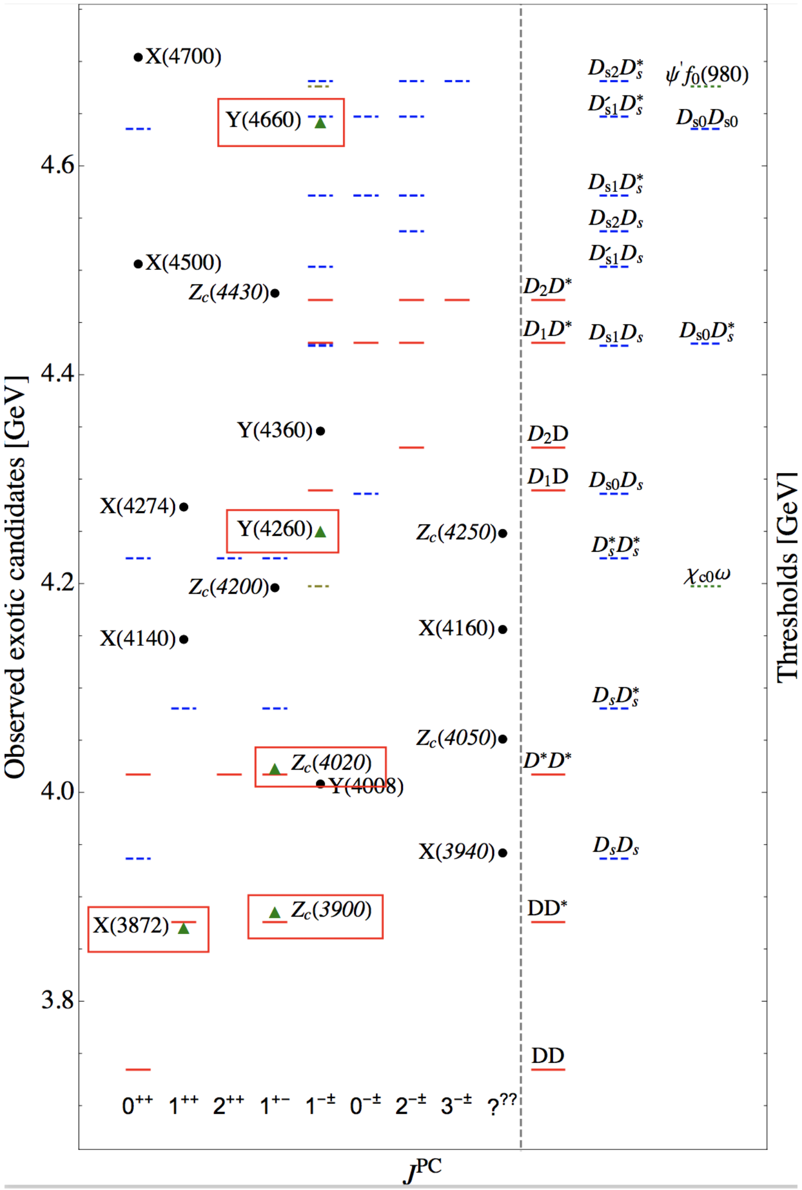
- Prominent examples in the $b\bar{b}$ spectrum:

  $Z_b(10610)$, $Z_b(10650)$

- and some examples of heavy baryons:

  $\Lambda_c(2595)$, $\Lambda_c(2940)$, $P_c(4312)$, $P_c(4557)$, . . .

- rich phenomenology, discuss just one apect




↪much more details in: Guo, Hanhart, UGM, Wang, Zhao, Zou, Rev. Mod. Phys. **90** (2018) 015004

# Misconceptions on hadroproduction



Albaladejo, Guo, Hanhart, UGM, Nieves, Nogga, Yang, Chin.Phys. C **41** (2017) 121001

- It is often claimed that molecules due to their large spatial extent can not be produced in high-energy collisions, say at the LHC $\rightarrow$ **this is wrong!**

Bignamini, Grinstein, Piccinini, Polosa, Sabelli, Phys. Rev. Lett. **103** (2009) 162001

$$\begin{aligned}
\sigma(\bar{p}p \to X) &\sim \left|\int d^3\mathrm{k} \langle X|D^0\bar{D}^{*0}(\mathrm{k})\rangle\langle D^0\bar{D}^{*0}(\mathrm{k})|\bar{p}p\rangle\right|^2 \\
&\simeq \left|\int_{\mathcal{R}} d^3\mathrm{k} \langle X|D^0\bar{D}^{*0}(\mathrm{k})\rangle\langle D^0\bar{D}^{*0}(\mathrm{k})|\bar{p}p\rangle\right|^2 \\
&\leq \int_{\mathcal{R}} d^3\mathrm{k}\, |\Psi(\mathrm{k})|^2 \int_{\mathcal{R}} d^3\mathrm{k} \left|\langle D^0\bar{D}^{*0}(\mathrm{k})|\bar{p}p\rangle\right|^2 \\
&\leq \int_{\mathcal{R}} d^3\mathrm{k} \left|\langle D^0\bar{D}^{*0}(\mathrm{k})|\bar{p}p\rangle\right|^2
\end{aligned}$$

- The result depends crucially on the value of $\mathcal{R}$ which specifies the region where the bound state wave function "$\Psi(\mathrm{k})$ is significantly different from zero"

- assumption by Bignamini et al: $\mathcal{R} \simeq 35$ MeV of the order of $\gamma$

  $\hookrightarrow$ $\sigma(\bar{p}p \to X) \simeq 0.07$ nb way smaller than experiment

  $\hookrightarrow$ the X(3872) can not be a molecule

  $\hookrightarrow$ so what goes wrong?

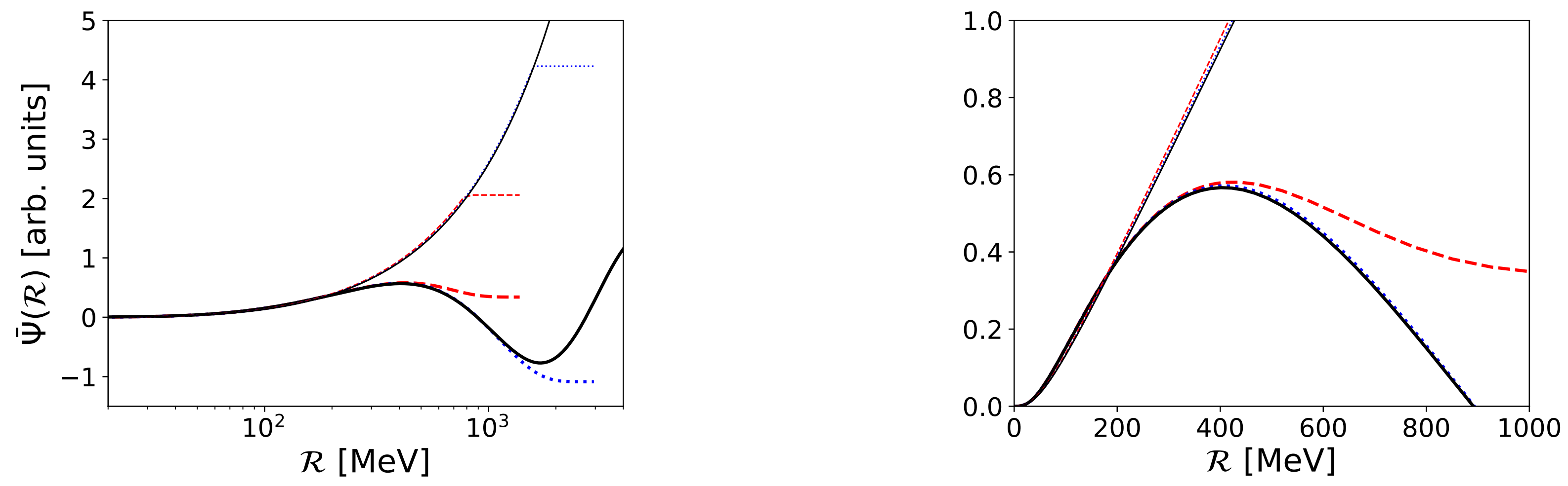
# Misconceptions on hadroproduction cont'd



Albaladejo, Guo, Hanhart, UGM, Nieves, Nogga, Yang, Chin.Phys. C **41** (2017) 121001

- Consider the relevant integral for the deuteron: $\bar{\Psi}_\lambda(\mathcal{R}) \equiv \int_{\mathcal{R}} d^3k\, \Psi_\lambda(k)$

- the binding momentum is $\gamma \simeq 45$ MeV, use that for the support $\mathcal{R}$:




$\hookrightarrow$ the integral is by far not saturated for $\mathcal{R} = \gamma$, need $\mathcal{R} \simeq 2M_\pi \simeq 300$ MeV

- Similar misconception: Molecules can not be produced at large $p_T$

$\hookrightarrow$ true for nuclei but not quarkonia and alike ($q$ versus $\bar{q}$)

# Hadroproduction of the X(3872)



- Nice example of a process involving short-distance physics

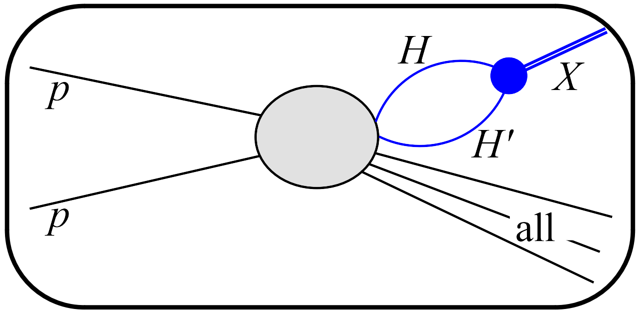
$\hookrightarrow$ still, factorization is at work, best seen using EFT

Artoisenet, Braaten, Phys. Rev. D **81** (2010) 114018

$\hookrightarrow$ consider production at the Tevatron and at LHC

$$\sigma[X] = \frac{1}{4 m_H m_{H'}} g^2 |G|^2 \left( \frac{d\sigma[HH'(k)]}{dk} \right)_{\text{MC}} \frac{4\pi^2 \mu}{k^2}$$

$$G(E,\Lambda) = -\frac{\mu}{\pi^2} \left[ \sqrt{2\pi}\, \frac{\Lambda}{4} + \sqrt{\pi}\, \gamma D \left( \frac{\sqrt{2}\gamma}{\Lambda} \right) - \frac{\pi}{2} \gamma\, e^{2\gamma^2/\Lambda^2} \right]$$




- typical results (using PYTHIA/HERWIG):

Guo, UGM, Wang, Yang, Eur. Phys. J. C **74** (2014) 3063

| $\sigma(pp/\bar{p} \to X(3872))$ | $\Lambda = 0.5 - 1.0$ GeV | Exp. |
|---|---|---|
| Tevatron | 5 - 29 [nb] | 37 - 115 [nb] |
| LHC7 | 4 - 55 [nb] | 13 - 39 [nb] |

$\Rightarrow$ not very precise, but perfectly consistent with the data!

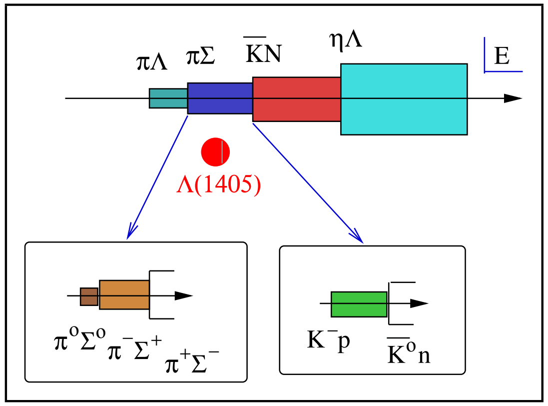
$\Rightarrow$ also predictions for the charm-strange mesons

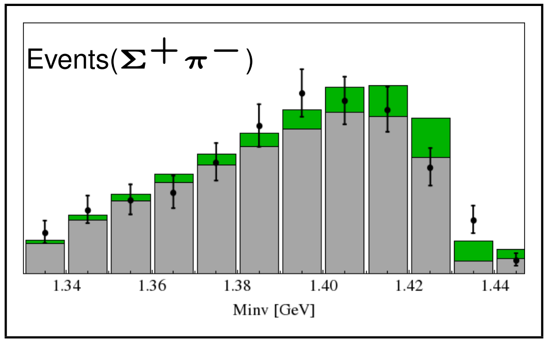
Guo, UGM, Wang, Yang, JHEP **1405** (2014) 138

# Lesson 5: A short tale of the two $\Lambda(1405)$ states

# The first exotic hadron – the story of the two $\Lambda(1405)$ 

- Quark model: $uds$ excitation with $J^P = \frac{1}{2}^-$ CLAS (2014)
  a few hundred MeV above the $\mathbf{\Lambda(1116)}$
  $m = 1405.1^{+1.3}_{-1.0}\,\mathrm{MeV}\,,\ \Gamma = 50.5 \pm 2.0\,\mathrm{MeV}$ [PDG 2015]

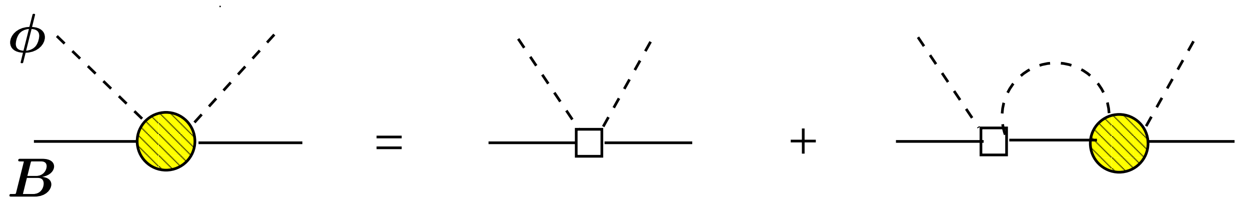



- Prediction as early as 1959 by Dalitz and Tuan:
  Resonance between the coupled $\pi\Sigma$ and $\bar{K}N$ channels
  Dalitz, Tuan, Phys. Rev. Lett. **2** (1959) 425; J.K. Kim, PRL **14** (1965) 29

- Clearly seen in $K^-p \to \Sigma 3\pi$ reactions at 4.2 GeV at CERN
  Hemingway, Nucl.Phys. B **253** (1985) 742

- An enigma: Too low in mass for the quark model,
  but well described in unitarized chiral perturbation theory: $\phi B \to \phi B$




Kaiser, Siegel, Weise, Ramos, Oset, Oller, UGM, Lutz, ...

# The two-pole scenario

- Detailed analysis found **two** poles in the complex energy plane

  Oller, UGM, Phys. Lett. B **500** (2001) 263

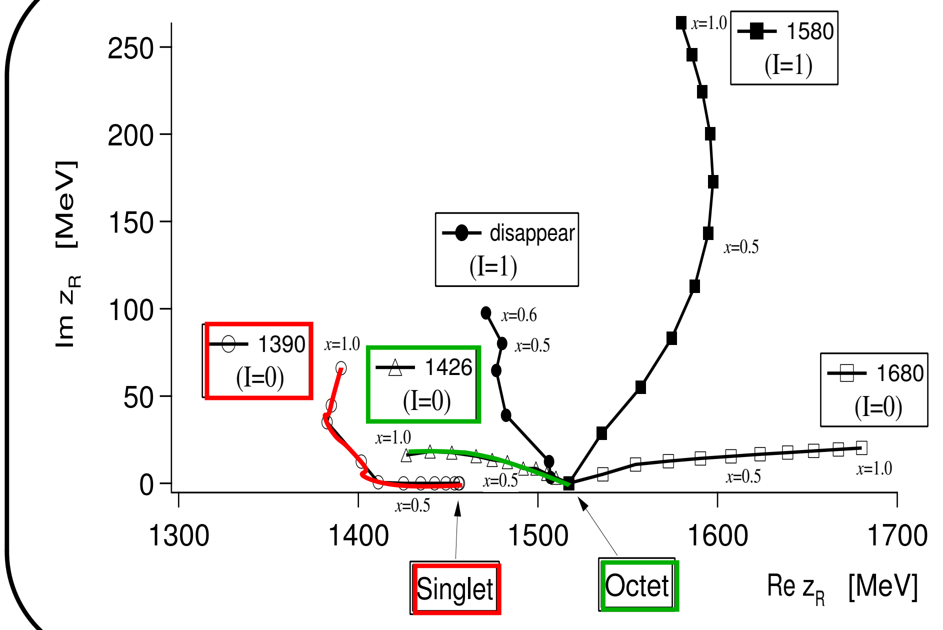
- Group theory:

$$\mathbf{8} \otimes \mathbf{8} = \underbrace{\mathbf{1} \oplus \mathbf{8}_s \oplus \mathbf{8}_a}_{\text{binding at LO}} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{27}$$

- Follow the pole movement from the SU(3) limit to the physical masses:

  Jido, Oller, Oset, Ramos, UGM,
  Nucl. Phys. A **725** (2003) 181

- Verified by various groups world-wide




- However: scattering and kaonic atom data alone do not lead to a unique solution (two poles, but spread in the complex plane)

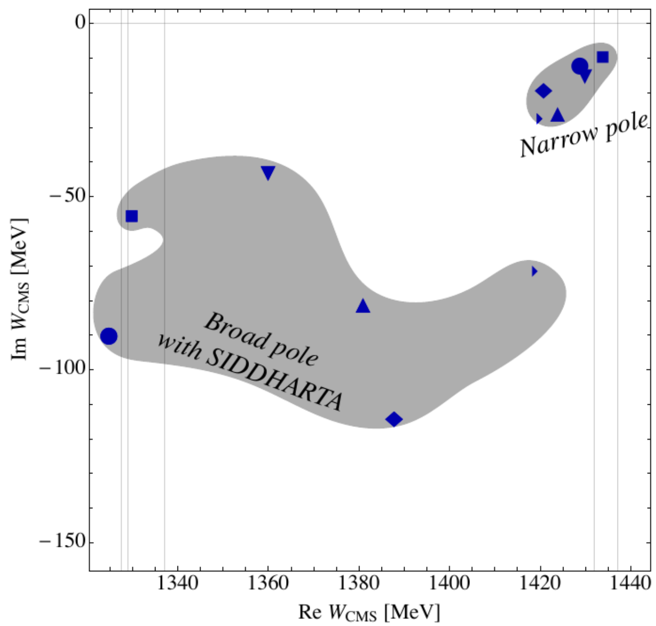
- Photoproduction to the rescue: $\gamma p \to K^+ \Sigma\pi$ CLAS, Phys. Rev. C **87**, 035206 (2013)

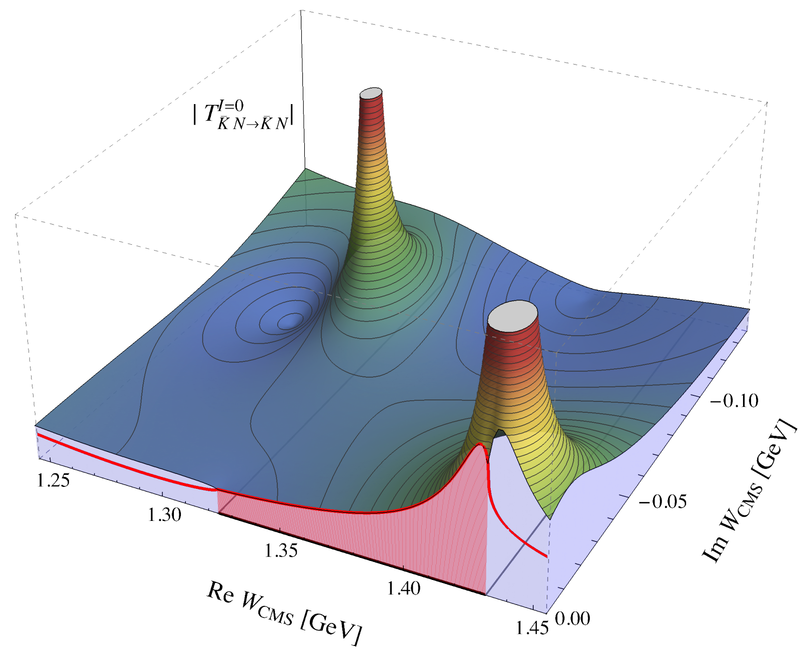
# Present status of the two-pole scenario



- Two poles from scattering plus CLAS data (one well, the other not-so-well fixed):

for details, see Mai, Eur. Phys. J. ST **230** (2021) 1593 [arXiv:2010.00056 [nucl-th]]




Figures courtesy Maxim Mai

Resonances are poles in the complex plane!

→ PDG 2016: **http://pdg.lbl.gov/2015/reviews/rpp2015-rev-lam-1405-pole-struct.pdf**

POLE STRUCTURE OF THE $\Lambda(1405)$ REGION
Written November 2015 by Ulf-G. Meißner and Tetsuo Hyodo

# Status in the Review of Particle Physics



- Two excited $\Lambda$ states listed in the 2020 RPP edition:

P. A. Zyla *et al.* [Particle Data Group], PTEP **2020** (2020) 083C01

Citation: P.A. Zyla *et al.* (Particle Data Group), Prog. Theor. Exp. Phys. **2020**, 083C01 (2020)

**$\Lambda(1380)\ 1/2^-$** $J^P = \frac{1}{2}^-$ Status: ∗∗

OMITTED FROM SUMMARY TABLE
See the related review on "Pole Structure of the $\Lambda(1405)$ Region."

Citation: P.A. Zyla *et al.* (Particle Data Group), Prog. Theor. Exp. Phys. **2020**, 083C01 (2020)

**$\Lambda(1405)\ 1/2^-$** $I(J^P) = 0(\frac{1}{2}^-)$ Status: ∗∗∗∗

In the 1998 Note on the $\Lambda(1405)$ in PDG 98, R.H. Dalitz discussed the S-shaped cusp behavior of the intensity at the $N$-$\overline{K}$ threshold observed in THOMAS 73 and HEMINGWAY 85. He commented that this behavior "is characteristic of $S$-wave coupling; the other below threshold hyperon, the $\Sigma(1385)$, has no such threshold distortion because its $N$-$\overline{K}$ coupling is $P$-wave. For $\Lambda(1405)$ this asymmetry is the sole direct evidence that $J^P = 1/2^-$."

A recent measurement by the CLAS collaboration, MORIYA 14, definitively established the long-assumed $J^P = 1/2^-$ spin-parity assignment of the $\Lambda(1405)$. The experiment produced the $\Lambda(1405)$ spin-polarized in the photoproduction process $\gamma p \to K^+ \Lambda(1405)$ and measured the decay of the $\Lambda(1405)$ (polarized) $\to \Sigma^+$(polarized)$\pi^-$. The observed isotropic decay of $\Lambda(1405)$ is consistent with spin $J = 1/2$. The polarization transfer to the $\Sigma^+$(polarized) direction revealed negative parity, and thus established $J^P = 1/2^-$.

**See the related review(s):**
Pole Structure of the $\Lambda(1405)$ Region

Hyodo, UGM

- a new two-star resonance at 1380 MeV
- still not in the summary table
- there are more such two-pole states!
- this is a fascinating phenomenon intimately tied to molecular structures
- for a review, see UGM, *Symmetry* **12** (2020) 981
- Two $\Lambda$s: recenty confirmed by lattice QCD Bulava et al., 2307.10413 [hep-lat]

# Lesson 5: The width of baryon resonances from EFT

# EFT including the $\Delta$



- Task: calculate the width of the $\Delta$ at two-loop order [one-loop too simple]

Gegelia, UGM, Siemens, Yao, Phys. Lett. B763 (2016) 1

- Consider the effective chiral Lagrangian of pions, nucleons and deltas:

$$\mathcal{L}^{(1)}_{\pi N} = \bar{\Psi}_N \left\{ i\not{D} - m + \tfrac{1}{2} g \not{u} \gamma^5 \right\} \Psi_N$$

$$\begin{aligned}\mathcal{L}^{(1)}_{\pi\Delta} = & -\bar{\Psi}^i_\mu \xi^{\frac{3}{2}}_{ij} \Big\{ \left( i\not{D}^{jk} - m_\Delta \delta^{jk} \right) g^{\mu\nu} - i \left( \gamma^\mu D^{\nu,jk} + \gamma^\nu D^{\mu,jk} \right) + i\gamma^\mu \not{D}^{jk} \gamma^\nu \\ & + m_\Delta \delta^{jk} \gamma^\mu \gamma^\nu + g_1 \tfrac{1}{2} \not{u}^{jk} \gamma_5 g^{\mu\nu} + g_2 \tfrac{1}{2} (\gamma^\mu u^{\nu,jk} + u^{\nu,jk} \gamma^\mu) \gamma_5 \\ & + g_3 \tfrac{1}{2} \gamma^\mu \not{u}^{jk} \gamma_5 \gamma^\nu \Big\} \xi^{\frac{3}{2}}_{kl} \Psi^l_\nu\end{aligned}$$

$$\mathcal{L}^{(1)}_{\pi N\Delta} = h\, \bar{\Psi}^i_\mu \xi^{\frac{3}{2}}_{ij} \Theta^{\mu\alpha}(z_1)\, \omega^j_\alpha \Psi_N + \text{h.c.}$$

$$\mathcal{L}^{(2)}_{\pi N\Delta} = \bar{\Psi}^i_\mu \xi^{\frac{3}{2}}_{ij} \Theta^{\mu\alpha}(z_2) \left[ i\, b_3 \omega^j_{\alpha\beta} \gamma^\beta + i\, b_8 \tfrac{1}{m} \omega^j_{\alpha\beta} i\, D^\beta \right] \Psi_N + \text{h.c.} + \ldots$$

$$\begin{aligned}\mathcal{L}^{(3)}_{\pi N\Delta} = & \bar{\Psi}^i_\mu \xi^{\frac{3}{2}}_{ij} \Theta^{\mu\nu}(z_3) \Big[ f_1 \tfrac{1}{m} [D_\nu, \omega^j_{\alpha\beta}] \gamma^\alpha i\, D^\beta - f_2 \tfrac{1}{2m^2} [D_\nu, \omega^j_{\alpha\beta}] \{D^\alpha, D^\beta\} + f_4 \omega^j_\nu \langle \chi_+ \rangle \\ & + f_5 [D_\nu, i\chi^j_-] \Big] \Psi_N + \text{h.c.} + \ldots\end{aligned}$$

- Power counting rests on $m_\Delta - m_N$ being a small quantity

- So many LECs, how can one possibly make a prediction?

# Complex-mass renormalization

- Method originally introduced for $W$, $Z$-physics, later transported to chiral EFT

  Stuart (1990), Denner, Dittmaier et al. (1999), Actis, Passarino (2007)

  Djukanovic, Gegelia, Keller, Scherer, Phys. Lett. B680 (2009) 235

- Evaluate the $\Delta$ self-energy on the complex pole:

$$z - m_\Delta^0 - \Sigma_1(z^2) - z\,\Sigma_6(z^2) \equiv z - m_\Delta^0 - \Sigma(z) = 0 \text{ with } \boxed{z = m_\Delta - i\,\frac{\Gamma_\Delta}{2}}$$

- Self-energy diagrams:

$\rightarrow$ one-loop easy

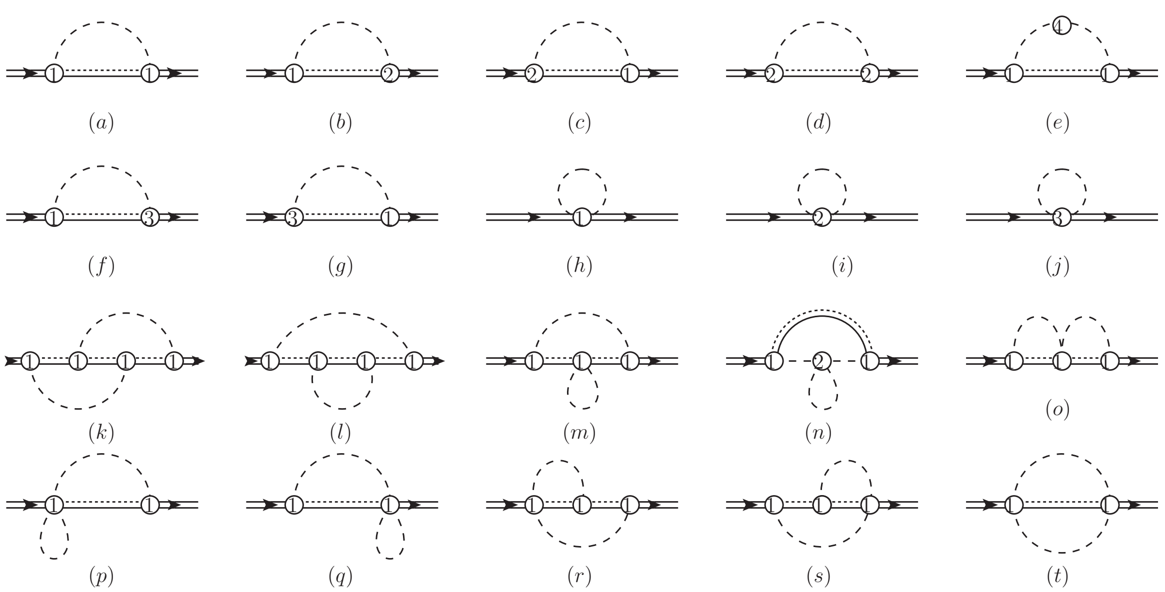
$\rightarrow$ two-loops:

use Cutkovsky rules for

instable particles

$\rightarrow$ width $\sim |A(\Delta \rightarrow N\pi)|^2$

Veltman, Physica 29 (1963) 186



# Calculation of the width

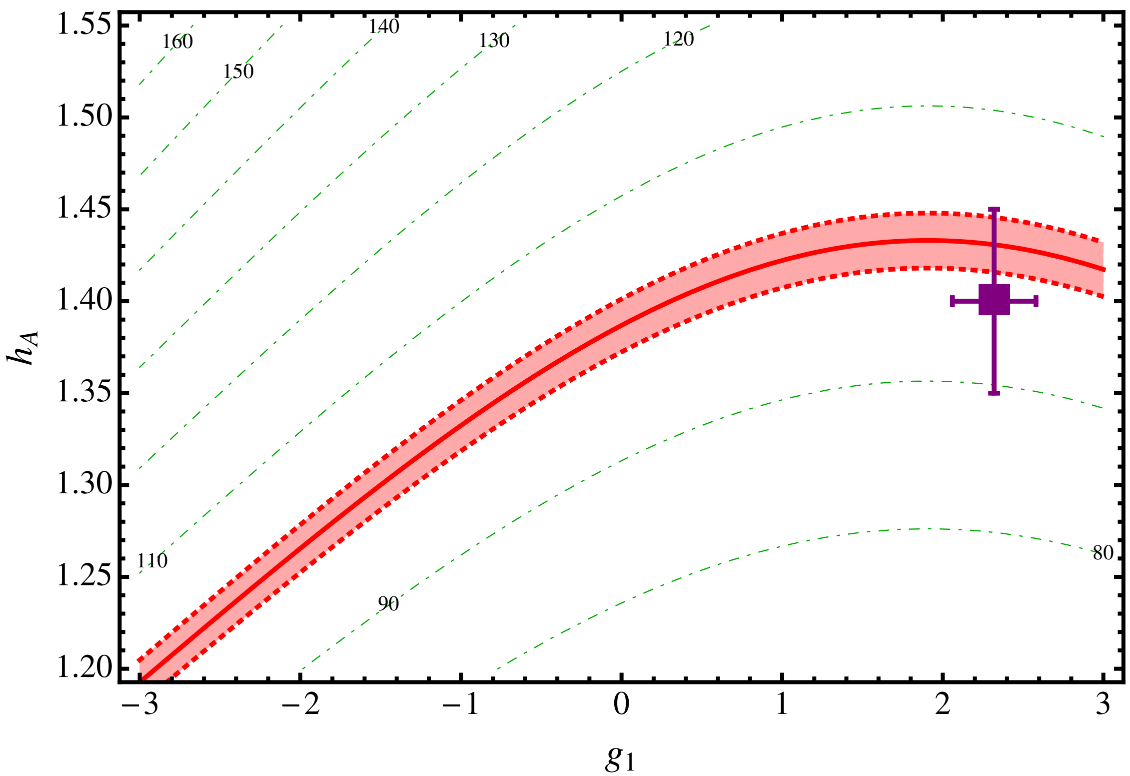
- Remarkable reduction of parameters: $\Delta_{23} = m_N - m_\Delta,\ \Delta_{123} = (M_\pi^2 + m_N^2 - m_\Delta^2)/(2m_N)$

$$h_A = h - (b_3\Delta_{23} + b_8\,\Delta_{123}) - (f_1\Delta_{23} + f_2\,\Delta_{123})\,\Delta_{123} + 2(2f_4 - f_5)M_\pi^2$$

- Very simple formula for the decay width $\Delta \to N\pi$:

$$\Gamma(\Delta \to N\pi) = (53.91\,h_A^2 + 0.87 g_1^2 h_A^2 - 3.31 g_1 h_A^2 - 0.99\,h_A^4)\ \mathrm{MeV}$$

- Correlation:




■ large $N_C$ w/ unc.

Siemens et al.,
Phys. Lett. B 770 (2017) 27

# EFT including the Roper-resonance

- Task: calculate the width of the Roper $N^*(1440)$ at two-loop order

Gegelia, UGM, Yao, Phys. Lett. B760 (2016) 736

- Remarkable feature: $\Gamma(R \to N\pi) \simeq \Gamma(R \to N\pi\pi)$

- Consider the effective chiral Lagrangian of pions, nucleons and deltas:

Borasoy et al., Phys. Lett. B641 (2006) 294, Djukanovic et al., Phys. Lett. B690 (2010) 123

Long, van Kolck, Nucl. Phys. A870-871 (2011) 72

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\pi\pi} + \mathcal{L}_{\pi N} + \mathcal{L}_{\pi\Delta} + \mathcal{L}_{\pi R} + \mathcal{L}_{\pi N\Delta} + \mathcal{L}_{\pi NR} + \mathcal{L}_{\pi\Delta R}$$

$$\mathcal{L}_{\pi R}^{(1)} = \bar{\Psi}_R \left\{ i\not{D} - m_R + \tfrac{1}{2} g_R \not{u} \gamma^5 \right\} \Psi_R$$

$$\mathcal{L}_{\pi R}^{(2)} = \bar{\Psi}_R \left\{ c_1^R \langle \chi_+ \rangle \right\} \Psi_R + \ldots$$

$$\mathcal{L}_{\pi NR}^{(1)} = \bar{\Psi}_R \left\{ \tfrac{1}{2} g_{\pi NR} \gamma^\mu \gamma_5 u_\mu \right\} \Psi_N + \text{h.c.}$$

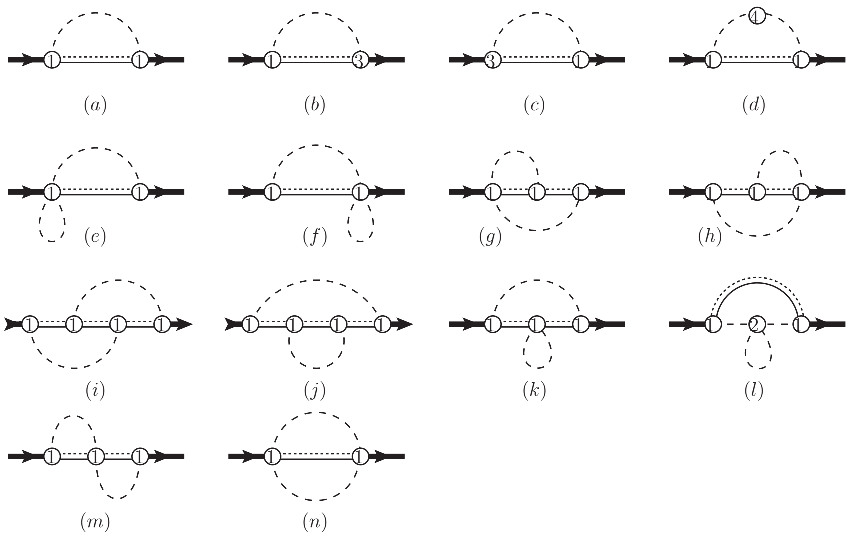
$$\mathcal{L}_{\pi\Delta R}^{(1)} = h_R \, \bar{\Psi}_\mu^i \xi_{ij}^{\frac{3}{2}} \Theta^{\mu\alpha}(\tilde{z}) \, \omega_\alpha^j \Psi_R + \text{h.c.}$$

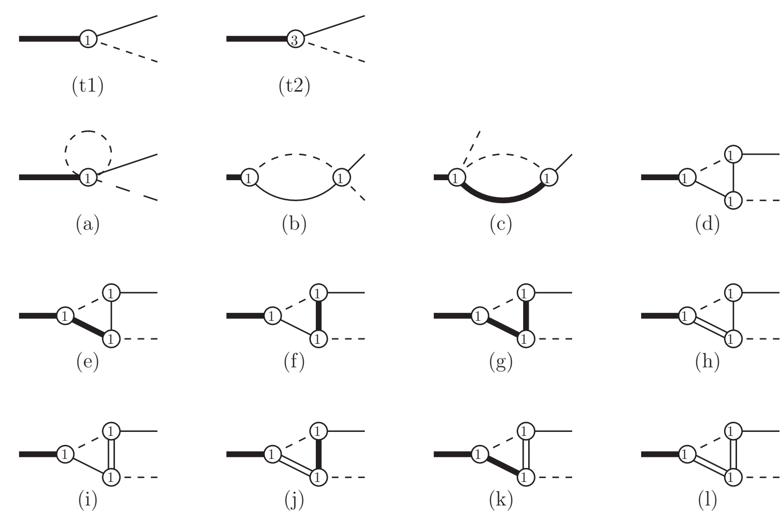
# EFT including the Roper-resonance cont'd

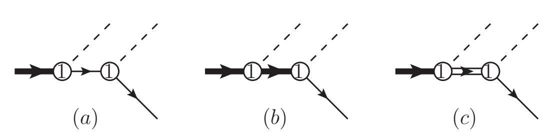
- The power counting is complicated, but can be set up around the complex pole:

$$m_R - m_N \sim \varepsilon\ ,\ \ m_R - m_\Delta \sim \varepsilon^2\ ,\ \ m_\Delta - m_N \sim \varepsilon^2\ ,\ \ M_\pi \sim \varepsilon^2$$

- Calculate the two-loop self-energy and the corresponding decay amplitudes



# Calculation of the width



- A lengthy calculation leads to:

$$\Gamma(R \to N\pi) = 550(58)\, g_{\pi NR}^2 \,\mathrm{MeV}$$

$$\Gamma(R \to N\pi\pi) = \Big(1.49(0.58)\, g_A^2\, g_{\pi NR}^2 - 2.76(1.07)\, g_A\, g_{\pi NR}^2\, g_R + 1.48(0.58)\, g_{\pi NR}^2\, g_R^2 + 2.96(0.94)\, g_A\, g_{\pi NR}\, h h_R - 3.79(1.37)\, g_{\pi NR}\, g_R\, h h_R + 9.93(5.45)\, h^2 h_R^2\Big) \,\mathrm{MeV}$$

- Fix $g_{\pi NR}$ from the PDG value: $g_{\pi NR} = \pm(0.47 \pm 0.05)$ PDG 2016

- Maximal mixing assumption: $g_R = g_A \,, \quad h_R = h$

Beane, van Kolck, J. Phys. G31 (2005) 921

↪ can make a prediction for the two-pion decay width of the Roper

$$\boxed{\Gamma(R \to N\pi\pi) = (41 \pm 22_{\mathrm{LECs}} \pm 17_{\mathrm{h.o.}})\,\mathrm{MeV}}$$

- consistent with the PDG value of $(67 \pm 10)$ MeV
- need an improved determination of the LECs $g_R$ and $h_R$

# Summary & Outlook

# Summary & outlook: Take home messages



The QCD spectrum is more than a collection of quark model states

Structure formation in QCD ties nuclear and hadron physics together

Lattice QCD is making progress in addressing complex resonance properties (must respect symmetries)

EFTs are of utmost importance in pushing this program forward

Forget Breit-Wigner if close to threshold / coupled-channels

# SPARES

# LATTICE QCD

- In principle ab initio calcs of non–pert. QCD on a discretized space–time

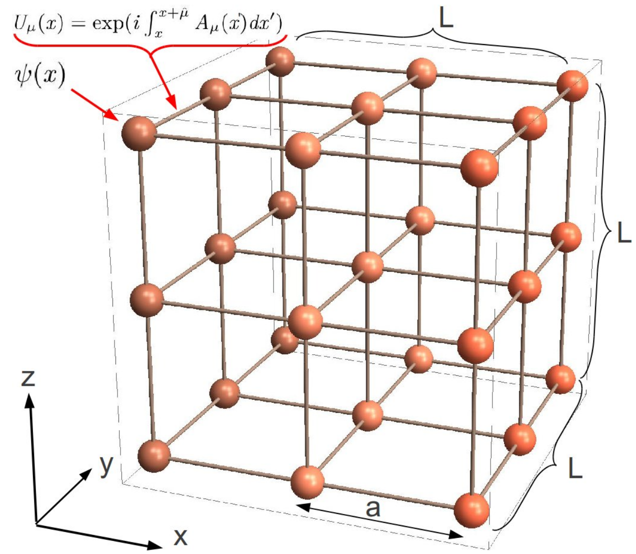
  $\hookrightarrow$ already some successes *but only now entering the chiral regime*

- Extrapolations neccessary:

  $\star$ finite volume $V = L^3 \times L_t \to \infty$

  $\star$ finite lattice spacing $a \to 0$

  $\star$ chiral extrapolation $m_q \to m_q^{\rm phys}$




- All these effects can be treated in suitably tailored EFTs

- how are resonances defined in such a finite space-time?

  $\Rightarrow$ consider finite volume effects for **low-lying hadron resonances**

# Some formalism



- Exact three-body unitarity via Khuri-Treiman equations: Khuri, Treiman (1960)

↪ write $\mathcal{A}_{+--}(B^- \to D^+\pi^-\pi^-)$ and $\mathcal{A}_{00-}(B^- \to D^0\pi^0\pi^-)$ as [reconstruction theorem]

$$\mathcal{A}_{+--}(s,t,u) = \mathcal{F}_0^{1/2}(s) + \frac{\kappa(s)}{4} z_s \mathcal{F}_1^{1/2}(s) + \frac{\kappa(s)^2}{16}(3z_s^2-1)\mathcal{F}_2^{1/2}(s) + (t \leftrightarrow s)$$

$$\mathcal{A}_{00-}(s,t,u) = -\frac{1}{\sqrt{2}}\mathcal{F}_0^{1/2}(s) - \frac{\kappa(s)}{4\sqrt{2}} z_s \mathcal{F}_1^{1/2}(s) - \frac{\kappa(s)^2}{16\sqrt{2}}(3z_s^2-1)\mathcal{F}_2^{1/2}(s) + \frac{\kappa_u(u)}{4} z_u \mathcal{F}_1^1(u)$$

$$z_s = \cos\theta_s = \frac{s(t-u)-\Delta}{\kappa(s)}, \; z_u = \cos\theta_u = \frac{t-s}{\kappa_u(u)}\,, \quad \Delta = (M_B^2 - M_\pi^2)(M_D^2 - M_\pi^2)$$

$$\kappa(s) = \lambda^{1/2}(s, M_D^2, M_\pi^2)\lambda^{1/2}(s, M_B^2, M_\pi^2), \; \kappa_u(u) = \lambda^{1/2}(u, M_B^2, M_D^2)\sqrt{1-4M_\pi^2/u}$$

$\mathcal{F}_\ell^I$: angular momentum $\ell \leq 2$, isospin $I < 3/2$

- Solve via the Omnès ansatz:

$$\mathcal{F}_\ell^I(s) = \Omega_\ell^I(s)\left\{Q_\ell^I(s) + \frac{s^n}{\pi}\int_{s_{\rm th}}^\infty \frac{ds'}{s'^n}\frac{\sin\delta_\ell^I(s')\hat{\mathcal{F}}_\ell^I(s')}{|\Omega_\ell^I(s')|(s'-s)}\right\}\,,$$

$Q_\ell^I(s)$ = polynom of degree zero (one subtraction suffices)

$$\Omega_\ell^I(s) = \exp\left\{\frac{s}{\pi}\int_{s_{\rm th}}^\infty ds' \frac{\delta_\ell^I(s')}{s'(s'-s)}\right\}$$