Lattice QCD meets effective field theories: two- and three-particle decays

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Plan

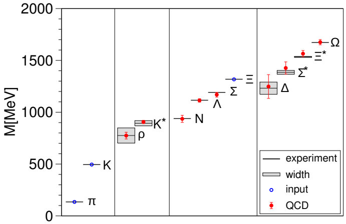
- Introduction
- Scale separation and the choice of the EFT
- Two-body sector: Lüscher equation, two-body decays
- Three (and more) particles
- Conclusions, outlook

Hadronic input in the studies of the BSM physics

Examples:

- Direct and indirect CP violation in $K \to 2\pi$ decays (lattice: RBC and UKQCD collaborations)
- CP violation in $K \to 3\pi$ decays, and so on...

In QCD, the structure of hadrons and their interactions at low energies cannot be studied in perturbation theory $\to$ **QCD on the lattice**

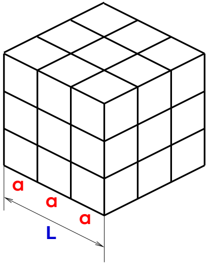


Meson and baryon spectrum in QCD, S. Dürr *et al.*, Science 322 (2008) 1224

"Scattering" in a finite volume

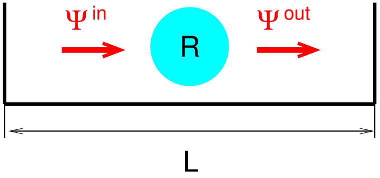
The scattering observables cannot be directly extracted from the amplitudes calculated on the lattice!

- (Periodic) boundary conditions imposed
- The spatial size of the box, L, is finite
- Assume the temporal size $L_t \gg L$, $L_t \to \infty$
- Three-momenta are quantized $\mathbf{p} = \dfrac{2\pi}{L}\mathbf{n}\,, \quad \mathbf{n} \in \mathbb{Z}^3$
- Discrete energy levels: $E_{n+1} - E_n = O(L^{-2})$



How does one extract the scattering observables:
phase shifts, cross sections, ... from the measured quantities
on the lattice?

EFT meets lattice



- When $R \ll L$, well-separated hadrons can be formed
- Natural scale separation
- Since $p \sim 1/L$ and $R \sim 1/m$, then $p \ll m$: non-relativistic EFT
  - Polarization effects, caused by creation/annihilation of the particles, are exponentially small and can be neglected

Scale separation: QCD (in a finite volume) $\Rightarrow$ EFT (in a finite volume)

Non-relativistic EFT: essentials

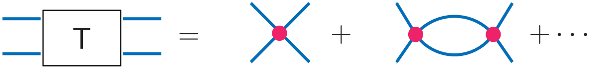
- Propagator:

$$\frac{1}{m^2-p^2} = \underbrace{\frac{1}{2w(\mathbf{p})(w(\mathbf{p})-p^0-i\varepsilon)}}_{particle} + \underbrace{\frac{1}{2w(\mathbf{p})(w(\mathbf{p})+p^0-i\varepsilon)}}_{anti-particle}$$

- The vertices in the Lagrangian conserve particle number:

$$\mathscr{L} = \phi^\dagger(i\partial_t - w)(2w)\phi + \underbrace{\frac{C_0}{4}\,\phi^\dagger\phi^\dagger\phi\phi + \frac{D_0}{36}\,\phi^\dagger\phi^\dagger\phi^\dagger\phi\phi\phi + \cdots}_{C_0,D_0 \text{ encode short-range physics}}$$

- Only bubble diagrams:



K-matrix

Matching of the EFT couplings

- Matching the EFT couplings in the two-body sector to the effective range expansion parameters:

$$K^{-1}(p) = p\cot\delta(p) = -\frac{1}{a} + \frac{1}{2}\,rp^2 + O(p^4)\,, \qquad C_0 \leftrightarrow a\,,\ldots$$

- Finite volume: $\mathbf{p} = \frac{2\pi}{L}\mathbf{n},\;\; \mathbf{n} \in \mathbb{Z}^3$, poles of the T-matrix $\Rightarrow$ spectrum
- The Lüscher equation (in the absence of partial-wave mixing) (Lüscher, 1991):

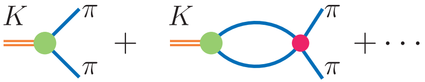
$$p\cot\delta(p) = \frac{2}{\sqrt{\pi}L}\,Z_{00}(1;q_0^2)$$

$\hookrightarrow$ measuring energy levels, one extracts phase shift at the same energy

NREFT serves as a bridge between finite and infinite volume

The Lellouch-Lüscher formula (Lellouch & Lüscher, 2001)

- Final-state interactions lead to an irregular L-dependence of the matrix element



- The non-relativistic Lagrangian

$$\begin{aligned}\mathscr{L} &= \phi^\dagger(i\partial_t - w)(2w)\phi + \frac{C_0}{4}\phi^\dagger\phi^\dagger\phi\phi + \cdots + K^\dagger(i\partial_t - w_K)(2w_K)K \\ &+ g(K^\dagger\phi\phi + \text{h.c.})\end{aligned}$$

- Calculate the decay matrix element in a finite and in the infinite volume, extract g
- Matrix elements are related through

$$|\langle n|H_W|K\rangle_L| = \underbrace{\Phi_2(L)}_{\textit{depends on phase shift}} |\langle \pi\pi; out|H_W|K\rangle_\infty|$$

Why three particles on the lattice?

- Three-pion decays of K, η, ω; $a_1(1260) \to \rho\pi \to 3\pi$; $a_1(1420) \to f_0(980)\pi \to 3\pi$
- Decays of exotica: $X(3872)$, $Y(4260)$, ...
- Roper resonance: πN and $\pi\pi N$ final states
- Few-body physics: reactions with the light nuclei

Lattice vs. infinite volume: observables

- Infinite volume: Three-particle bound states; Elastic scattering; Rearrangement reactions; Breakup; Three-particle resonances; Decay matrix elements (complex): e.g., $\langle \pi\pi\pi | H_W | K \rangle$
- Finite volume: Two- and three-particle energy levels; Matrix elements between eigenstates (real)

How does one connect these two sets? EFT serves as a bridge!

Three-particle quantization condition

- Is the three-particle spectrum determined solely in terms of the S-matrix?

  K. Polejaeva and AR, 2012: Yes!

- Three different but equivalent formulations of the three-particle quantization condition are available
  - RFT (Relativistic Field Theory): Hansen & Sharpe, 2014
  - NREFT (Non-Relativistic Effective Field Theory): Hammer, Pang & AR, 2017
  - FVU (Finite-Volume Unitarity): Mai & Döring, 2017

- Enables one to extract scattering observables in the three-body sector from the measured finite-volume spectrum

Particle-dimer picture

- Dimer: an alternative description of an infinite bubble sum; dummy field in the path integral
- Mathematically equivalent to the standard treatment – not an approximation

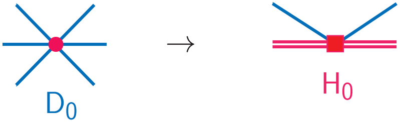
dimer :

- Particle-dimer Lagrangian:

$$\mathscr{L} = \phi^\dagger (i(v\cdot\partial) - w_v)(2w_v)\phi + \sigma T^\dagger T + \left(T^\dagger \left[\frac{f_0}{2}\phi\phi + \cdots \right] + \text{h.c.} \right)$$

- Matching: $f_0, \ldots \leftrightarrow C_0, \ldots \leftrightarrow a, r, \ldots, \quad \sigma = \pm 1.$
- v^μ is a unit vector in the direction of the total four-momentum of the three-particle system

Particle-dimer picture in the three-particle sector

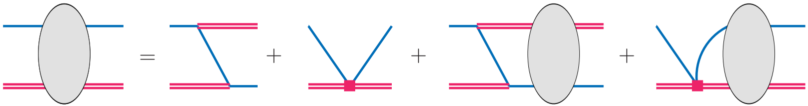


- The particle-dimer Lagrangian in the three-particle sector

$$\mathscr{L}_3 = h_0 T^\dagger T \phi^\dagger \phi + \cdots$$

- Matching: $h_0, \ldots \leftrightarrow D_0, \ldots$
- Terms with higher derivatives, higher dimer spin and orbital momentum should be added

The scattering equation in the infinite volume



Bethe-Salpeter equation

$$\mathcal{M}(p,q) = Z(p,q) + 8\pi \int \frac{d^3 k_\perp}{(2\pi)^3 2 w_v(k)} \theta(\Lambda^2 + m^2 - (vk)^2) Z(p,k) \tau(K-k) \mathcal{M}(k,q)$$

$$\tau(P) = \frac{2\sqrt{P^2}}{k^* \cot\delta(k^*) - ik^*} \qquad k^* = \sqrt{\frac{P^2}{4} - m^2}$$

$$Z(p,q) = \frac{1}{2w_v(K-p-q)(w_v(p) + w_v(q) + w_v(K-p-q) - (vK) - i\varepsilon)} + \tilde{H}_0 + \cdots$$

Relativistic invariant QC in the three-body sector

(F. Müller, J.-Y. Pang, AR and J.-J. Wu, JHEP 02 (2022) 158)

$$\mathcal{M}_L(p,q) = Z(p,q) + \frac{8\pi}{L^3}\sum_{\mathbf{k}} \theta(\Lambda^2 + m^2 - (vk)^2) Z(p,k) \frac{\tau_L(K-k)}{2w(\mathbf{k})} \mathcal{M}_L(k,q)$$

$$\tau_L(P) = \frac{2\sqrt{K^2}}{k^* \cot\delta(k^*) - \dfrac{2}{\sqrt{\pi}L\gamma} Z_{00}^{\mathbf{P}}(1; q_0^2)}\,, \qquad q_0 = \frac{k^* L}{2\pi}$$

$$Z(p,q) = \frac{1}{2w_v(K-p-q)(w_v(p)+w_v(q)+w_v(K-p-q)-(vK)-i\varepsilon} + \tilde{H}_0 + \cdots$$

- Quantization condition:

$$\det \mathscr{A} = 0\,, \qquad \mathscr{A}_{pq} = L^3 2w(\mathbf{p})\delta^3_{\mathbf{pq}}(8\pi\tau_L(K-p))^{-1} - Z(p,q)$$

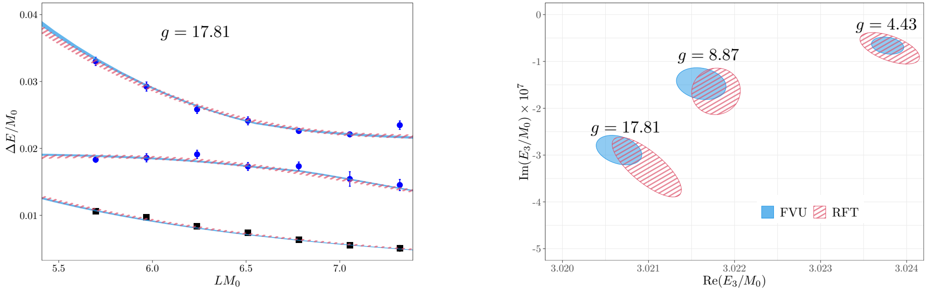
Quantization condition: essentials

- Two-body interactions as an input: $k^* \cot \delta(k^*)$ fitted in the two-particle sector
- Extracting short-range quantities encoded in the three-body couplings $\tilde{H}_0, \ldots$
  – should be fitted to the three-particle energies
- Finally, solve the equations in the infinite volume to arrive at the S-matrix elements!

Three-particle resonances

(M. Garofalo, M. Mai, F. Romero-Lopez, AR and C. Urbach, arXiv:2211.05605)

$$\mathscr{L} = \frac{1}{2}\sum_{i=0,1}\left(\partial_\mu\varphi_i\partial_\mu\varphi_i + m_i^2\varphi_i\varphi_i + 2\lambda_i(\varphi_i\varphi_i)^2\right) + \frac{g}{2}\left(\varphi_1^\dagger\varphi_0^3 + \text{h.c.}\right)$$



Three-particle decays

(F. Müller and AR, JHEP 03 (2021) 152, F. Müller, J.-Y. Pang, AR and J.-J.Wu, arXiv:2211.10126)

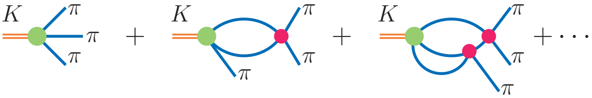
a) Decays through the weak or electromagnetic interactions; isospin-breaking decays: pole on the real axis

Example: $K \to 3\pi$

b) Decays through strong interactions, the pole moves into the complex plane

Example: $N(1440) \to \pi\pi N$

- Final-state interactions lead to the irregular volume-dependence in the matrix element



An analog of the LL formula in the three-particle sector?

The 3-particle LL factor

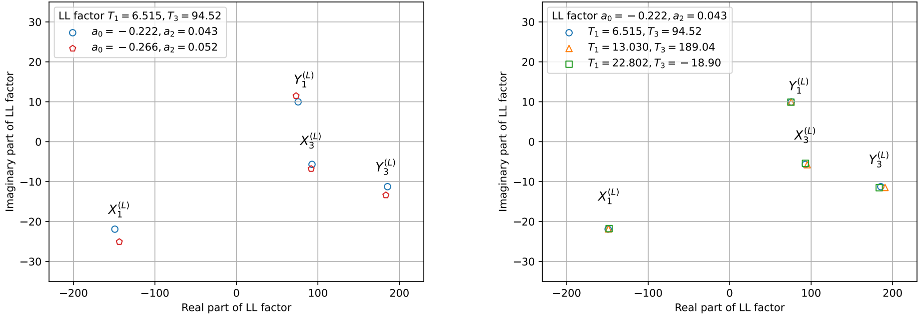
$$\langle \pi(k_1)\pi(k_2)\pi(k_3); out|H_W|K\rangle_\infty = \Phi_3(\{k\})L^{3/2}\langle n|H_W|K\rangle_L$$

- The factor $\Phi_3(\{k\})$ depends on the $\pi\pi$, $\pi\pi\pi$ interactions and on L, but not on the couplings that describe the short-range part of the $K \to 3\pi$ amplitude!
- The derivative couplings emerge at higher orders; decay amplitudes into different final states mix. The three-particle LL factor becomes a matrix

Apart from the two-body scattering parameters, the LL factor depends on the three-body force, which should be determined prior to calculating the matrix element. Is this dependence significant numerically?

Role of the three-body force in the LL factor

(J.-Y. Pang, R. Bubna, F. Müller, AR and J.-J. Wu, in preparation)



- 2×2 LL factor, corresponding to $K^+ \to \pi^+\pi^+\pi^-$ and $K^+ \to \pi^0\pi^0\pi^+$ decays
- Sensitive to the values of a_0, a_2, very little dependence on the three-body force!

Conclusions & outlook

- In the analysis of lattice data, EFT can be used to systematically relate the finite- and infinite-volume observables. This facilitates the extraction of scattering observables from lattice data
- The crucial point: decoupling of short- and long-range physics
- The quantization condition and an analog of the LL formula in the three-particle sector is derived
- Decays of K^+ into three pions: LL factor worked out explicitly to lowest order, Very little dependence on the three-body input!
- Outlook
  - Long-range forces in a finite volume: one-pion exchange, Coulomb force
  - Nuclear physics on the lattice
  - The Roper resonance
  - Boxed exotica