The relevance of surface terms in the action principle

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but the action is much more than that.

Its value has physical significance, and since surface terms might be non-zero, care must be exercised when dealing with them.

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I will work with the Hamiltonian form of the action $\int dt(p_i\dot{q}^i - H)+$ surface terms.

(In the case of field theory, $p_i\dot{q}^i = \int d^dx\pi_A\dot{\phi}^A$ and $H = \int d^dx\mathcal{H}$.)

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This is important quantum-mechanically, but can already be understood classically.

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$$\delta S = \int_{t_1}^{t_2} dt \left(\delta p_i(\dot{q}^i - \frac{\partial H}{\partial p_i}) + \delta q^i(-\dot{p}_i - \frac{\partial H}{\partial q^i})\right) + \left[p_i \delta q^i\right]_{t_1}^{t_2}$$

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If one were to fix the p_i's at the time boudaries ($\delta p_i = 0$), the action would not be an extremum unless $p_i(t_1) = p_i(t_2) = 0$, which eliminates most initial conditions !

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The boundary term $\left[p_i q^i \right]_{t_1}^{t_2}$ by which S and S' differ is in general non-zero and makes the difference.

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One goes from one to the other by Fourier transform, which is exactly what the surface term $e^{-\frac{i}{\hbar}\left((p_i q^i)(t_2)\right)}\,e^{\frac{i}{\hbar}\left((p_i q^i)(t_1)\right)}$ is needed for.

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We recall that in classical mechanics, the action $S[q(t)] = \int_{t_1}^{t_2} dt(p_i\dot{q}^i - H)$ is equal to $S = -E(t_2 - t_1)$ for a time-independent solution to the equations of motion ($\dot{q}^i = 0 = \dot{p}_i$; we assume the Hamiltonian to be time-independent).

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The problem is that we are using an action which is incorrect because it misses important surface terms at the space boundaries.

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In the Schwarzschild metric, this term involves the mass M, which is thus varied.

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It is equal to the mass M when evaluated on the Schwarzshild solution (for which the linear momentum P^k vanishes).

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If one were to fix M and leave $t_2 - t_1$ free, the original action would be ok.

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All the corresponding charge-generators are given by surface integrals at infinity.

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This fixes the black hole temperature to be $\frac{1}{8\pi M}$.

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THANK YOU !