Rapidity Renormalization Group
... and its Applications

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Outline

1. Motivation - Event Shapes
2. Soft Collinear Effective Field Theory (SCET)
3. $\eta$-Regulator and $\nu$-Renormalization Group
4. Numerics and Data
5. Other Applications
6. Conclusion
Angularities

... as a subclass of event shape observable

- Berger, Kucs, Sterman, 03

\[ \tau_a = \frac{1}{\sqrt{s}} \sum_{i \in X} E_i (\sin \theta_i)^a (1 - |\cos \theta_i|)^{1-a} \]

\[ = \frac{1}{\sqrt{s}} \sum_{i \in X} |\mathbf{p}_{i,\perp}| e^{-|\eta_i|(1-a)} \]

where rapidity \( \eta = \frac{1}{2} \log \left( \frac{E+\parallel}{E-\parallel} \right) \)

- Infrared safety: \(-\infty < a < 2\)

- Factorizability: \(-\infty < a < 1\?) (Hornig, Lee, Ovanesyan)

For \( a \ll 1 \), \( |\mathbf{p}_{\text{soft},\perp}|/|\mathbf{p}_{\text{coll.},\perp}| \sim e^{-\eta_{\text{coll.}}} \ll 1 \)

E.g., \( a=0 \), thrust distribution, \( \tau = 1 - T \).

For \( a \sim 1 \), \( |\mathbf{p}_{\text{soft},\perp}| \sim |\mathbf{p}_{\text{coll.},\perp}| \).

E.g., \( a=1 \), jet broadening observable, \( B \).
Factorization of Angularities

- Factorization Theorem in QCD
  - Collins, Soper, Sterman,...
  - Berger, Kucs, Sterman (03)
    Angularities for $a < 1$ been calculated to NLL/LO

- Factorization Theorem in Traditional SCET (SCET I)
  - Bauer, Manohar, Wise, Lee, Sterman, Becher, Schwartz, Fleming, Hoang, Mantry, Stewart, ...
  - Hornig, Lee, Ovanesyan Calculated angularities in SCET to NLL/LO for $a < 1$.

- Fail at $a=1$?! 
  - Spurious divergences in each sector ($a \geq 1$), yet disappear after summing over sectors as long as $a \leq 2$. 
  - Rapidity divergence

- Other cases with divergence in similar nature?
Soft Collinear Effective Theory (SCET I)
(Luke, Bauer, Fleming, Pirjol, Stewart)

- Describe interactions between energetic particles $E \sim Q$.
- Fluctuations, ($\Lambda_{QCD}$ or other low energy scales, about light cone coordinate $n = (1, 0, 0, 1)$

\[ p = (p \cdot \bar{n}, p \cdot n, p_{\perp}) \]
\[ = (p^-, p^+, p_{\perp}) \]

\[ k_s = (k_s^-, k_s^+, k_s,_{\perp}) \]
\[ \sim (\Lambda_{QCD}, \Lambda_{QCD}, \Lambda_{QCD}) \]

- Integrate out “far offshell” degrees of freedom.
  - soft-collinear decoupling

\[ p_{\perp} \sim \sqrt{Q\Lambda_{QCD}} \]
\[ p^+ \sim \Lambda_{QCD} \]
\[ p^- \sim Q \]
SCET degrees of freedom (modes)

\[ p^\mu = (p^-, p^+, p^\perp); \quad p^2 = p^+ p^- + p^\perp \]

- Light Cone Coordinates:
  \[ n = (1, \bar{n}) \sim (1, 0, 0, 1) \]

- Power counting parameter
  \[ \lambda \equiv \frac{\Lambda_{QCD}}{Q} \]

- Hard modes: \( p^2 \sim Q^2 \)
  integrated out

- \( n \)-collinear
  \[ p^\mu \sim Q(1, \lambda^2, \lambda) \]

- \( \bar{n} \)-collinear
  \[ p^\mu \sim Q(\lambda^2, 1, \lambda) \]

- Ussoft (\( p^2 \sim Q^2 \lambda^4 \))
  \[ p^\mu = Q(\lambda^2, \lambda^2, \lambda^2) \]
Matching from full QCD onto SCET

\[ j_i^\mu(x) = \sum_n \sum_{\tilde{p}_n, \tilde{p}_{\bar{n}}} C_{n\bar{n}}(\tilde{p}_n, \tilde{p}_{\bar{n}}; \mu) \mathcal{O}_{n,\bar{n}}(x; \tilde{p}_n, \tilde{p}_{\bar{n}}; \mu), \]

in which,

\[ \mathcal{O}_{n,\bar{n}}(x; \tilde{p}_n, \tilde{p}_{\bar{n}}; \mu) = e^{i(\tilde{p}_n - \tilde{p}_{\bar{n}})} \bar{\chi}_{n, p_n}(x) Y_n(x) \Gamma_i^\mu \bar{Y}_{\bar{n}}(x) \chi_{\bar{n}, p_{\bar{n}}}(x) \]

Write the event shape distribution in SCET in a factorized form

\[ \frac{d\sigma}{de} = \frac{1}{6Q^2} \sum_n |C_{n\bar{n}}(\tilde{p}_n, \tilde{p}_{\bar{n}}; \mu)|^2 \int dx \int de_n de_{\bar{n}} de_s \delta(e - e_n - e_{\bar{n}} - e_s) \frac{1}{N_C^2} \]

\[ \times \text{Tr} \langle 0 | \chi_{n, Q(x)}(x) \delta(e_n - \hat{e}_n) \bar{\chi}_{n, Q(0)} | 0 \rangle \text{Tr} \langle 0 | \bar{\chi}_{n, Q(0)}(x) \delta(e_{\bar{n}} - \hat{e}_{\bar{n}}) \chi_{n, Q(0)}(x) | 0 \rangle \]

\[ \times \text{Tr} \langle 0 | \bar{Y}_{\bar{n}}^\dagger(x) Y_n^\dagger(x) \delta(e_s - \hat{e}_s) Y_n \bar{Y}_{\bar{n}}(0) | 0 \rangle \sum_{i=V, A} L^i \left( (\Gamma_i^\mu) \right)_{\alpha\beta} (\Gamma_i^\mu)_{\gamma\delta} \]

Final result for differential event shape distribution

\[ \frac{1}{\sigma_0} \frac{d\sigma}{de} = H(s; \mu) \int de_n de_{\bar{n}} de_s \delta(e - e_n - e_{\bar{n}} - e_s) J_n(e_n; \mu) J_{\bar{n}}(e_{\bar{n}}; \mu) S(e_s; \mu) \]
When $a \rightarrow 1$

$$\tau_a = \frac{1}{\sqrt{s}} \sum_{i \in X} |p_{i,\perp}| e^{-|\eta_i|(1-a)}$$

- For $a \ll 1$, $e_s \sim e_n \sim e_{\bar{n}}$, but $|p^s_\perp| \ll |p^n_\perp| \sim |p^{\bar{n}}_\perp|$ and $|\eta^s_s| \ll |\eta^n_n| \sim |\eta^{\bar{n}}_{\bar{n}}|$.

  Soft radiation decoupled from the collinear radiation.

- For $a \rightarrow 1$, $\tau_{a=1} = \frac{1}{\sqrt{s}} \sum |p^i_\perp|$ is independent of the rapidity of the rapidity of each state. For the $e \sim e_s \sim e_n$, we need $|p^s_\perp| \sim |p^n_\perp|$.

\begin{align*}
S & : k_s \sim (p_T, p_T, p_T) \\
J_{\bar{n}} & : k_{\bar{n}} \sim (\sqrt{p^2_{\bar{T}}/Q}, Q, p_T) \\
J_n & : k_n \sim (Q, \sqrt{p^2_T/Q}, p_T)
\end{align*}
SCET Modes
Jet Broadening Event Shape

\[ B_T = \frac{1}{2} \tau_{a=1} = \frac{1}{2} \sum \frac{|\vec{k}_i,\perp|}{Q} \]

Demanding \( B_T \ll 1 \) for dijet events

Relevant on-shell modes with \(|\vec{p}_t| \sim \lambda Q\)

soft modes: \( k_s \sim Q(\lambda, \lambda, \lambda) \)
collinear modes: \( k_n \sim Q(1, \lambda^2, \lambda) \)
\( k_{\bar{n}} \sim Q(\lambda^2, 1, \lambda) \)

Invariant massess of soft and collinear modes are on the same order \( \mathcal{O}(Q^2 \lambda^2) \)

Rapidity divergence arises as jets go soft or soft radiations go collinear since they are all on the same parabola.
Factorization Theorem for Jet Broadening $B_T = \frac{1}{2} \tau_{a=1}$

Define $e \equiv \tau_{a=1} = 2B_T$ from now on to simplify the notification

$$
\frac{d\sigma}{de} = \sigma_0 H(s) \int d\epsilon_n d\epsilon_{\bar{n}} d\epsilon_s \delta(e - \epsilon_n - \epsilon_{\bar{n}} - \epsilon_s) \\
\times \int d\vec{p}_{1t} d\vec{p}_{2t} J_n(Q_-, e_n, \vec{p}_{1t}) J_{\bar{n}}(Q_+, e_{\bar{n}}, \vec{p}_{2t}) S(\epsilon_s, \vec{p}_{1t}, \vec{p}_{2t}),
$$

where in covariant guages

$$(\bar{d} \equiv 2 - 2\epsilon)$$

$$J_n = \frac{2\pi \Omega d}{N_c} \langle 0 | \chi_n \delta(\hat{P}^- - Q^-) \delta(\hat{\epsilon} - \epsilon_n) \delta(\hat{P}_\perp + \vec{p}_1) \frac{\vec{n}}{2} \chi_n | 0 \rangle,$$

$$J_{\bar{n}} = \frac{2\pi \Omega d}{N_c} \langle 0 | \frac{\vec{n}}{2} \chi_{\bar{n}} \delta(\hat{P}^+ - Q^+) \delta(\hat{\epsilon} - \epsilon_{\bar{n}}) \delta(\hat{P}_\perp + \vec{p}_2) \bar{\chi}_{\bar{n}} | 0 \rangle,$$

$$S = \rho_1^{1-2\epsilon} \rho_2^{1-2\epsilon} \Omega d \int \frac{d\Omega_{12}}{N_c} \times
\langle 0 | S_n^\dagger S_{\bar{n}} \delta(\hat{\epsilon} - \epsilon_s) \delta(\hat{P}_{n\perp} - \vec{p}_1) \delta(\hat{P}_{\bar{n}\perp} - \vec{p}_2) S_n^\dagger S_n | 0 \rangle.$$

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Bare jet function:

\[ J_n(e_n, p_i, t = 0) = \frac{\alpha_s C_F}{\pi} \left( \frac{\mu^2}{Q^2 e_n^2} \right)^\epsilon \frac{1}{e_n} \int_0^1 \frac{1 + (1 - z)^2}{z} \, dz, \]

where \( z \equiv l^-/Q \), and \( l \) is the momentum of the gluon going across the cut.

Integral ill-defined as \( z \to 1 \), the soft region.

Leftover \( \frac{1}{\epsilon} \) divergence multiplies non-zero \( e_n \) terms that virtual diagrams, which are always proportional to \( \delta(e_n) \), cannot cancel.

Traditional dim-reg regulating the \( \vec{k}_\perp \) part of the real radiation dose not regulate the phase space integral while \( p_T \) is fixed.
New Regulator and $\nu$-Renormalization Group

Goal:

- Multiplicatively Renormalizable
- In the spirit of dimensional regularization
- Does not introduce new dimensionful scales in the integrants, and maintains manifest power counting in the effective theory.

$\eta$-regulator

$$W_n = \left[ \sum_{\text{perm}} \exp \left( -g \frac{\vec{n} \cdot \hat{P}}{\vec{n} \cdot \hat{A}_{n,\vec{q}(0)}} \right) \right]$$

$$S_n = \left[ \sum_{\text{perm}} \exp \left( -g \frac{\vec{n} \cdot \hat{P}}{\vec{n} \cdot \hat{A}_{s,\vec{q}(0)}} \right) \right]$$
\[\eta \text{-Regulator}\]

\[
\mu^\eta \frac{g^m}{m!} \prod_{i=1}^{m} |\vec{n} \cdot q_i|^{-\eta} \sum_{\text{perms}} \frac{\mu^\kappa T^A \ldots \mu_1 T^{A_1}}{[\vec{n} \cdot q_1][\vec{n} \cdot (q_1 + q_2)] \ldots [\vec{n} \cdot \sum_{j=1}^{m} q_j]}
\]

- Regulates the energy of each gluon coming off Wilson Line.
- Preserves Exponentiation Theorems.
- Preserves modes and their power counting.
- Dim-reg. style evolution equations.
- Does not hide soft functions, as analytic regulators do.
- \(\eta\) contribution spontaneously goes to zero when rapidity divergence is not present.
Regulator Comparison

- **Analytic regulator**
    - Hides soft contributions (although still get fix-order matrix element correct after summing different sectors).
    - Each collinear sector has complicated regulator dependent and is meaningless before summing all sectors together...
      ⇒ breaks factorization
    - Does not exponentiation.

- **$\triangle$-regulator**
  - JC, Fuhrer, Hoang, Kelley, Manohar, 09
    - Introduces additional scales into integrals
    - Exponentiates after proper zero-bin subtraction
    - No known evolutions equation.

- **Off the light cone**
  - Collins, Soper, 81
    - Introduces more scales into integrals
    - Proof of exponentiation straightforward
    - Introduces gauge modes that are not appropriate by strict power counting.
    - Understood evolution equation.
Jet Function Calculation with $\eta$-regulator

Total jet function in $n$-direction including real and virtual yields

$$J_n = \frac{\alpha_s C_F}{\pi} \left\{ \frac{1}{2} \frac{1 - \epsilon}{\Gamma(1 - \epsilon)} \left( \frac{\mu^2 e^{\gamma_E}}{Q^2} \right)^\epsilon \frac{1}{2} \left[ \left( \frac{e_n}{2} \right)^{-1-2\epsilon} \right]^{(+\infty)} \right. $$

$$+ 2 \frac{\Gamma(-\eta)}{\Gamma(1 - \epsilon)\Gamma(2 - \eta)} \left( \frac{\mu^2 e^{\gamma_E}}{Q^2} \right)^\epsilon \left( \frac{\nu}{Q} \right) \eta \frac{1}{2} \left[ \left( \frac{e_n}{2} \right)^{-1-2\epsilon} \right]^{(+\infty)} \right\}$$
Soft Function Calculation with \( \eta \)-regulator

\[
S = \frac{\alpha_s(\mu) C_F}{\pi} \frac{\sqrt{\pi} \Gamma(\eta) e^{\gamma_E}}{2^{2\eta} \Gamma(1/2 + \eta) \Gamma(1 - \epsilon)} \left( \frac{\mu}{Q} \right)^{2\epsilon} \left( \frac{\nu}{Q} \right)^{2\eta} \left[ \frac{1}{\epsilon_s^{1+2\epsilon+2\eta}} \right]^{(\infty)} + \times \left\{ \delta(p_1 t - Qe_s)\delta(p_2 t) + \delta(p_1 t)\delta(p_2 t - Qe_s) \right\}.
\]

\[
V_s(e, \vec{p}, \mu, \nu, \nu_0) = (4\pi) \theta(e) \theta(eQ - p_t) \frac{\zeta e^{\gamma_E \zeta}}{\Gamma(-\zeta)} \frac{Q}{\mu(eQ)^2} \left( \frac{\mu eQ}{e^2 Q^2 - p_t^2} \right)^{1+\zeta}
\]
Jet and Soft Function Contributions to the cross-section

For integrated cross-section with a cut on jet broadening, define

\[ \Sigma(e_c \ll 1) \equiv \int_0^{e_c} de \frac{d\sigma}{de}. \]

We have

- Contribution from jet functions

\[ \frac{1}{\sigma_0} \Sigma_{jet}(e_c) = -\frac{\alpha_s C_F}{2\pi} \ln \left( \frac{\sqrt{se_c}}{2\mu} \right) \left[ 3 + 4 \ln \left( \frac{\nu}{Q^{\pm}} \right) \right] \]

  ▶ To minimize the log, the natural scales for jet functions are: \( \mu_J = \frac{1}{2} \sqrt{se_c} \), and \( \nu_J = Q^{\pm} \).

- Contribution from soft function

\[ \frac{1}{\sigma_0} \Sigma_{soft}(e_c) = \frac{\alpha_s C_F}{\pi} \left[ -2 \ln^2 \left( \frac{\sqrt{se_c}}{2\mu} \right) + 4 \ln \frac{\nu}{\mu} \ln \left( \frac{\sqrt{se_c}}{2\mu} \right) \right] \]

  ▶ Natural scales for soft functions are: \( \nu_s = \mu_s = \frac{1}{2} \sqrt{se_c} \)
Running Strategy: 2-Parameter RG
\( \eta \)-divergences and \( \nu \)-anomalous dimensions cancels when we sum up the contributions from the jet and soft functions.

Individual \( J \) and \( S \) are multiplicatively renormalizable.

\( \eta \)-divergence are absorbed in the renormalization constants, \( Z_{J,S} \), such that

\[
J_{n}^{(0)} J_{\bar{n}}^{(0)} S^{(0)} = Z_{J_{n}}(\mu, \nu) J_{\bar{n}}^{R}(\mu, \nu) \left[ Z_{J_{\bar{n}}}^{R}(\mu, \nu) J_{\bar{n}}^{R}(\mu, \nu) \right] Z_{S}(\mu, \nu) S^{R}(\mu, \nu),
\]

where

\[
Z_{J_{n}}(\mu, \nu) Z_{J_{\bar{n}}}^{R}(\mu, \nu) Z_{S}(\mu, \nu) = Z_{H}^{-1}(\mu).
\]
The rapidity divergences for the jet and soft functions introduce a new set of anomalous dimension \( (\gamma_J^\nu, \gamma_S^\nu) \):

\[
\nu \frac{d}{d\nu} S^R(\mu, \nu) = \gamma_S^\nu S^R(\mu, \nu),\quad \nu \frac{d}{d\nu} J_n^R(\mu, \nu) = \gamma_J^\nu J_n^R(\mu, \nu)
\]

Just like the traditional \( \mu \) anomalous dimension:

\[
\mu \frac{d}{d\mu} S^R(\mu, \nu) = \gamma_S^\mu S^R(\mu, \nu),\quad \text{and} \quad \mu \frac{d}{d\mu} J_n^R(\mu, \nu) = \gamma_J^\mu J_n^R(\mu, \nu)
\]

Since the cross-section is invariant under \( \mu \) and \( \nu \) variation, and that the hard function itself is free from rapidity divergence (and therefore \( \gamma_H^\nu = 0 \)), we must have the relations

\[
\gamma_H^\mu + \gamma_{J_n}^\mu + \gamma_{J_n}^\mu + \gamma_S^\mu = 0, \quad \text{and} \quad \gamma_J^\nu + \gamma_{J_n}^\nu + \gamma_S^\nu = 0.
\]
Running Strategy

- **Natural scales:**
  - hard function: independent of \( \nu, \mu_H = \sqrt{s} \)
  - soft function \( (\nu_S, \mu_S) = (\sqrt{s}e, \sqrt{s}e) \)
  - jet functions \( (\nu_J, \mu_J) = (\sqrt{s}, \sqrt{s}e) \)

- **Running**
  - In \( \mu \): Evolve hard function from high scale \( \mu_H = \sqrt{s} \) to common low scale \( \mu_J = \mu_S = \sqrt{se} \)
  - In \( \nu \): Evolve soft function from \( \nu_S = \sqrt{se} \) to jet scale \( \nu_J = \sqrt{s} \)
Solution to the $\mu$-RGE for the hard function

$$H(s, \mu) = U_H(s; \mu_H, \mu)H(s; \mu_H)$$

with

$$U_H(s; \mu_H, \mu) = \left| e^{K_H(\mu_H, \mu)} \left( \frac{-s - i0}{\mu_H^2} \right)^{\eta_H(\mu_H, \mu)} \right|^2$$

Solution to the $\nu$-RGE for the soft function

$$S(\mu, \nu) = V_s \left( \mu, \frac{\nu}{\nu_s} \right) \otimes S(\mu, \nu_S)$$

with

$$V_s(e, \vec{p}, \mu, \nu) = (4\pi)\theta(e)\theta(eQ - p_t) \frac{-\omega_s e^{-\gamma E\omega_s}}{\Gamma(-\zeta)} \frac{Q}{\mu(eQ)^2} \left( \frac{\mu eQ}{e^2Q^2 - p^2} \right)^{1-\omega_s}$$

and

$$\omega_s \left( \mu, \frac{\nu}{\nu_s} \right) = 2\Gamma_{cusp}[\alpha_s(\mu)] \log \frac{\nu}{\nu_s}.$$
NLO singular cross-section ($e \equiv 2B_T$)

$$\frac{1}{\sigma_0} \frac{d\sigma}{de} = \frac{1}{e} \frac{\alpha_s(\mu)}{\pi} C_F (-3 - 4 \log \frac{e}{2})$$

Resummed cross-section up to NLL

$$\frac{1}{\sigma_0} \frac{d\sigma}{de} = \frac{1}{e} \frac{U_H(s, \mu_H, \mu)}{\Gamma(2\omega_S) 4e^{2\gamma_E\omega_S}} \left( \frac{eQ}{\mu} \right)^{2\omega_S} \left[ 1 - 2F_1(1, 1; 1 + \omega_S; -1) \right]^2$$
Comparison with literature...

QCD calculations with resummation in the literature:

1. Catani, Turnock and Webber (1992) ⇒ Correct up to NLL. Agree with our NLL result in ref.

2. Dokshitzer, Lucenti, Marchesini and Salam (1998) ⇒ Corrections to previous result in [1] at NLL’ or NNLL

SCET resummation for $B_T$ after our letter...

Comparison with data

\[ Q = 130 \text{ GeV} \]

\[ \frac{d\sigma}{dB_T} \text{ (nbarn)} \]

Data from L3

- NLL
- LL
- NLO sing.

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Rapidity divergences do not only appear when observing Jet Broadening...

- $p_T$ spectrum for Higgs/Drell-Yan Production
  - JC’s talk in the Higgs session
- TMD-PDF, Generalized Parton Distribution
- Electroweak corrections to high energy process
- ...

In general, processes or observables involving collinear and soft mode with similar transverse momentum or off-shellness.
Conclusion

- When measuring transverse momentum related observables...
  - Soft contributions are important
  - Uncanceled divergences remain in each sector, rapidity divergence.
  - New kind of logarithms to resum, yet related to the cups angle (the high scale).
- There are other cases with rapidity divergence such as, higgs $p_T$ distribution, and electroweak corrections to LHC processes.
- Rapidity RG making use of the $\eta$-regulator provides controllable form to divergences, and a way to resum the log systematically.